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On pseudo-injective and pseudo-projective modules

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Abstract. In this work we obtain characterizations of QI rings and semisimple rings using quasi-injective and pseudo-injective modules respectively. We define and construct the pseudo-injective hull of a module and we give sufficient conditions on a ring to have the following properties: every pseudo-injective module is pseudo-projective and every pseudo-projective module is pseudo-injective. We also give some properties of the big lattice of classes of modules being closed under submodules and quasi-injective hulls.

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1 Introduction

In this paper R denotes an associative ring with unitary element 1 and R-Mod denotes the category of left unital modules over R, to which all "R-modules" and "modules" will belong. Recall that a ring R is called *quasi-Frobeinus* if it is left noetherian and left self-injective and is called a left *V-ring* if and only if every simple R-module is injective. Also, an *artinian principal ideal ring* is a left and right artinian, left and right principal ideal ring.

A module M is called *quasi-injective* if for every submodule N of M, every morphism $f : N \to M$ can be extended to a morphism $g : M \to M$. Quasi-injective modules have been extensively studied in [13, 15, 16]. It is well known that a module M is quasi-injective if and only if M is invariant under any endomorphism of its injective hull. Also, a ring R is a QI ring if each quasi-injective module is injective, see [11]. In this paper we explore those rings more closely.

A module M is called *pseudo-injective* if for every submodule N of M, every *monomorphism* $f : N \to M$ can be extended to a morphism $g : M \to M$. Pseudo-injective modules have been studied in [9, 18–21]. Recently, in [10] the authors proved that a module M is pseudo-injective if and only if M is invariant under any automorphism of its injective hull. Here, we explore rings for which every module is pseudo-injective and the property: every pseudo-injective module is injective.

In studying rings and modules, lattices of module classes have been used to obtain information about the internal structure of the ring and about the module categories associated, see for example [14] and [8]. Let $p_1, \ldots, p_n$ be symbols denoting closure properties, such as $\leq$ (submodules), $/$ (quotients), ext (extensions), Π (direct products), $\oplus$ (direct sums) or $E(\_)$ (injective hulls). By $\mathscr{L}_{p_1,\ldots,p_n}$ we mean the (big) lattice[1] of non-empty abstract[2] module classes closed under each of $p_1, \ldots, p_n$, ordered by class inclusion. Note that infima in $\mathscr{L}_{p_1,\ldots,p_n}$ are intersections. As usual, for an arbitrary class $\mathcal{A}$ of modules, $\xi_{p_1,\ldots,p_n}(\mathcal{A})$ will denote the least element of $\mathscr{L}_{p_1,\ldots,p_n}$ containing $\mathcal{A}$.

Thus, *R-her* $= \mathscr{L}_{\leq}$ (the big lattice of *hereditary* classes), *R-quot* $= \mathscr{L}_{/}$ (the big lattice of *cohereditary* classes), *R-open* $= \mathscr{L}_{\leq,/}$ (the big lattice of *open* classes), *R-wis* $= \mathscr{L}_{\leq,/,\oplus}$ (the lattice of *hereditary pretorsion*, or *Wisbauer*, classes), *R-serre* $= \mathscr{L}_{\leq,/,\mathrm{ext}}$ (the big lattice of *Serre subcategories*) and *R-nat* $= \mathscr{L}_{\leq,\oplus,E(\_)}$ (the lattice of *natural* classes, see [8]). Information about these lattices can be found in [1] (R-her, R-quot, R-nat and R-conat), [3] (more on R-conat), [5, Section 2.3] (R-nat), [5, Section 2.5] (R-prenat) and [8] (R-wis and the module class σ[M]).

