BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(88), 2018, Pages 32–56
ISSN 1024–7696

Regularized gradient-projection algorithm for solving one-parameter nonexpansive semigroup, constrained convex minimization and generalized equilibrium problems

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Abstract. Our purpose in this paper is to propose an iterative algorithm for finding a common element of the fixed points set of common solutions of a one-parameter nonexpansive semigroup, the set of solutions of constrained convex minimization problem and the set of solutions of generalized equilibrium problem in a real Hilbert space using the idea of regularized gradient-projection algorithm under suitable conditions. Finally, we give an application.

Mathematics subject classification: 47H09; 47H10; 49J20; 49J40.
Keywords and phrases: Nonexpansive semigroup; generalized equilibrium problem; constrained convex minimization problem; regularization method; fixed point problem; Hilbert space.

1 Introduction

Let H be a real Hilbert space with norm $\|.\|$ and inner product $\langle ., . \rangle$ and let C be a nonempty, closed and convex subset of H. A mapping $G : C \to C$ is called nonexpansive if

$$\|Gx - Gy\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1}$$

One parameter family of mappings $\mathcal{T} := \{G(t) : 0 \leq t < \infty\}$ is called a continuous Lipschitzian semigroup on C if the following conditions are satisfied:

(1) $G(0)x = x$ for all $x \in C$;

(2) $G(s + t) = G(s)G(t)$ for all $s, t \geq 0$;

(3) for each $t > 0$, there exists a bounded measurable function $L_t : (0, \infty) \to [0, \infty)$ such that $\|G(t)x - G(t)y\| \leq L_t\|x - y\|,\ x, y \in C$;

(4) for each $x \in C$, the mapping $G(.)x$ from $[0, \infty)$ into C is continuous.

A Lipschitzian semigroup $\mathcal{T}$ is called nonexpansive if $L_t = 1$ for all $t > 0$ and asymptotically nonexpansive if $\limsup_{t\to\infty} L_t \leq 1$. Let $F(\mathcal{T})$ denote the common fixed point set of the semigroup $\mathcal{T}$ i.e. $F(\mathcal{T}) := \{x \in C : G(t)x = x, \forall t > 0\}$.

We say that a mapping $G : C \to C$ is said to be

(i) k-Lipschitz continuous if

$$\|Gx - Gy\| \leq k\|x - y\|, \quad \forall x, y \in C, \; k > 0;$$

(ii) monotone if

$$\langle Gx - Gy, x - y\rangle \geq 0, \quad \forall x.y \in C;$$

(iii) α - strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Gx - Gy, x - y\rangle \geq \alpha\|x - y\|, \quad \forall x, y \in C;$$

(iv) η- inverse strongly monotone if there exists a constant $\eta > 0$ such that

$$\langle Gx - Gy, x - y\rangle \geq \eta\|Gx - Gy\|^2, \quad \forall x, y \in C.$$

Let $F : C \times C \to \mathbb{R}$ be a bifunction and $\psi : C \to H$ be a nonlinear mapping. The generalized equilibrium problem is to find $x \in C$ such that

$$F(x, y) + \langle \psi x, y - x\rangle \geq 0, \quad \forall\; y \in C. \tag{2}$$

The set of solutions of generalized equilibrium problem (2) is denoted by $GEP(F, \psi)$. Thus

$$GEP(F, \psi) := \{x \in C : F(x, y) + \langle \psi x, y - x\rangle \geq 0, \quad \forall\; y \in C\}.$$

In the case of $\psi \equiv 0$, problem (2) reduces to an equilibrium problem, which is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall\; y \in C \tag{3}$$

and the set of solutions is denoted by $EP(F)$. Problem (2) includes, as a special case, optimization problems, variational inequalities, minimax problem, Nash equilibrium problem in noncooperative games, etc, (see for example, [2,12,14]). Several problems in physics, optimization and economics can be reduced to generalized problem (2). Some methods have been proposed to solve the generalized equilibrium problem, equilibrium problem and related optimization problems (see, for example, [1,5,6,14, 16,18,19,22] and the references contained therein).

For solving the generalized equilibrium problem, the bifunction $F : C \times C \to \mathbb{R}$ is assumed to satisfy the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e. $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$, $\lim_{t\downarrow 0} F(tz + (1 - t)x) \leq F(x, y)$;

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

Let $A : H \to H$ be a single-valued nonlinear mapping and let $M : H \to 2^H$ be a set-valued mapping. The variational inclusion problem is to find $u \in H$ such that

$$\theta \in A(u) + M(u), \tag{4}$$

where θ is the zero vector in H. The set of solutions to variational inclusion (4) is denoted by $I(A, M)$. Consider the following constrained minimization problem:

$$\min_{x \in C} g(x) \tag{5}$$

where $g : C \to \mathbb{R}$ is a real-valued convex and continuously Fréchet differentiable functional. Assume that the constrained convex minimization problem (5) has a solution, we denote the set of solutions of (5) by Γ. The Gradient-Projection Algorithm (GPA) generates a sequence $\{x_n\}$ according to the recursive formula

$$x_{n+1} = P_C(I - \gamma_n \nabla g)x_n, \quad \forall n \geq 0, \tag{6}$$

where the parameters γ_n are real positive numbers, and P_C is the metric projection from H onto C. It is well known that the convergence of the algorithms (6) is determined by the gradient ∇g and the metric projection onto C. If the gradient ∇g is only assumed to be inverse strongly monotone, then the sequence $\{x_n\}$ defined by the algorithm (6) can only converge weakly to a minimizer of (5). If the gradient ∇g is Lipschitz continuous and strongly monotone, then the sequence generated by (6) can converge strongly to a unique minimizer of (5) provided the parameters γ_n satisfy appropriate conditions.

In 2011, Xu [24] proposed average mappings to GPA, and he constructed a counter-example which shows that the GPA does not have strong convergence in an infinite-dimensional space. Moreover, he provided two convergent modifications of GPA which are shown to converge in norm.

Also, in 2011 motivated by Xu, Cent *et al.*[4] presented the following iterative algorithm:

$$x_{n+1} = P_C\left[\theta_n \gamma f(x_n) + (I - \theta_n \mu F)T_n(x_n)\right], \quad n \geq 0, \tag{7}$$

where $f : C \to H$ is an l-Lipschitzian mapping with constant $l > 0$, and $F : C \to H$ is a k-Lipschitzian and η-strongly monotone operator with constants $k, \eta > 0$. Let $0 < \mu < 2\eta/k^2$, $0 \leq \gamma l < \tau$ and $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)}$. Let T_n and θ_n satisfy $\theta_n = \frac{2 - \lambda_n L}{4}$, $P_C(I - \lambda_n \nabla g) = \theta_n I + (1 - \theta_n T_n)$. Under suitable conditions, it is proved that the sequence $\{x_n\}$ generated by (7) converges strongly to a minimizer x^* of (6).

In 2012, Tian and Liu [11] introduced the following iterative method in a Hilbert space: $x_1 \in C$ and

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall\, y \in C, \\ x_{n+1} = \alpha_n \gamma f(u_n) + (1 - \alpha_n A)T_n(u_n), & \forall\, n \in \mathbb{N}, \end{cases} \tag{8}$$

where $F : C \times C \to \mathbb{R}$, $u_n = Q_{r_n}(x_n)$, $P_C(I - \lambda_n \nabla g) = \theta_n I + (I - \theta_n)T_n \theta_n = \frac{2-\lambda_n L}{4}$, and $\{\lambda_n\} \subset (0, 2/L$, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\theta_n\}$ satisfy appropriate conditions. Furthermore, they proved that the sequence $\{x_n\}$ converges strongly to a point $q \in \Gamma \cap EP(F)$, which solves the variational inequality

$$\langle (A - \gamma f)q, q - z\rangle \leq 0, \quad z \in \Gamma \cap EP(F).$$

However, it is known that the minimization problem (5) has more than one solution, so regularization is needed to find a unique solution. Now, consider the following regularized minimization problem:

$$\min_{x \in C} g_\alpha(x) := \min_{x \in C} \left\{ g(x) + \frac{\alpha}{2}\|x\|^2 \right\},$$

where $\alpha > 0$ is the regularization parameter, g is a convex function with a $1/L$-ism continuous gradient ∇g. Then the Regularized Gradient Projection Algorithm (RGPA) generates a sequence $\{x_n\}$ by the following recursive formula:

$$x_{n+1} = P_C(I - \gamma \nabla g_{\alpha_n})x_n = P_C\left[x_n - \gamma(\nabla g + \alpha_n I)(x_n)\right], \tag{9}$$

where the parameter $\alpha_n > 0$, γ is a constant with $0 < \gamma < 2/L$, and P_C is the metric projection from H onto C. It is well known that the sequence $\{x_n\}$ generated by algorithm (9) converges weakly to a minimizer of (5) in the setting of infinite-dimensional space (see [25]).

In 2010, Tian [23] combined the iterative methods of [13,26] to propose a general iterative method for approximating a fixed point of a nonexpansive mapping T defined on a real Hilbert space. Let f be a l-contraction on C with $0 < l < 1$, and let S be η-strongly monotone and k-Lipschitzian. For a constant μ satisfying $0 < \mu < 2\eta/k^2$, a constant t satisfying $0 < t < \mu(\eta - \frac{\mu k^2}{2})/l = \tau/l$, then for $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = \alpha_n t f(x_n) + (1 - \alpha_n \mu S)T x_n, \quad n \geq 0, \tag{10}$$

where $F(T)$ denotes the fixed points of mapping T, i.e, $F(T) = \{x \in H : x = Tx\}$.

Recently, motivated by the works of Tian [23], Tian and Liu [11], Ming Tian and Si-Wen Jiao [22] introduced a new iterative algorithm: $x_1 \in C$ and

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall\, y \in C, \\ x_{n+1} = \alpha_n t f(x_n) + (I - \alpha_n \mu S)T_{\lambda_n}(u_n), & \forall\, n \in \mathbb{N}, \end{cases} \tag{11}$$

for finding an element of $\Gamma \cap EP(F)$, where $F : C \times C \to \mathbb{R}$, $u_n = Q_{r_n}(x_n)$, $P_C(I - \gamma \nabla g_{\lambda_n}) = T_{\lambda_n}$, $\nabla g_{\lambda_n} = \nabla g + \lambda_n I$, $\gamma \in (0, 2/L)$. Under appropriate conditions, they proved that the sequence $\{x_n\}$ generated by (11) converges strongly to a point $q \in \Gamma \cap EP(F)$, which is also a solution to the variational inequality

$$\langle (\mu S - t f)q, q - z\rangle \leq 0, \quad \forall\, z \in \Gamma \cap EP(F).$$

In [18], Y. Shehu introduced an iterative scheme for finding a common element of the set of common fixed points of a nonexpansive semigroup, the set of solutions of a generalized equilibrium problem and the set of solutions of a variational inclusion problem in a real Hilbert space. In particular, they proved the following theorem:

Theorem 1. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4), ψ a μ- inverse-strongly monotone mapping from C into H, A an α-inverse strongly monotone mapping from C into H and $M : H \to 2^H$ a maximal monotone mapping. Let $\mathcal{T} : \{G(u) : 0 \leq u < \infty\}$ be a one-parameter nonexpansive semigroup on H such that $\Omega := F(\mathcal{T}) \cap I(A, M) \cap GEP(F, \psi) \neq \emptyset$ and suppose $f : H \to H$ is a contraction mapping with a constant $\gamma \in (0, 1)$. Let $\{t_n\} \subset (0, \infty)$ be a real sequence such that $\lim_{n\to\infty} t_n = \infty$. Suppose $\{x_n\}$ and $\{u_n\}$ are generated by $x_1 \in H$,*

$$\begin{cases} F(u_n, y) + \langle \psi x_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall\, y \in C \quad \textit{and} \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)\left(\frac{1}{t_n}\int_0^{t_n} G(u)[\alpha_n f(x_n) + (1 - \alpha_n)J_{M,\lambda}(u_n - \lambda A u_n)]du\right), \end{cases} \tag{12}$$

for all $n \neq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences in $(0, 1)$ and $\{r_n\}_{n=1}^{\infty} \subset (0, \infty)$ satisfying:

(i) $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;

(ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(iii) $\lambda \in (0, 2\alpha]$;

(iv) $0 < a \leq r_n \leq b < 2\mu$, $\liminf_{n\to\infty} r_n > 0$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;

(v) $\lim_{n\to\infty} \frac{t_n - t_{n-1}}{t_n}\frac{1}{\alpha_n(1-\beta_n)} = 0$.

Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to z, where $z := P_\Omega f(z)$.

In this paper, motivated by the works of Y. Shehu [18], Ming Tian and Si-Wen Jiao [22] and ongoing results, we prove strong convergence theorems for finding a common element of the set of common fixed points of a nonexpansive semigroup, the set of solutions of a generalized equilibrium problem and the set of solutions of a constrained convex minimization problem in a real Hilbert space. Our contribution lies in the fact that our iterative method solves fixed point problem for nonexpansive semigroup, generalized equilibrium problem and constrained convex minimization problem at same time.

2 Preliminaries

Let H be a real Hilbert space and let C be a nonempty, closed and convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \quad \forall\, y \in C.$$

Such a P_C is called the metric projection of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies the following:

$$\begin{array}{rcl}\langle x-y, P_Cx-P_Cy\rangle & \geq & \|P_Cx-P_Cy\|^2, \ \ \forall\, x,y\in H; \\ \langle x-P_Cx, y-P_Cx\rangle & \leq & 0; \\ \|x-y\|^2\geq\|x-P_Cx\|^2 & + & \|y-P_Cx\|^2, \ \ \forall\, x\in H, y\ \in C.\end{array} \tag{13}$$

Lemma 1. *[4] The following inequality holds in an inner product space X :*

$$\|x+y\|^2\leq\|x\|^2+2\langle y, x+y\rangle, \ \ \forall\, x,y\in X.$$

Lemma 2. *[9] Let $T:C\to C$ be nonexpansive mapping with $F(T)\neq\emptyset$. If $\{x_n\}_{n=1}^{\infty}$ is a sequence in C that converges weakly to x and if $\{(I-T)x_n\}_{n=1}^{\infty}$ converges strongly to y, then $(I-T)x=y$. In particular, if $y=0$, then $x\in F(T)$.*

Lemma 3. *[20]. Let D be a nonempty, bounded, closed and convex subset of a real Hilbert space H and let $\mathcal{T}:=\{G(u):0\leq u<\infty\}$ a nonexpansive semigroup on D, then for any $h\geq 0$,*

$$\lim_{t\to\infty}\sup_{x\in D}\left\|G(h)\left(\frac{1}{t}\int_0^t G(u)x du\right)-\left(\frac{1}{t}\int_0^t G(u)x du\right)\right\|=0.$$

Lemma 4. *[24]. Assume that $\{a_n\}_{n=0}^{\infty}$ is a sequence of non-negative real numbers such that*

$$a_{n+1}\leq(1-\delta_n)a_n+\delta_n\sigma_n+b_n, \ \ n\geq 0,$$

where $\{\gamma_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are sequences in $(0,1)$ and $\{\delta_n\}_{n=0}^{\infty}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=0}^{\infty}\delta_n=\infty$;

(ii) either $\limsup_{n\to\infty}\sigma_n\leq 0$ *or* $\sum_{n=0}^{\infty}\delta_n|\sigma_n|<0$;

(iii) $\sum_{n=0}^{\infty}b_n<\infty$.

Then $\lim_{n\to\infty}a_n=0$.

Lemma 5. *[2]. Let $F:C\times C\to\mathbb{R}$ be a bifunction satisfying (A1)-(A4). Let $r>0$ and $x\in H$. Then there exists $z\in C$ such that*

$$F(z,y)+\frac{1}{r}\langle y-z, z-x\rangle\geq 0, \ \ \forall\, y\in C.$$

Lemma 6. *[8] Assume that $F:C\times C\to\mathbb{R}$ is a bifunction satisfying (A1)-(A4). For $r>0$ and $x\in H$, define a mapping $Q_r:H\to C$ as follows:*

$$Q_r(x)=\left\{z\in C:F(z,y)+\frac{1}{r}\langle y-z, z-x\rangle\geq 0, \ \ \forall\, y\in C\right\}.$$

Then the following hold:

(1) Q_r *is single-valued;*

(2) Q_r *is firmly nonexpansive, i.e.* $\|Q_r - Q_r y\|^2 \leq \langle Q_r x - Q_r y, x - y\rangle$ *for any* $x, y \in H$;

(3) $F(Q_r) = EP(F)$;