Recall that a big lattice is *bounded* if and only if it has a least and a greatest element, and that b is a *pseudocomplement* (respectively, a *strong pseudocomplement*) of a in a big lattice $\mathscr{L}$ with least element 0 if and only if b is maximal (resp., greatest) in $\mathscr{L}$ with respect to $a \wedge b = 0$. Note that, if a has a strong pseudocomplement b in $\mathscr{L}$, then b is the only pseudocomplement in $\mathscr{L}$ of a. If $\mathcal{C} \in \mathscr{L}_{p_1,\ldots,p_n}$, we denote by $\mathcal{C}^{\perp_{p_1,\ldots,p_n}}$ a pseudocomplement of $\mathcal{C}$ in $\mathscr{L}_{p_1,\ldots,p_n}$. Furthermore, a big lattice is called *pseudocomplemented* (respectively, *strongly pseudocomplemented*) if and only if every one of its elements has a pseudocomplement (resp., a strong pseudocomplement). The class of pseudocomplements of a bounded big lattice $\mathscr{L}$ is called the *skeleton* of $\mathscr{L}$ and is denoted by Skel($\mathscr{L}$). It is known for example that *R-nat* $=$ Skel(*R-her*). In this article we study the lattices $\mathscr{L}_{\leq,E}$ and $\mathscr{L}_{\leq,E_Q}$, that is, the lattices of module classes closed under taking submodules and injective hulls and under taking submodules and quasi-injective hulls respectively. Finally, we study when $\mathscr{L}_{\leq,E} = \mathscr{L}_{\leq,E_Q}$.

In the sequel by QF we mean quasi-Frobenius. Also, $N \leq_e M$ will stand for N is essential in M and $E(M)$ denotes the injective hull of M.

2 The big lattices $\mathscr{L}_{\leq,E}$ and $\mathscr{L}_{\leq,E_Q}$

Recall that a module M is called quasi-injective if for each submodule N of M and each morphism $f : N \to M$ there exists $g \in End(M)$ such that the following diagram commutes:

$$\begin{tikzcd} N \arrow[r, hook, "i"] \arrow[d, "f"'] & M \arrow[dl, "g"] \\ M & \end{tikzcd}$$

[1]The only difference between a lattice and a *big lattice* is that the first one is a set, or at least cardinable (that is, bijectable with some set), while the second one is not necessarily so.

[2]That is, closed under isomorphisms.

Following [17] we say that a submodule $N \leq M$ is fully invariant in M if $f(N) \subseteq N$, for every $f \in End(M)$.

The following is a well known lemma due to Johnson and Wong:

Lemma 1. *A module M is quasi-injective if and only if it is fully invariant in its injective hull $E(M)$.*

It is known that every module M has a quasi-injective hull denoted by $E_Q(M)$. That is that $E_Q(M)$ is the minimal essential extension of M which is quasi-injective. In fact $E_Q(M)$ can be described as $E_Q(M) = \sum\{f(M) \mid f \in End(E(M))\}$. Thus $M \leq_e E_Q(M) \leq_e E(M)$ and $E(E_Q(M)) = E(M)$.

Let $\mathscr{C} \subseteq R - Mod$ be a non-empty abstract class of modules. $\mathscr{C}$ belongs to $\mathscr{L}_{\leq, E_Q}$ if $\mathscr{C}$ is closed under taking submodules and quasi-injective hulls.

Observation 1. Note that $\mathscr{L}_{\leq, E} \subset \mathscr{L}_{\leq, E_Q}$ and the inclusion can be proper. For instance, in $\mathbb{Z} - Mod$, for each prime number p, the class $\{N \in \mathbb{Z} - Mod \mid$ there exists a monomorphism $G \rightarrowtail \mathbb{Z}_p\} \in \mathscr{L}_{\leq, E_Q}$, but it does not belong to $\mathscr{L}_{\leq, E}$.

Example 1. Since every semisimple module M is quasi-injective, $\mathscr{C} = \{M \mid M$ is a semisimple module$\} \in \mathscr{L}_{\leq, E_Q}$.

Example 2. Each natural class and therefore each hereditary torsion free class belongs to $\mathscr{L}_{\leq, E_Q}$.

Lemma 2. *If $\mathscr{C} \subseteq R - Mod$, then:*

1) $\xi_{\leq, E}(\mathscr{C}) = \{M \mid$ *there exists a monomorphism* $M \rightarrowtail E(C)$ *for some* $C \in \mathscr{C}\}$;

2) $\xi_{\leq, E_Q}(\mathscr{C}) = \{M \mid$ *there exists a monomorphism* $M \rightarrowtail E_Q(C)$ *for some* $C \in \mathscr{C}\}$.

Proof. 1) See [2, Remark 2.5].