(4) $EP(F)$ *is closed and convex.*

Lemma 7. *[22] Let* H *be a real Hilbert space and* C *be a nonempty, closed and convex subset of* H*. Let* $f : H \to H$ *be a contraction with constant* $l \in (0,1)$*, and* $S : C \to H$ *be a* k*-Lipschitzian and* η*-strongly monotone operator with* $k > 0,\ \eta > 0$*. Suppose that* ∇g *is* $1/L$*-ism continuous. Let* Q_{r_n} *be sequence of mappings defined as in Lemma 2.6. Consider the following mapping* X_n *on* H *defined by*

$$X_n(x) = \alpha_n t f(x) + (I - \alpha_n \mu S)T_{\lambda_n} Q_{r_n}(x), \quad \forall x \in H,\ n \in \mathbb{N},$$

where $P_C(I - \gamma\nabla g_{\lambda_n}) = T_{\lambda_n}$*,* $\nabla g_{\lambda_n} = \nabla g + \lambda_n I$*,* $\gamma \in (0, 2/L)$*,* $\{\alpha_n\} \subset (0,1)$*,* $\mu \in (0, 2\eta/k^2)$*,* $0 < t < \mu(\eta - \frac{\mu k^2}{2})/l = \tau/l$*. Then* X_n *is a contraction. i.e.*

$$\|X_n(x) - X_n(y)\| \leq (1 - \alpha_n(\tau - tl))\|x - y\|.$$

3 Main Results

Lemma 8. *Let* C *be a nonempty, closed and convex subset of a real Hilbert space* H*. Let* $F : C \times C \to \mathbb{R}$ *be a bifunction satisfying (A1)-(A4),* $\psi : C \to H$ *be a monotone mapping and let* $g : C \to \mathbb{R}$ *be a real-valued convex function, and assume that the gradient* ∇g *is* $1/L$*-ism with a constant* $L > 0$*. Let* $f : H \to H$ *be a contraction with the constant* $0 < l < 1$ *and let* $S : C \to H$ *be* η*-strongly monotone and* k*-Lipschitzian. Fix a constant* μ *satisfying* $0 < \mu < 2\eta/k^2$*, a constant* t *satisfying* $0 < t < \mu(\eta - \frac{\mu k^2}{2})/l = \tau/l$*. Let* $\mathcal{T} := \{G(u) : 0 \leq u < \infty\}$ *be a one-parameter nonexpansive semigroup on* H *such that* $\Upsilon := F(\mathcal{T}) \cap \Gamma \cap GEP(F, \psi) \neq \emptyset$*, and* $\{t_n\} \subset (0, \infty)$ *be a sequence such that* $\lim_{n\to\infty} t_n = \infty$*. Suppose* $\{x_n\}_{n=1}^{\infty}$ *and* $\{u_n\}_{n=1}^{\infty}$ *are generated by* $x_1 \in H$ *as follows:*

$$\begin{cases} F(u_n, y) + \langle \psi x_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall\ y \in C; \\ T_{\lambda_n}(u_n) = P_C(I - \gamma\nabla g_{\lambda_n})u_n; \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)\Big(\frac{1}{t_n}\int_0^{t_n} G(u)[\alpha_n t f(x_n) + (1 - \alpha_n \mu S)T_{\lambda_n}(u_n)]du\Big), \end{cases} \tag{14}$$

where $u_n = Q_{r_n}(x_n)$*,* $\nabla g_{\lambda_n} = \nabla g + \lambda_n I$*,* $T_{\lambda_n} = P_C(I - \gamma\nabla g_{\lambda_n})$*,* $\gamma \in (0, 2/L)$*. Let* $\{\beta_n\}$*,* $\{r_n\}$*,* $\{\alpha_n\}$*,* $\{\lambda_n\}$ *satisfy the following conditions:*

(i) $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;

(ii) $\alpha_n \subset (0,1)$ $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(iii) $\{\lambda_n\} \subset (0, 2/\gamma - L)$, $\lambda_n = o(\alpha_n)$, $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$;

(iv) $\{r_n\} \subset (0, \infty)$, $\liminf_{n\to\infty} r_n > 0$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;

(v) $\lim_{n\to\infty} \frac{t_n - t_{n-1}}{t_n} \frac{1}{\alpha_n(1-\beta_n)} = 0$.

Then $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}$, $\{u_n\}$ *and* $\left\{\frac{1}{t_n}\int_0^{t_n} G(u) y_n du\right\}$ *are bounded.*

Proof. First, we show that $(I - \gamma\nabla g_{\lambda_n})$ is nonexpansive. For all $x, y \in C$ and $\gamma \in (0, 2/L)$, we have

$$\begin{aligned} \|(I-\gamma\nabla g_{\lambda_n})x - (I-\gamma\nabla g_{\lambda_n})y\|^2 &= \|(x-y) - \gamma(\nabla g_{\lambda_n}x - \nabla g_{\lambda_n}y)\|^2 \\ &= \|x-y\|^2 - 2\gamma\langle x-y, \nabla g_{\lambda_n}x - \nabla g_{\lambda_n}y\rangle \\ &\quad +\gamma^2\|\nabla g_{\lambda_n}x - \nabla g_{\lambda_n}y\|^2 \\ &\leq \|x-y\|^2 - \frac{2\gamma}{L}\|\nabla g_{\lambda_n}x - \nabla g_{\lambda_n}y\|^2 \\ &\quad +\gamma^2\|\nabla g_{\lambda_n}x - \nabla g_{\lambda_n}y\|^2 \\ &= \|x-y\|^2 \\ &\quad + \gamma\left(\gamma - \frac{2}{L}\right)\|\nabla g_{\lambda_n}x - \nabla g_{\lambda_n}y\|^2 \\ &\leq \|x-y\|^2. \end{aligned} \tag{15}$$

Next, we show that $\{x_n\}$ is bounded. Let $p \in F(\mathcal{T}) \cap \Gamma \cap GEP(F, \psi) \neq \emptyset$ and by Lemma 6, we know that

$$\|u_n - p\| = \|Q_{r_n}(x_n) - Q_{r_n}(p)\| \leq \|x_n - p\|. \tag{16}$$

Now, let $y_n := \alpha_n t f(x_n) + (1 - \alpha_n \mu S) T_{\lambda_n}(u_n)$, $n \geq 1$. So

$$\begin{aligned} \|y_n - p\| &= \|\alpha_n t f(x_n) + (I - \alpha_n\mu S)T_{\lambda_n}(u_n) - p\| \\ &\leq \|(I-\alpha_n\mu S)T_{\lambda_n}(u_n) - (I-\alpha_n\mu S)T_{\lambda_n}(p)\| \\ &\quad +\|(I-\alpha_n\mu S)T_{\lambda_n}(p) - (I-\alpha_n\mu S)T(p)\| \\ &\quad +\alpha_n t\|f(x_n) - f(p)\| + \alpha_n\|tf(x_n) - \mu S(p)\| \\ &\leq (1-\alpha_n\tau)\|u_n - p\| + \|T_{\lambda_n}(p) - T(p)\| \\ &\quad +\|\alpha_n\mu S T_{\lambda_n}(p) - \alpha_n\mu S T(p)\| \\ &\quad + \alpha_n t l\|x_n - p\| + \alpha_n\|tf(p) - \mu S(p)\| \\ &\leq (1 - \alpha_n(\tau - tl))\|x_n - p\| \\ &\quad + (\alpha_n\mu k + 1)\|T_{\lambda_n}(p) - T(p)\| + \alpha_n\|tf(p) - \mu S(p)\|. \end{aligned} \tag{17}$$

For $x \in C$, note that

$$P_C(I - \gamma\nabla g_{\lambda_n})x = T_{\lambda_n}x$$

and

$$P_C(I - \gamma\nabla g)x = Tx.$$

Then we get

$$\begin{aligned} \|T_{\lambda_n}x - Tx\| &= \|P_C(I-\gamma\nabla g_{\lambda_n})x - P_C(I-\gamma\nabla g)x\| \\ &\le \lambda_n\gamma\|x\|. \end{aligned} \tag{18}$$

It follows from (17) and (18) that

$$\begin{aligned} \|y_n - p\| &\le (1-\alpha_n(\tau - tl))\|x_n - p\| \\ &\quad +\alpha_n(\tau - tl)\left[\frac{\lambda_n}{\alpha_n}.\frac{(\alpha_n\mu k+1)\gamma}{\tau - tl}\|p\| + \frac{\|tf(p)-\mu S(p)\|}{\tau - tl}\right]. \end{aligned} \tag{19}$$

From (14), we obtain

$$\begin{aligned} \|x_{n+1} - p\| &= \left\|\beta(x_n - p) + (1-\beta_n)\left(\frac{1}{t_n}\int_0^{t_n}[G(u)y_n - G(u)p]du\right)\right\| \\ &\le \beta_n\|x_n - p\| + (1-\beta_n)\|y_n - p\| \\ &\le \beta_n\|x_n - p\| + (1-\beta_n)(1-\alpha_n(\tau - tl))\|x_n - p\| \\ &\quad +\alpha_n(1-\beta_n)(\tau - tl)\left[\frac{\lambda_n}{\alpha_n}.\frac{(\alpha_n\mu k+1)\gamma}{\tau - tl}\|p\| + \frac{\|tf(p)-\mu S(p)\|}{\tau - tl}\right] \\ &= [1-\alpha_n(\tau - tl)(1-\beta_n)]\,\|x_n - p\| \\ &\quad +\alpha_n(1-\beta_n)(\tau - tl)\left[\frac{\lambda_n}{\alpha_n}.\frac{(\alpha_n\mu k+1)\gamma}{\tau - tl}\|p\| + \frac{\|tf(p)-\mu S(p)\|}{\tau - tl}\right]. \end{aligned}$$

Since $\lambda_n = o(\alpha_n)$, there exists a real number $M_1 > 0$ such that $\frac{\lambda_n}{\alpha_n} \le M_1$. Thus,

$$\begin{aligned} \|x_{n+1} - p\| &\le [1-\alpha_n(\tau - tl)(1-\beta_n)]\,\|x_n - p\| \\ &\quad +\alpha_n(1-\beta_n)(\tau - tl)\frac{M_1(\alpha_n\mu k+1)\gamma\|p\| + \|tf(p)-\mu S(p)\|}{\tau - tl} \\ &\le \max\left\{\|x_n - p\|, \frac{1}{\tau - tl}\left(M_1(\alpha_n\mu k+1)\gamma\|p\| + \|tf(p)-\mu S(p)\|\right)\right\} \\ &\ \ \vdots \\ &\le \max\left\{\|x_1 - p\|, \frac{1}{\tau - tl}\left(M_1(\alpha_n\mu k+1)\gamma\|p\| + \|tf(p)-\mu S(p)\|\right)\right\} \end{aligned}$$

$n \ge 1$.