2) Let $A = \{M \mid$ there exists a monomorphism $M \rightarrowtail E_Q(C)$ for some $C \in \mathscr{C}\}$. It is clear that A is closed under taking submodules. Now, take $M \in A$. Then there exists a monomorphism $M \rightarrowtail E_Q(C)$, for some $C \in \mathscr{C}$. Since $E_Q(M)$ is the least quasi-injective module containing M and $E_Q(C)$ being quasi-injective, there exists a monomorphism $E_Q(M) \to E_Q(C)$ with $C \in \mathscr{C}$. Therefore $E_Q(M) \in A$, and $A \in \mathscr{L}_{\leq, E_Q}$. Clearly $\mathscr{C} \subseteq A$. Finally, consider $\mathscr{D} \in \mathscr{L}_{\leq, E_Q}$ such that $\mathscr{C} \in \mathscr{D}$. If $M \in A$, then there exists a monomorphism $M \rightarrowtail E_Q(C)$, for some $C \in \mathscr{C}$. As $C \in \mathscr{D}$, then $E_Q(C) \in \mathscr{D}$. Hence $M \in \mathscr{D}$. Therefore $A = \xi_{\leq, E_Q}(\mathscr{C})$.

$\square$

The following theorem can be found in [3].

Theorem 1. *Let P, Q be sets of closure properties such that $\mathscr{L}_P$ and $\mathscr{L}_Q$ are strong pseudocomplemented (big) lattices. If $Skel(\mathscr{L}_P) \subseteq \mathscr{L}_Q \subseteq \mathscr{L}_P$ then $Skel(\mathscr{L}_Q) = \mathscr{L}_P$.*

As observed in [1] $Skel(\mathscr{L}_{\{\leq\}}) = \mathscr{L}_{\leq,\oplus,E(\_)} = R - nat$ whose elements are called natural classes, which are also known to be classes closed under extensions. The following proposition is a consequence of Theorem 1.

Proposition 1. *$\mathscr{L}_{\{\leq\}}, \mathscr{L}_{\{\leq,E\}}, \mathscr{L}_{\{\leq,E_Q\}}$, all have the same skeleton: R-nat. Moreover, each pseudocomplement in these lattices is strong. Thus the skeletons coincide with the strong skeletons. In each one of these lattices, the strong pseudocomplement is given by the class of modules which has no non-zero submodules in $\mathscr{C}$.*

Note that in these lattices, $\mathscr{L}_{\leq,E}, \mathscr{L}_{\leq,E_Q}$, the suprema are given by class union. Thus both lattices are frames and co-frames. Unfortunately, both of them are boolean lattices only when the ring is trivial. In [4, Theorem 2.10] the authors proved the above assertion for the lattice $\mathscr{L}_{\leq,E}$, with a similar argument it can be proved for the lattice $\mathscr{L}_{\leq,E_Q}$.

Example 3. For the rings $\mathbb{Z}$ of integers, the class of torsion abelian groups $\mathbb{T}$ and the class of torsion free abelian groups $\mathbb{F}$ are strong pseudocomplements one of each other in $\mathscr{L}_{\leq,E_Q}$.

Example 4. Let R be a commutative domain. Then all non-zero ideals of R are essential. Therefore for any module M, its singular submodule is precisely its torsion submodule. In particular M is singular if and only if M is torsion, and M is nonsingular if and only if M is torsion free. Thus, for this type of ring the Goldie torsion theory $(\mathbb{T}_{\tau_G}, \mathbb{F}_{\tau_G})$ is given by $\mathbb{T}_{\tau_G} = \{M \mid M \text{ is singular}\}$ and $\mathbb{F}_{\tau_G} = \{M \mid M \text{ is nonsingular}\}$. Since the Goldie torsion theory is stable both classes are natural classes. Note that $\mathbb{F}_{\tau_G}, \mathbb{T}_{\tau_G} \in \mathscr{L}_{\leq,E_Q}$ and each one is the strong pseudocomplement of the other in $\mathscr{L}_{\leq,E_Q}$. Since $\mathbb{F}_{\tau_G} \cup \mathbb{T}_{\tau_G} \subsetneq R - Mod$ none of them has a complement.

3 QI-Rings

Recall that a ring R is called a left QI ring provided that every quasi-injective left module is injective. As every semisimple module is quasi-injective, then a left QI ring is a left noetherian V-ring. See [11].

Example 5. Clearly any semisimple ring is QI.

In [6] the author proved the following proposition:

Proposition 2. *If R is a left V-ring, a left noetherian ring for which left ideals are principal and if R has the ascending chain condition on principal right ideals then R is a left QI ring.*

Example 6. Let k be a universal differential field with derivation D. We denote $R = k[y, D]$ the ring of differential polynomials in the indeterminate y with coefficients in k. Then in [7] the author proved that R is a simple noncommutative domain and right V-ring having all its one-sided ideals principal, which is not a division ring. Now take R^* the opposite ring of R. Then R^* is a left V-ring which satisfies the hypothesis of Proposition 2. Therefore R^* is a left QI ring.