So, $\{x_n\}$ is bounded. Hence, $\{y_n\}$, $\{u_n\}$ and $\left\{\frac{1}{t_n}\int_0^{t_n} G(u)y_n du\right\}$ are also bounded. □

Lemma 9. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4), $\psi : C \to H$ be a monotone mapping and let $g : C \to \mathbb{R}$ be a real-valued convex function, and assume that the gradient ∇g is $1/L$-ism with a constant $L > 0$. Let $f : H \to H$*

be a contraction with the constant $0 < l < 1$ and let $S : C \to H$ be η-strongly monotone and k-Lipschitzian. Fix a constant μ satisfying $0 < \mu < 2\eta/k^2$, a constant t satisfying $0 < t < \mu(\eta - \frac{\mu k^2}{2})/l = \tau/l$. Let $\mathcal{T} := \{G(u) : 0 \leq u < \infty\}$ be a one-parameter nonexpansive semigroup on H such that $\Upsilon := F(\mathcal{T}) \cap \Gamma \cap GEP(F, \psi) \neq \emptyset$, and $\{t_n\} \subset (0, \infty)$ be a sequence such that $\lim_{n\to\infty} t_n = \infty$. Suppose $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ are generated by (14). Then $\lim\limits_{n\to\infty} \|x_{n+1} - x_n\| = 0$ and $\lim\limits_{n\to\infty} \|u_{n+1} - u_n\| = 0$.

Proof. From (17), we have

$$\begin{aligned}
\|y_n - p\| &\leq (1 - \alpha_n(\tau - tl))\|x_n - p\| \\
&\quad + \alpha_n(\tau - tl)\frac{1}{\tau - tl}\left(M_1(\alpha_n \mu k + 1)\gamma\|p\| + \|tf(p) - \mu S(p)\|\right) \\
&\leq \|x_n - p\| + \frac{1}{\tau - tl}\left(M_1(\alpha_n \mu k + 1)\gamma\|p\| + \|tf(p) - \mu S(p)\|\right) \\
&\leq \max\left\{\|x_1 - p\|, \frac{1}{\tau - tl}\left(M_1(\alpha_n \mu k + 1)\gamma\|p\| + \|tf(p) - \mu S(p)\|\right)\right\} \\
&\quad + \frac{1}{\tau - tl}\left(M_1(\alpha_n \mu k + 1)\gamma\|p\| + \|tf(p) - \mu S(p)\|\right) \\
&\leq \|x_1 - p\| + \frac{2}{\tau - tl}\left(M_1(\alpha_n \mu k + 1)\gamma\|p\| + \|tf(p) - \mu S(p)\|\right).
\end{aligned}$$

Put $D = \{w \in H : \|w - p\| \leq \|x_1 - p\| + \frac{2}{\tau - tl}\left(M_1(\alpha_n \mu k + 1)\gamma\|p\| + \|tf(p) - \mu S(p)\|\right)\}$. Then D is a nonempty, bounded, closed and convex subset of H. Since $G(u)$ is nonexpansive for any $u \in [0, \infty)$, D is $G(u)$-invariant for each $u \in [0, \infty)$ and contains $\{y_n\}$. Without loss of generality, we may assume that $\mathcal{T} := \{G(u) : 0 \leq u < \infty\}$ is a nonexpansive semigroup on D. By Lemma 3, we get

$$\lim_{n\to\infty} \left\| \left(\frac{1}{t_n}\int_0^{t_n} G(u)y_n du\right) - G(h)\left(\frac{1}{t_n}\int_0^{t_n} G(u)y_n du\right)\right\| = 0, \tag{20}$$

for every $h \in [0, \infty)$. Furthermore, observe that

$$\begin{aligned}
\|x_{n+1} - G(h)x_{n+1}\| &\leq \left\|x_{n+1} - \frac{1}{t_n}\int_0^{t_n} G(u)y_n du\right\| \\
&\quad + \left\|\left(\frac{1}{t_n}\int_0^{t_n} G(u)y_n du\right) - G(h)\left(\frac{1}{t_n}\int_0^{t_n} G(u)y_n du\right)\right\| \\
&\quad + \left\|G(h)\left(\frac{1}{t_n}\int_0^{t_n} G(u)y_n du\right) - G(h)x_{n+1}\right\| \\
&\leq 2\left\|x_{n+1} - \frac{1}{t_n}\int_0^{t_n} G(u)y_n du\right\| \\
&\quad + \left\|\left(\frac{1}{t_n}\int_0^{t_n} G(u)y_n du\right) - G(h)\left(\frac{1}{t_n}\int_0^{t_n} G(u)y_n du\right)\right\| \\
&\leq 2\beta_n\left\|x_n - \frac{1}{t_n}\int_0^{t_n} G(u)y_n du\right\|
\end{aligned}$$

$$+\left\|\left(\frac{1}{t_n}\int_0^{t_n} G(u)y_n du\right) - G(h)\left(\frac{1}{t_n}\int_0^{t_n} G(u)y_n du\right)\right\|,$$

from $\lim_{n\to\infty}\beta_n = 0$ and (20), we get $\lim_{n\to\infty}\|x_{n+1} - G(h)x_{n+1}\| = 0$ and hence

$$\lim_{n\to\infty}\|x_n - G(h)x_n\| = 0. \tag{21}$$

Next, we show that $\|x_{n+1} - x_n\| \to 0,\ n\to\infty$. Since ∇g is $1/L$-ism, $P_C(I - \nabla g_{\lambda_n}) = T_{\lambda_n}$, so we have that

$$\begin{aligned}\|T_{\lambda_n}(u_{n-1}) - T_{\lambda_{n-1}}(u_{n-1})\| &= \|P_C(I-\gamma\nabla g_{\lambda_n})u_{n-1} - (I-\gamma\nabla g\lambda_{n-1})u_{n-1}\|\\ &\leq \|(I-\gamma\nabla g_{\lambda_n})u_{n-1} - (I-\gamma\nabla g_{\lambda_n})u_{n-1}\|\\ &= \gamma\|\nabla g(u_{n-1}) + \lambda_{n-1}u_{n-1} - \nabla g(u_{n-1}) - \lambda_n u_{n-1}\|\\ &= \gamma|\lambda_n - \lambda_{n-1}|\|u_{n-1}\|.\end{aligned}$$

Thus, we get

$$\begin{aligned}\|y_n - y_{n-1}\| &= \|(\alpha_n t f(x_n) + (I-\alpha_n\mu S)T_{\lambda_n}(u_n) - (\alpha_{n-1}tf(x_{n-1})\\ &\quad +(I-\alpha_{n-1}\mu S)T_{\lambda_{n-1}}(u_{n-1})\|\\ &\leq \|\alpha_n tf(x_n) - \alpha_n tf(x_{n-1})\| + \|\alpha_n tf(x_{n-1}) - \alpha_{n-1}tf(x_{n-1})\|\\ &\quad +\|(I-\alpha_n\mu S)T_{\lambda_n(u_n)} - (I-\alpha_n\mu S)T_{\lambda_n}(u_{n-1})\|\\ &\quad +\|(I-\alpha_n\mu S)T_{\lambda_n}(u_{n-1}) - (I-\alpha_{n-1}\mu S)T_{\lambda_{n-1}}(u_{n-1})\|\\ &\leq \alpha_n tl\|x_n - x_{n-1}\| + t|\alpha_n - \alpha_{n-1}|.\|f(x_{n-1})\| + (1-\alpha_n\tau)\|u_n - u_{n-1}\|\\ &\quad +\|T_{\lambda_n}(u_{n-1}) - T_{\lambda_{n-1}}(u_{n-1})\|\\ &\quad +\|\alpha_{n-1}\mu ST_{\lambda_{n-1}}(u_{n-1}) - \alpha_n\mu ST_{\lambda_n}(u_{n-1})\|\\ &\leq \alpha_n tl\|x_n - x_{n-1}\| + t|\alpha_n - \alpha_{n-1}|.\|f(x_{n-1})\|\\ &\quad +(1+\alpha_{n-1}\mu k)\|T_{\lambda_n}(u_{n-1}) - T_{\lambda_{n-1}}(u_{n-1})\|\\ &\quad +|\alpha_n - \alpha_{n-1}|.\|ST_{\lambda_n}(u_{n-1})\| + (1-\alpha_n\tau)\|u_n - u_{n-1}\|\\ &\leq \alpha_n tl\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|.\left(t\|f(x_n)\| + \mu\|ST_{\lambda_n}(u_{n-1})\|\right)\\ &\quad +(1-\alpha_n\tau)\|u_n - u_{n-1}\|\\ &\quad +\gamma(1+\mu k)|\lambda_n - \lambda_{n-1}|.\|u_{n-1}\|.\end{aligned}$$

Since from Lemma 8, $\{u_n\}$, $\{f(x_n)\}$ and $\{ST_{\lambda_n}(u_n)\}$ are bounded, then there exists a constant $M_2 > 0$ such that

$$M_2 \geq \max\left\{\gamma(1+\mu k)\|u_{n-1}\|, t\|f(x_n)\| + \mu\|ST_{\lambda_n}(u_{n-1})\right\},\quad \forall\ n\geq 1.$$

Hence

$$\begin{aligned}\|y_n - y_{n-1}\| &\leq \alpha_n tl\|x_n - x_{n-1}\| + (1-\alpha_n\tau)\|u_n - u_{n-1}\|\\ &\quad +M_2(|\alpha_n - \alpha_{n-1}| + |\lambda_n - \lambda_{n-1}|).\end{aligned} \tag{22}$$

From $u_{n+1} = Q_{r_{n+1}}(x_{n+1})$ and $u_n = Q_{r_n}(x_n)$, we note that

$$F(u_n, y) + \langle \psi x_n, y - u_n \rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C \tag{23}$$

and

$$F(u_{n+1}, y) + \langle \psi x_{n+1}, y - u_{n+1} \rangle + \frac{1}{r_{n+1}}\langle y - u_{n+1}, u_{n+1} - x_{n+1} \rangle \geq 0, \quad \forall y \in C. \tag{24}$$

Putting $y = u_n$ in (23) and $y = u_{n+1}$ in (24), we have

$$F(u_n, u_{n+1}) + \langle \psi x_n, u_{n+1} - u_n \rangle + \frac{1}{r_n}\langle u_{n+1} - u_n, u_n - x_n \rangle \geq 0$$

and

$$F(u_{n+1}, u_n) + \langle \psi x_{n+1}, u_n - u_{n+1} \rangle + \frac{1}{r_{n+1}}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1} \rangle \geq 0.$$

So, from (A2), we have

$$\langle \psi x_{n+1} - \psi x_n, u_n - u_{n+1} \rangle + \left\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}} \right\rangle \geq 0$$

and hence,

$$\begin{aligned} 0 &\leq \left\langle u_n - u_{n+1}, r_n(\psi x_{n+1} - \psi x_n) + \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1}) - (u_n - x_n) \right\rangle \\ &= \Big\langle u_{n+1} - u_n, u_n - u_{n+1} + \left(1 - \frac{r_n}{r_{n+1}}\right) u_{n+1} + (x_{n+1} - r_n \psi x_{n+1}) \\ &\quad -(x_n - r_n \psi x_n) - x_{n+1} + \frac{r_n}{r_{n+1}} x_{n+1} \Big\rangle \\ &= \Big\langle u_{n+1} - u_n, u_n - u_{n+1} + \left(1 - \frac{r_n}{r_{n+1}}\right)(u_{n+1} - x_{n+1}) + (x_{n+1} - r_n \psi x_{n+1}) \\ &\quad -(x_n - r_n \psi x_n) \Big\rangle. \end{aligned}$$

It then follows that

$$\|u_{n+1} - u_n\|^2 \leq \|u_{n+1} - u_n\| \left\{ \left|1 - \frac{r_n}{r_{n+1}}\right| \|u_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| \right\}$$

and so we have

$$\|u_{n+1} - u_n\| \leq \left|1 - \frac{r_n}{r_{n+1}}\right| \|u_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\|.$$

Without loss of generality, we assume that there exists $w \in \mathbb{R}$ such that $r_n > w > 0$, $\forall n \geq 1$. Then

$$\|u_{n+1} - u_n\| \leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}}|r_{n+1} - r_n| \|u_{n+1} - x_{n+1}\|$$

$$\leq \quad \|x_{n+1} - x_n\| + \frac{1}{w}|r_{n+1} - r_n|M_3, \tag{25}$$

where $M_3 := \sup_{n\geq 1} \|u_n - x_n\|$.
From (22) and (25), we get

$$\begin{aligned}
\|y_n - y_{n-1}\| \quad &\leq \quad \alpha_n tl\|x_n - x_{n-1}\| + (1-\alpha_n\tau)\|u_n - u_{n-1}\| \\
&\quad +M_2(|\lambda_n - \lambda_{n-1}| + |\alpha_n - \alpha_{n-1}|) \\
&\leq \quad \alpha_n tl\|x_n - x_{n-1}\| \\
&\quad +(1-\alpha_n\tau)\left(\|x_n - x_{n-1}\| + \frac{1}{w}|r_n - r_{n-1}|M_3\right) \\
&\quad +M_2(|\lambda_n - \lambda_{n-1}| + |\alpha_n - \alpha_{n-1}|) \\
&\leq \quad (1-\alpha_n(\tau - tl))\|x_n - x_{n-1}\| + \frac{M_3}{w}|r_n - r_{n-1}| \\
&\quad +M_2(|\lambda_n - \lambda_{n-1}| + |\alpha_n - \alpha_{n-1}|).
\end{aligned} \tag{26}$$

Let $z_n := \frac{1}{t_n}\int_0^{t_n} G(u)y_n du; \quad n \geq 1$. Then we have

$$\|z_n - z_{n-1}\| =$$

$$\left\|\frac{1}{t_n}\int_0^{t_n}[G(u)y_n - G(u)y_{n-1}]du + \left(\frac{1}{t_n} - \frac{1}{t_{n-1}}\right)\int_0^{t_{n-1}} du + \frac{1}{t_n}\int_{t_{n-1}}^{t_n} G(u)y_{n-1}du\right\|.$$

Given that
$\left(\frac{1}{a} - \frac{1}{b}\right) b = -\frac{a-b}{b}, \quad a, b \neq 0$; if $p \in \Omega$, we can write

$$\|z_n - z_{n-1}\| = \left\|\frac{1}{t_n}\int_0^{t_n}[G(u)y_n - G(u)y_{n-1}]du\right.$$

$$\left. + \left(\frac{1}{t_n} - \frac{1}{t_{n-1}}\right)\int_0^{t_{n-1}}[G(u)y_{n-1} - G(u)p]du + \frac{1}{t_n}\int_{t_{n-1}}^{t_n}[G(u)y_{n-1} - G(u)p]du\right\|.$$

Thus,

$$\|z_n - z_{n-1}\| \leq \|y_n - y_{n-1}\| + \left(\frac{2|t_n - t_{n-1}|}{t_n}\right)\|y_{n-1} - p\|. \tag{27}$$

Substituting (26) into (27), we obtain

$$\begin{aligned}
\|z_n - z_{n-1}\| \quad &\leq \quad (1-\alpha_n(\tau - tl))\|x_n - x_{n-1}\| + \frac{M_3}{w}|r_n - r_{n-1}| \\
&\quad +M_2(|\lambda_n - \lambda_{n-1}| + |\alpha_n - \alpha_{n-1}|) \\
&\quad +\left(\frac{2|t_n - t_{n-1}|}{t_n}\right)\|y_{n-1} - p\|.
\end{aligned} \tag{28}$$

From (14), we have $x_{n+1} = \beta_n x_n + (1-\beta_n)z_n$ and this implies that

$$\|x_{n+1} - x_n\| \quad = \quad \|\beta_n x_n + (1-\beta_n)z_n - \beta_{n-1}x_{n-1} - (1-\beta_{n-1})z_{n-1}\|$$

$$\begin{aligned} &= \|\beta_n x_n - \beta_{n-1}x_{n-1} + \beta_n x_{n-1} - \beta_n x_{n-1}(1-\beta_n)z_n \\ &\quad -(1-\beta_{n-1})z_{n-1} + (1-\beta_n)z_{n-1} - (1-\beta_n)z_{n-1}\| \\ &\leq \beta_n\|x_n - x_{n-1}\| + (1-\beta_n)\|z_n - z_{n-1}\| \\ &\quad +|\beta_n - \beta_{n-1}|(\|x_{n-1}\| + \|z_{n-1}\|). \end{aligned} \tag{29}$$