Definition 1. A module M is called *fully invariant semisimple ($f.i.$-semisimple)* if every fully invariant submodule N of M is a direct summand.

Example 7. Every semisimple module is $f.i.$-semisimple.

Definition 2. A non-zero module M is called *fully invariant simple ($f.i.$-simple)* if the only fully invariant submodules are trivial.

Observation 2. If every injective module is $f.i$ simple, then R is a V-ring.

Proof. Let S be a simple module. As S is quasi-injective, then it is fully invariant in $E(S)$. Therefore $S = E(S)$. □

Note that every $f.i.$-simple module is $f.i.$-semisimple.

Theorem 2. *The following statements are equivalent:*

1) R is a QI ring.

2) Direct sum of two quasi-injective modules is quasi-injective.

3) Every injective module is $f.i$-semisimple.

Proof. 1) $\Rightarrow$ 2) Let M and N be two quasi-injective modules. By hypothesis they are injective. Thus $M \oplus N$ is injective and therefore quasi-injective.

2) $\Rightarrow$ 1) See [16, Theorem 2.2].

1) $\Rightarrow$ 3) Let M be an injective module and N a fully invariant submodule of M. By hypothesis, it suffices to show that N is a quasi-injective module. Indeed, let f be an endomorphism of $E(N)$. Then f can be extended to the following commutative diagram:

$$\begin{tikzcd} E(N) \arrow[r, hook] \arrow[d, "f"'] & E(M) = M \arrow[d, "g"] \\ E(N) \arrow[r, hook] & E(M) = M \end{tikzcd}$$

As N is fully invariant in M, then $f(N) = g(N) \leq N$. Therefore by Lemma 1, N is quasi-injective.

3) $\Rightarrow$ 1) Let M be a quasi-injective module. As M is fully invariant in $E(M)$, then M is a direct summand of $E(M)$ by hypothesis. Hence $M = E(M)$. Therefore R is a QI ring. □

Lemma 3. *The following statements are equivalent:*

1) $\mathscr{L}_{\leq,E} = \mathscr{L}_{\leq,E_Q}$.

2) $\xi_{\leq,E}(M) = \xi_{\leq,E_Q}(M)$ for each module M.

Proof. 1) $\Rightarrow$ 2) Let M be a module. As $\xi_{\leq,E} \in \mathscr{L}_{\leq,E_Q}$ and contains M, then $\xi_{\leq,E}(M)$. Symmetrically $\xi_{\leq,E(M)} \leq \xi_{\leq,E_Q}(M)$.

2) $\Rightarrow$ 1) Let $\mathscr{C}$ be a class in $\mathscr{L}_{\leq,E}$. If $M \in \mathscr{C}$, then $\xi_{\leq,E}(M) \in \mathscr{L}_{\leq,E}$ and contains M. Thus $\xi_{\leq,E_Q}(M) \subseteq \mathscr{C}$. Hence $E_Q(M) \in \mathscr{C}$. Therefore $\mathscr{C} \in \mathscr{L}_{\leq,E_Q}$, concluding $\mathscr{L}_{\leq,E} \subseteq \mathscr{L}_{\leq,E_Q}$. Similarly we have the other inclusion. □

Corollary 1. *If R is a QI ring, then $\mathscr{L}_{\leq,E} = \mathscr{L}_{\leq,E_Q}$.*

Proof. By Lemmas 2 and 3, it suffices to show that $E(M) = E_Q(M)$ for each module M. Indeed $E_Q(M)$ is fully invariant in $E(E_Q(M)) = E(M)$. By Theorem 2, $E_Q(M)$ is a direct summand of $E(M)$. Since $E_Q(M) \leq_e E(M)$, then $E_Q(M) = E(M)$. □

4 Pseudo-injective modules

Recall that a module M is called pseudo-injective if for each submodule N of M and each monomorphism $f : N \to M$ there exists $g \in End(M)$ such that the following diagram commutes:

$$\begin{tikzcd} N \arrow[r, hook, "i"] \arrow[d, "f"'] & M \arrow[dl, "g"] \\ M & \end{tikzcd}$$

Following [10], we say that a module M is an automorphism-invariant module if M is invariant under any automorphism of its injective hull.