Using (28) in (29), we obtain

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \beta_n\|x_n - x_{n-1}\| \\ &\quad +(1-\beta_n)\Big[(1-\alpha_n(\tau - tl))\|x_n - x_{n-1}\| + \frac{M_3}{w}|r_n - r_{n-1}| \\ &\quad +M_2(|\lambda_n - \lambda_{n-1}| + |\alpha_n - \alpha_{n-1}|) + \left(\frac{2|t_n - t_{n-1}|}{t_n}\right)\|y_{n-1} - p\|\Big] \\ &\quad +|\beta_n - \beta_{n-1}|(\|x_{n-1}\| + \|z_{n-1}\|) \\ &\leq [1-\alpha_n(\tau - tl)(1-\beta_n)]\,\|x_n - x_{n-1}\| \\ &\quad +\frac{M_3}{w}|r_n - r_{n-1}| + |\beta_n - \beta_{n-1}|(\|x_{n-1}\| + \|z_{n-1}\|) \\ &\quad +M_2(|\lambda_n - \lambda_{n-1}| + |\alpha_n - \alpha_{n-1}|) + \left(\frac{2|t_n - t_{n-1}|}{t_n}\right)\|y_{n-1} - p\| \\ &\leq [1-\alpha_n(\tau - tl)(1-\beta_n)]\,\|x_n - x_{n-1}\| \\ &\quad +D\Big[|r_n - r_{n-1}| + |\beta_n - \beta_{n-1}| + (|\lambda_n - \lambda_{n-1}| + |\alpha_n - \alpha_{n-1}|) \\ &\quad +\frac{2|t_n - t_{n-1}|}{t_n}\Big], \end{aligned}$$

where $D := \max\{\sup_{n\geq 1}(\|x_n\| + \|z_n\|), \sup_{n\geq 1}\|y_n - p\|, \frac{M_3}{w}, M_2\}$. From Lemma 4 taking $\delta_n = \alpha_n(\tau - tl)(1-\beta_n)$, $b_n = \frac{2D|t_n - t_{n-1}|}{t_n}$ and
$\sigma_n = D(|r_n - r_{n-1}| + |\beta_n - \beta_{n-1}| + |\lambda_n - \lambda_{n-1}| + |\alpha_n - \alpha_{n-1}|)$, by using conditions (i)-(v), it follows that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{30}$$

Since $\lim_{n\to\infty}|r_{n+} - r_n| = 0$, then from (25) and (30), we have that

$$\lim_{n\to\infty}\|u_{n+1} - u_n\| = 0. \tag{31}$$

□

Lemma 10. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4), $\psi : C \to H$ be a monotone mapping and let $g : C \to \mathbb{R}$ be a real-valued convex function, and assume that the gradient ∇g is $1/L$-ism with a constant $L > 0$. Let $f : H \to H$ be a contraction with the constant $0 < l < 1$ and let $S : C \to H$ be η-strongly monotone and k-Lipschitzian. Fix a constant μ satisfying $0 < \mu < 2\eta/k^2$, a constant t satisfying $0 < t < \mu(\eta - \frac{\mu k^2}{2})/l = \tau/l$. Let $\mathcal{T} := \{G(u) : 0 \leq u < \infty\}$ be a one-parameter nonexpansive semigroup on H such that $\Upsilon := F(\mathcal{T}) \cap \Gamma \cap GEP(F,\psi) \neq \emptyset$,* *and* $\{t_n\} \subset (0,\infty)$ *be a sequence such that* $\lim_{n\to\infty} t_n = \infty$. *Suppose* $\{x_n\}_{n=1}^{\infty}$ *and* $\{u_n\}_{n=1}^{\infty}$ *are generated by* (14). *Then* $\lim\limits_{n\to\infty}\|u_n - T_{\lambda_n}(u_n)\| = 0$ *and* $\lim\limits_{n\to\infty}\|y_n - u_n\| = 0$.

Proof. From (14), (15) and (16), we obtain

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \left\|\beta_n(x_n-p)+(1-\beta_n)\left(\frac{1}{t_n}\int_0^{t_n}[G(u)y_n - G(u)p]du\right)\right\|^2\\
&\leq \beta_n\|x_n-p\|^2+(1-\beta_n)\|y_n-p\|^2 = \beta_n\|x_n-p\|^2\\
&\quad +(1-\beta_n)\|\alpha_n(tf(x_n)-\mu Sp)+(1-\alpha_n\mu S)(T_{\lambda_n}(u_n)-p)\|^2\\
&\leq \beta_n\|x_n-p\|^2+(1-\beta_n)\Big[\alpha_n\|tf(x_n)-\mu Sp\|^2\\
&\quad +(1-\alpha_n\mu S)\|T_{\lambda_n}(u_n)-p\|^2\Big]\\
&\leq \beta_n\|x_n-p\|^2+(1-\beta_n)\Big[\alpha_n\|tf(x_n)-\mu Sp\|^2\\
&\quad +(1-\alpha_n\mu S)(\|u_n-p\|^2+\gamma\left(\gamma-\frac{2}{L}\right)\|\nabla g_{\lambda_n}(u_n)-\nabla g_{\lambda_n}p\|^2)\Big]\\
&\leq \beta_n\|x_n-p\|^2+(1-\beta_n)\Big[\alpha_n\|tf(x_n)-\mu Sp\|^2\\
&\quad +(1-\alpha_n\mu S)(\|x_n-p\|^2+\gamma\left(\gamma-\frac{2}{L}\right)\|\nabla g_{\lambda_n}(u_n)-\nabla g_{\lambda_n}p\|^2)\Big]\\
&\leq \|x_n-p\|^2+\alpha_n\|tf(x_n)-\mu Sp\|^2\\
&\quad +\gamma\left(\gamma-\frac{2}{L}\right)\|\nabla g_{\lambda_n}(u_n)-\nabla g_{\lambda_n}p\|^2.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
-\gamma\left(\gamma-\frac{2}{L}\right)\|\nabla g_{\lambda_n}(u_n)-\nabla g_{\lambda_n}p\|^2 &\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2\\
&\quad +\alpha_n\|tf(x_n)-\mu Sp\|^2\\
&\leq \|x_{n+1}-x_n\|(\|x_n-p\|+\|x_{n+1}-p\|)\\
&\quad +\alpha_n\|tf(x_n)-\mu Sp\|^2.
\end{aligned}$$

Since $\lim\limits_{n\to\infty}\alpha_n = 0$ and $\lim\limits_{n\to\infty}\|x_{n+1}-x_n\| = 0$ by Lemma 9, we obtain

$$\lim_{n\to\infty}\|\nabla g_{\lambda_n}(u_n)-\nabla g_{\lambda_n}p\| = 0.$$

From (14), we obtain (noting that $(I-\gamma\nabla g_{\lambda_n})$ is nonexpansive)

$$\begin{aligned}
\|T_{\lambda_n}(u_n)-p\|^2 &= \|P_C(I-\gamma\nabla g_{\lambda_n})u_n - P_C(I-\gamma\nabla g\lambda_n)p\|^2\\
&\leq [\langle(u_n-\gamma\nabla g_{\lambda_n}u_n)-(p-\gamma\nabla g_{\lambda_n}p), T_{\lambda_n}(u_n)-p\rangle]\\
&= \frac{1}{2}\Big[\|(u_n-\gamma\nabla g_{\lambda_n}u_n)-(p-\gamma\nabla g_{\lambda_n}p)\|^2+\|T_{\lambda_n}(u_n)-p\|^2\\
&\quad -\|(u_n-\gamma\nabla g_{\lambda_n})-(p-\gamma\nabla g_{\lambda_n}p)-(T_{\lambda_n}(u_n)-p)\|^2\Big]
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2}\Big[\|u_n-p\|^2+\|T_{\lambda_n}(u_n)-p\|^2 \\
&\quad +\|(u_n-T_{\lambda_n}(u_n))-\gamma(\nabla g_{\lambda_n}u_n-\nabla g_{\lambda_n}p)\|^2\Big] \\
&\leq \frac{1}{2}\Big[\|u_n-p\|^2+\|T_{\lambda_n}(u_n)-p\|^2-\|u_n-T_{\lambda_n}(u_n)\|^2 \\
&\quad +2\gamma\langle u_n-T_{\lambda_n}(u_n),\nabla g_{\lambda_n}u_n-\nabla g_{\lambda_n}p\rangle-\gamma^2\|\nabla g_{\lambda_n}u_n-\nabla g_{\lambda_n}p\|^2\Big].
\end{aligned}$$

So, we have

$$\begin{aligned}
\|T_{\lambda_n}(u_n)-p\|^2 &\leq \|u_n-p\|^2-\|T_{\lambda_n}(u_n)-u_n\|^2 \qquad (32)\\
&\quad +2\gamma\langle u_n-T_{\lambda_n}(u_n),\nabla g_{\lambda_n}u_n-\nabla g_{\lambda_n}p\rangle-\gamma^2\|\nabla g_{\lambda_n}u_n-\nabla g_{\lambda_n}p\|^2.
\end{aligned}$$