In [10], the authors proved the following theorem:

Theorem 3. *A module M is automorphism-invariant if and only if it is pseudo-injective.*

Definition 3. A *pseudo-injective hull* of a module M is a pseudo-injective module $P \supseteq M$ which is minimal with respect to these properties.

Observation 3. The class of pseudo-injective modules is an abstract class.

Theorem 4. *Any module M has a pseudo-injective hull.*

Proof. Let M be a module and

$$P = \sum\{f(M) \mid f \in Aut(E(M))\}.$$

As $M \leq_e P \leq_e E(M)$, then $E(P) = E(M)$. Let $g \in Aut(E(M))$, then

$$\begin{aligned} g(P) &= g\left(\sum\{f(M) \mid f \in Aut(E(M))\}\right) \\ &= \sum\{(gf)(M) \mid f \in Aut(E(M))\} \leq P. \end{aligned}$$

Thus, by Theorem 3 P is pseudo-injective. We now take a pseudo-injective module N containing M. Then $E(M) \leq E(N)$. Therefore there exists a submodule $L \leq E(N)$ such that $E(M) \oplus L = E(N)$. Then for each automorphism f of $E(M)$ we obtain an automorphism $g : E(N) \to E(N)$ defined by $g(m + l) = f(m) + l$. So we have the following commutative diagram:

$$\begin{tikzcd} E(M) \arrow[r, hook] \arrow[d, "f"'] & E(M) \oplus L = E(N) \arrow[d, "g"] \\ E(M) \arrow[r, hook] & E(M) \oplus L = E(N) \end{tikzcd}$$

Hence $f(M) = g(M) \subseteq g(N) \leq N$. Therefore $P \leq N$. □

Definition 4. A module M is called *automorphism invariant semisimple (a.i.-semisimple)* if every automorphism invariant submodule N of M is a direct summand.

The following Lemma can be found in [15] and [9].

Lemma 4. *A module M is quasi-injective if and only if $M \oplus M$ is pseudo-injective.*

Theorem 5. *The following statements are equivalent:*

1) Every pseudo-injective module is injective.

2) Direct sum of two pseudo-injective modules is pseudo-injective.

3) Every injective module is a.i-semisimple.

Proof. 1) $\Rightarrow$ 2) Let M and N be two pseudo-injective modules and by hypothesis injective. Thus $M \oplus N$ is injective and therefore pseudo-injective.

2) $\Rightarrow$ 1) First we note that under 2), every pseudo-injective module is quasi-injective. Indeed, let M be a pseudo-injective module. Then $M \oplus M$ is pseudo-injective. Therefore M is quasi-injective by Lemma 4. Notice that we only need to prove that R is a QI ring. Let M and N be two quasi-injective modules, thus pseudo-injective. Hence $M \oplus N$ is pseudo-injective by hypothesis. Therefore $M \oplus N$ is quasi-injective by the above argument. Then by Theorem 2, R is a QI ring.

1) $\Rightarrow$ 3) Let M be an injective module and N an automorphism invariant submodule of M. By hypothesis it suffices to show that N is a pseudo-injective module. Indeed, let f be an automorphism of $E(N)$. Then f can be extended to the following commutative diagram:

$$\begin{tikzcd} E(N) \arrow[r, hook] \arrow[d, "f"'] & E(N) \oplus L = M \arrow[d, "g"] \\ E(N) \arrow[r, hook] & E(M) \oplus L = M \end{tikzcd}$$

where $g \in Aut(M)$. As N is automorphism invariant in M, then $f(N) = g(N) \leq N$. Therefore by Theorem 3, N is pseudo-injective.

3) $\Rightarrow$ 1) Let M be a pseudo-injective module. By Theorem 3, M is automorphism invariant in $E(M)$. Hence M is a direct summand of $E(M)$. Therefore M is injective. $\square$

Corollary 2. *The following statements are equivalent:*

1. *R is semisimple.*
2. *R is pseudo-injective and every pseudo-injective module is injective.*
3. *Every R-module is pseudo-injective.*
4. *Every finitely generated R-module is pseudo-injective.*
5. *Every R-module generated by four elements is pseudo-injective.*

Proof. 1) $\Rightarrow$ 2), 3) $\Rightarrow$ 4) and 4) $\Rightarrow$ 5) They are clear.