From (14) and (32), we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \left\|\beta_n(x_n-p)+(1-\beta_n)\left(\frac{1}{t_n}\int_0^{t_n}[G(u)y_n-G(u)p]du\right)\right\|^2 \\
&\leq \beta_n\|x_n-p\|^2+(1-\beta_n)\|y_n-p\|^2 \\
&\leq \beta_n\|x_n-p\|^2+(1-\beta_n)\left(\alpha_n\|tf(x_n)-\mu Sp\|^2+(1-\alpha_n\mu S)\|T_{\lambda_n}(u_n)-p\|^2\right) \\
&\leq \beta_n\|x_n-p\|^2+\alpha_n\|tf(x_n)-\mu Sp\|^2 \\
&\quad +(1-\beta_n)[\|u_n-p\|^2-\|T_{\lambda_n}(u_n)-u_n\|^2 \\
&\quad +2\gamma\langle u_n-T_{\lambda_n}(u_n),\nabla g_{\lambda_n}u_n-\nabla g_{\lambda_n}p\rangle-\gamma^2\|\nabla g_{\lambda_n}u_n-\nabla g_{\lambda_n}p\|^2] \\
&\leq \beta_n\|x_n-p\|^2+\alpha_n\|tf(x_n)-\mu Sp\|^2 \\
&\quad +(1-\beta_n)[\|x_n-p\|^2-\|T_{\lambda_n}(u_n)-u_n\|^2 \\
&\quad +2\gamma\langle u_n-T_{\lambda_n}(u_n),\nabla g_{\lambda_n}u_n-\nabla g_{\lambda_n}p\rangle-\gamma^2\|\nabla g_{\lambda_n}u_n-\nabla g_{\lambda_n}p\|^2] \\
&\leq \|x_n-p\|^2+\alpha_n\|tf(x_n)-\mu Sp\|^2-(1-\beta_n)\|u_n-T_{\lambda_n}(u_n)\|^2 \\
&\quad +2(1-\beta_n)\gamma\|u_n-T_{\lambda_n}(u_n)\|\|\nabla g_{\lambda_n}u_n-\nabla g_{\lambda_n}p\|. \qquad (33)
\end{aligned}$$

From (33), we have

$$\begin{aligned}
(1-\beta_n)\|u_n-T_{\lambda_n}(u_n)\|^2 &\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n\|tf(x_n)-\mu Sp\|^2 \\
&\quad +2\gamma\|u_n-T_{\lambda_n}(u_n)\|\|\nabla g_{\lambda_n}u_n-\nabla g_{\lambda_n}p\|.
\end{aligned}$$

Since $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$ by Lemma 9, $\lim_{n\to\infty}\alpha_n=0$ and $\lim_{n\to\infty}\|\nabla g_{\lambda_n}u_n-\nabla g_{\lambda_n}p\|=0$, we obtain

$$\lim_{n\to\infty}(1-\beta_n)\|u_n-T_{\lambda_n}(u_n)\|=0.$$

Since $\lim_{n\to\infty}\beta_n=0$, we obtain

$$\lim_{n\to\infty}\|u_n-T_{\lambda_n}(u_n)\|=0. \qquad (34)$$

From $y_n=\alpha_n tf(x_n)+(1-\alpha_n\mu S)T_{\lambda_n}(u_n)$, we obtain $y_n-T_{\lambda_n}(u_n)=\alpha_n(tf(x_n)-\mu ST_{\lambda_n}(u_n))$. So,

$$\|y_n-T_{\lambda_n}(u_n)\|=\alpha_n\|(tf(x_n)-\mu ST_{\lambda_n}(u_n))\|\to 0, \quad as \quad n\to\infty. \qquad (35)$$

Next we show that $\lim_{n\to\infty}\|u_n - x_n\| = 0$. Indeed, for any $p \in F(\mathcal{T}) \cap \Gamma \cap EP(F)$, by Lemma 6 we have

$$\begin{aligned}\|u_n - p\|^2 &= \|Q_{r_n}(x_n) - Q_{r_n}(p)\|^2 \\ &\leq \langle x_n - p, u_n - p\rangle \\ &= \frac{1}{2}(\|x_n - p\|^2 + \|u_n - p\|^2 - \|u_n - x_n\|^2).\end{aligned}$$

This implies that

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|u_n - x_n\|^2. \tag{36}$$

Then, from (18) and (36), we derive that

$$\begin{aligned}\|y_n - p\|^2 &= \|\alpha_n t f(x_n) + (I - \alpha_n \mu S)T_{\lambda_n}(u_n) - p\|^2 \\ &\leq [((1-\alpha_n\tau)\|u_n - p\| + \alpha_n tl\|x_n - p\|) \\ &\quad +(\lambda_n\gamma(1+\alpha_n\mu k)\|p\| + \alpha_n\|tf(p) - \mu S(p)\|)]^2 \\ &\leq \|u_n - p\|^2 + (\alpha_n^2 t^2 l^2 + 2\alpha_n tl)\|x_n - p\|^2 \\ &\quad +\lambda_n^2\gamma^2(1+\alpha_n\mu k)^2\|p\|^2 + \alpha_n^2\|tf(p) - \mu S(p)\|^2 \\ &\quad +2\lambda_n\gamma(1+\alpha_n\gamma)\|p\|.\|tf(x_n) - \mu S(p)\| \\ &\quad +2\lambda_n\gamma(1+tl)(1+\alpha_n\mu k)\|x_n - p\|.\|p\| \\ &\quad +2\alpha_n(1+tl)\|x_n - p\|.\|tf(p) - \mu S(p)\| \\ &\leq (1+\alpha_n tl)^2\|x_n - p\|^2 - \|u_n - x_n\|^2 \\ &\quad +\lambda_n^2\gamma^2(1+\alpha_n\mu k)^2\|p\|^2 + \alpha_n^2\|tf(p) - \mu S(p)\|^2 \\ &\quad +2\lambda_n\gamma(1+\alpha_n\gamma)\|p\|.\|tf(x_n) - \mu S(p)\| \\ &\quad +2\lambda_n\gamma(1+tl)(1+\alpha_n\mu k)\|x_n - p\|.\|p\| \\ &\quad +2\alpha_n(1+tl)\|x_n - p\|.\|tf(p) - \mu S(p)\|. \end{aligned} \tag{37}$$

From (14) ,(37) and by the convexity of $\|.\|^2$, we obtain

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \left\|\beta_n(x_n - p) + (1-\beta_n)\left(\frac{1}{t_n}\int_0^{t_n}[G(u)y_n - G(u)p]du\right)\right\|^2 \\ &\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)\|y_n - p\|^2 \\ &\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)\Big[(1+\alpha_n tl)^2\|x_n - p\|^2 \\ &\quad -\|u_n - x_n\|^2 + \lambda_n^2\gamma^2(1+\alpha_n\mu k)^2\|p\|^2 \\ &\quad +\alpha_n^2\|tf(p) - \mu S(p)\|^2 + 2\lambda_n\gamma(1+\alpha_n\gamma)\|p\|.\|tf(x_n) - \mu S(p)\| \\ &\quad +2\lambda_n\gamma(1+tl)(1+\alpha_n\mu k)\|x_n - p\|.\|p\| \\ &\quad +2\alpha_n(1+tl)\|x_n - p\|.\|tf(p) - \mu S(p)\|\Big]. \end{aligned} \tag{38}$$

Thus, we get

$$(1-\beta_n)\|u_n - x_n\|^2 \leq \beta_n\|x_n - p\|^2 + (1-\beta_n)\Big[(1+\alpha_n tl)^2\|x_n - p\|^2 - \|x_{n+1} - p\|^2$$

$$+\lambda_n^2\gamma^2(1+\alpha_n\mu k)^2\|p\|^2+\alpha_n^2\|tf(p)-\mu S(p)\|^2$$
$$+2\lambda_n\gamma(1+\alpha_n\gamma)\|p\|.\|tf(x_n)-\mu S(p)\|$$
$$+2\lambda_n\gamma(1+tl)(1+\alpha_n\mu k)\|x_n-p\|.\|p\|$$
$$+2\alpha_n(1+tl)\|x_n-p\|.\|tf(p)-\mu S(p)\|\Big].$$

Since $\{x_n\}$ is bounded by Lemma 8, $\alpha_n \to 0$, $\beta_n \to 0$, $\lambda_n \to 0$, $n \to \infty$ and $\|x_{n+1}-x_n\| \to 0$ by Lemma 9, we have

$$\lim_{n\to\infty}\|u_n-x_n\|=0. \tag{39}$$

Furthermore, from (34), (35), (37) and for every $h \in [0,\infty)$, we have that

$$\begin{aligned}\|G(h)y_n-G(h)x_n\| &\leq \|y_n-x_n\| \\ &\leq \|u_n-x_n\|+\|u_n-T_{\lambda_n}(u_n)\|+\|T_{\lambda_n}(u_n)-y_n\|\to 0, \end{aligned} \tag{40}$$

as $n\to\infty$. Hence, from (21) and (40), we obtain

$$\|G(h)y_n-x_n\|\leq\|G(h)y_n-G(h)x_n\|+\|G(h)x_n-x_n\|\to 0,\quad n\to\infty.$$

Also, we have that

$$\|x_n-T_{\lambda_n}(u_n)\|\leq\|x_n-u_n\|+\|u_n-T_{\lambda_n}(u_n)\|\to 0,\quad n\to\infty.$$