2) $\Rightarrow$ 3) As R is pseudo-injective, then it is self-injective. Also R is a QI ring by hypothesis. Then R is a left V-ring left and noetherian ring. Hence, R is a QF ring, thus left artinian with $J(R) = \{0\}$. Therefore it is a semisimple ring.

5) $\Rightarrow$ 1) Let M be a cyclic module. Then $(R \oplus M)^{(2)}$ is pseudo injective by hypothesis. Hence $R \oplus M$ is quasi-injective by Lemma 4. Then by [13, Lemma 1.1], M is injective. Therefore every cyclic module is injective and this is well known to be equivalent to R being semisimple. $\square$

5 Some relations between pseudo-injective and pseudo-projective modules

Recall that a module M is called quasi-projective if for each quotient N of M and each morphism $f : M \to N$ there exists $g \in End(M)$ such that the following diagram commutes:

$$\begin{tikzcd} & M \arrow[dl, "g"'] \arrow[d, "f"] \\ M \arrow[r, "q"] & N \end{tikzcd}$$

In [13] and [6] Fuller and Byrd demonstrated the following theorem:

Theorem 6. *The following conditions are equivalent:*

1. *R is an artinian principal ideal ring.*
2. *Every quasi-injective module is quasi-projective.*
3. *Every quasi-projective module is quasi-injective.*

Definition 5. A module M is called pseudo-projective if for each quotient N of M and each epimorphism $f : M \to N$ there exists $g \in End(M)$ such that the following diagram commutes:

$$\begin{tikzcd} & M \arrow[dl, "g"'] \arrow[d, "f"] \\ M \arrow[r, "q"] & N \end{tikzcd}$$

So it is natural to ask about the same condition for pseudo-injective and pseudo-projective modules. In this section we give sufficient conditions on a ring to have the following properties: every pseudo-injective module is pseudo-projective and every pseudo-projective module is pseudo-injective. But first we give the following lemma:

Lemma 5. *If $M \oplus M$ is pseudo-projective, then M is quasi-projective.*

Proof. Let N be a submodule of M, $f : M \to M/N$ a morphism and $\pi : M \to M/N$ the natural epimorphism. We define $g : M \oplus M \to M/N$ by $g((m_1, m_2)) = \pi(m_1) + f(m_2)$. It is clear that g is an epimorphism and by hypothesis there exists $h \in End(M \oplus M)$ such that the following diagram is commutative:

$$\begin{tikzcd} & & M \oplus M \arrow[dll, "h"'] \arrow[d, "g"] \\ M \oplus M \arrow[r, "\pi_2"] & M \arrow[r, "\pi"] & M/N \end{tikzcd}$$

where $\pi_2 : M \oplus M \to M$ is the second projection. If $i_2 : M \to M \oplus M$ is the second inclusion, then $\overline{h} = \pi_2 h i_2$ is an endomorphism of M such that the following diagram is commutative:

$$\begin{tikzcd} & M \arrow[dl, "\overline{h}"'] \arrow[d, "f"] \\ M \arrow[r, "\pi"] & M/N \end{tikzcd}$$

Therefore M is quasi-projective. □

Theorem 7. *If every pseudo-injective module is pseudo-projective, then R is an artinian principal ideal ring.*

Proof. Let M be an injective module. Then $(M \oplus M)^{(2)}$ is injective, thus pseudo-injective. Hence $(M \oplus M)^{(2)}$ is pseudo-projective by hypothesis. Then by Lemma 5, $(M \oplus M)$ is quasi-projective and therefore M is projective by [5, Proposition 2.1]. Thus R is QF. Let I be any two-sided ideal of R. It is straightforward to verify that the ring R/I also satisfies the hypothesis. It follows that R/I is QF. By [11, Proposition 25.4.6B], R is an artinian principal ideal ring. □

Theorem 8. *If every pseudo-projective module is pseudo-injective, then R is an artinian principal ideal ring.*

Proof. Let F be a free R-module. Then $F \oplus F$ is projective and therefore pseudo-projective. Hence $F \oplus F$ is pseudo-injective by hypothesis. Then by Lemma 4, F is quasi-injective and contains a copy of R, so by [13, Lemma 1.1], F is injective. Hence every direct summand of a free module is injective. Therefore every projective module is injective. Thus R is QF. Let I be any two-sided ideal of R. Again, the ring R/I also satisfies the hypothesis. It follows that R/I is QF. By [11, Proposition 25.4.6B], R is an artinian principal ideal ring. $\square$

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