Hence, for every $h\in[0,\infty)$ we have that

$$\begin{aligned}\|G(h)y_n-y_n\| &\leq \|G(h)y_n-x_n\|+\|x_n-T_{\lambda_n}(u_n)\| \\ &+ \|T_{\lambda_n}(u_n)-y_n\|\to 0,\quad n\to\infty. \end{aligned} \tag{41}$$

Next, we show that $\|x_n-T_{\lambda_n}(x_n)\|\to 0$, $n\to\infty$.

$$\begin{aligned}\|x_n-T_{\lambda_n}(x_n)\| &= \|x_n-u_n+u_n-T_{\lambda_n}(u_n)+T_{\lambda_n}(u_n)-T_{\lambda_n}(u_n)-T_{\lambda_n}(x_n)\| \\ &\leq \|x_n-u_n\|+\|u_n-T_{\lambda_n}(u_n)\|+\|T_{\lambda_n}(u_n)-T_{\lambda_n}(x_n)\| \\ &\leq \|x_n-u_n\|+\|u_n-T_{\lambda_n}(u_n)\|+\|u_n-x_n\|. \end{aligned}$$

From (34) and (39), we have

$$\|x_n-T_{\lambda_n}(x_n)\|\to 0,\quad n\to\infty.$$

From (34) and (35), we obtain that

$$\|y_n-u_n\|\leq\|u_n-T_{\lambda_n}(u_n)\|+\|y_n-T_{\lambda_n}(u_n)\|\to 0,\quad n\to\infty. \tag{42}$$

□

Lemma 11. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4), $\psi : C \to H$ be a monotone mapping and let $g : C \to \mathbb{R}$ be a real-valued convex function, and assume that the gradient ∇g is $1/L$-ism with a constant $L > 0$. Let $f : H \to H$ be a contraction with the constant $0 < l < 1$ and let $S : C \to H$ be η-strongly monotone and k-Lipschitzian. Fix a constant μ satisfying $0 < \mu < 2\eta/k^2$, a constant t satisfying $0 < t < \mu(\eta - \frac{\mu k^2}{2})/l = \tau/l$. Let $\mathcal{T} := \{G(u) : 0 \leq u < \infty\}$ be a one-parameter nonexpansive semigroup on H such that $\Upsilon := F(\mathcal{T}) \cap \Gamma \cap GEP(F, \psi) \neq \emptyset$, and $\{t_n\} \subset (0, \infty)$ be a sequence such that $\lim_{n\to\infty} t_n = \infty$. Suppose $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ are generated by (14). Then*

$$\limsup_{n\to\infty} \langle y_n - z, -(\mu S - tf)z\rangle \leq 0,$$

where $z = P_\Upsilon(I - \mu S + tf)z$.

Proof. Now if we take a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty} \langle y_n - z, -(\mu S - tf)z\rangle = \limsup_{n\to\infty} \langle y_{n_k} - z, -(\mu S - tf)z\rangle, \tag{43}$$

by (42) and $y_{n_k} \rightharpoonup q$, we have that $u_{n_k} \rightharpoonup q$. Note that

$$\begin{aligned} \|u_n - T(u_n)\| &\leq \|u_n - T_{\lambda_n}(u_n)\| + \|T_{\lambda_n}(u_n) - T(u_n)\| \\ &\leq \|u_n - T_{\lambda_n}(u_n)\| + \lambda_n \gamma \|u_n\|. \end{aligned}$$

Hence, by using the fact that $\|u_n - T_{\lambda_n}(u_n)\| \to 0$ by Lemma 10 and $\lambda_n \to 0$, we get $\|u_n - T(u_n)\| \to 0$. From Lemma 2 we get $q \in F(T) = \Gamma$. Next, we show that $q \in GEP(F, \psi)$. Since $u_n = Q_{r_n} x_n$, for any $y \in C$, we obtain

$$F(u_n, y) + \langle \psi x_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0.$$

Furthermore, replacing n by n_j in the last inequality and using (A2), we obtain

$$\langle \psi x_{n_j}, y - u_{n_j}\rangle + \frac{1}{r_{n_j}}\langle y - u_n, u_n - x_n\rangle \geq F(y, u_{n_j}). \tag{44}$$

Let $z_t := ty + (1-t)q$ for all $t \in (0, 1]$ and $y \in C$. This implies that $z_t \in C$. Then, by (44), we have

$$\begin{aligned} \langle z_t - u_{n_j}, \psi z_t\rangle &\geq \langle z_t - u_{n_j}, \psi z_t\rangle - \langle z_t - u_{u_j}, \psi x_{n_j}\rangle - \left\langle z_t - u_{n_j}, \frac{u_{n_j} - x_{n_j}}{r_{n_j}}\right\rangle + F(z_t, u_{n_j}) \\ &= \langle z_t - u_{n_j}, \psi z_t - \psi u_{n_j}\rangle + \langle z_t - u_{n_j}, \psi u_{n_j} - \psi x_{n_j}\rangle \\ &\quad - \left\langle z_t - u_{n_j}, \frac{u_{n_j} - x_{n_j}}{r_{n_j}}\right\rangle + F(z_t, u_{n_j}). \end{aligned} \tag{45}$$

Since $\lim_{j\to\infty} \|x_{n_j} - u_{n_j}\| = 0$, we have $\lim_{j\to\infty} \|\psi x_{n_j} - \psi u_{n_j}\| = 0$. Furthermore, by the monotonicity of ψ, we obtain $\langle z_t - u_{n_j}, \psi z_t - \psi u_{n_j}\rangle \geq 0$.

Since $\lim_{j\to\infty}\|y_{n_j}-u_{n_j}\|=0$ and $\lim_{j\to\infty}y_{n_j}=q$, we obtain that $\lim_{j\to\infty}u_{n_j}=q$. Then, using assumption (A4) in (45), we obtain

$$\langle z_t-q,\psi z_t\rangle\geq F(z_t,q),\quad j\to\infty. \tag{46}$$

Using (A1), (A4) and (46), we also obtain

$$\begin{aligned}0 &= F(z_t,z_t)\leq tF(z_t,y)+(1-t)F(z_t,q)\\ &\leq tF(z_t,y)+(1-t)\langle z_t-q,\psi\rangle\\ &\leq tF(z_t,y)+(1-t)t\langle y-q,\psi z_t\rangle\end{aligned}$$

and hence

$$0\leq F(z_t,y)+(1-t)\langle y-q,\psi z_t\rangle.$$

Letting $t\to 0$ and using assumption (A3), we have, for each $y\in C$,

$$0\quad\leq\quad F(q,y)+\langle y-q,\psi q\rangle. \tag{47}$$

Hence $q\in GEP(F,\psi)$.

Next, we show that $q\in F(\mathcal{T})$. Assume that $q\neq G(h)q$ for some $h\in[0,\infty)$. Then by Opial's condition, we obtain from (41) that

$$\begin{aligned}\liminf_{j\to\infty}\|y_{n_j}-q\| &< \liminf_{j\to\infty}\|y_{n_j}-G(h)q\|\\ &\leq \liminf_{j\to\infty}(\|y_{n_j}-G(h)y_{n_j}\|+\|G(h)y_{n_j}-G(h)q\|)\\ &\leq \liminf_{j\to\infty}\|y_{n_j}-q\|.\end{aligned}$$

This is a contradiction. Hence, $q\in F(\mathcal{T})$. Thus $q\in\Upsilon:=F(\mathcal{T})\cap\Gamma\cap GEP(F,\psi)$. By (43) and property of metric projection, we obtain

$$\begin{aligned}\limsup_{n\to\infty}\langle y_n-z(\mu S-tf)z\rangle &= \lim_{j\to\infty}\langle y_{n_j}-z,(\mu S-tf)z\rangle\\ &= \langle y-z,(\mu S-tf)z\rangle\leq 0. \end{aligned}\tag{48}$$

□

Theorem 2. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $F:C\times C\to\mathbb{R}$ be a bifunction satisfying (A1)-(A4), $\psi:C\to H$ be a monotone mapping and let $g:C\to\mathbb{R}$ be a real-valued convex function, and assume that the gradient ∇g is $1/L$-ism with a constant $L>0$. Let $f:H\to H$ be a contraction with the constant $0<l<1$ and let $S:C\to H$ be η-strongly monotone and k-Lipschitzian. Fix a constant μ satisfying $0<\mu<2\eta/k^2$, a constant t satisfying $0<t<\mu(\eta-\frac{\mu k^2}{2})/l=\tau/l$. Let $\mathcal{T}:=\{G(u):0\leq u<\infty\}$ be a one-parameter nonexpansive semigroup on H such that $\Upsilon:=F(\mathcal{T})\cap\Gamma\cap GEP(F,\psi)\neq\emptyset$, and $\{t_n\}\subset(0,\infty)$ be a sequence such that $\lim_{n\to\infty}t_n=\infty$. Suppose $\{x_n\}_{n=1}^\infty$ and $\{u_n\}_{n=1}^\infty$ are generated by (14). Then $\{x_n\}_{n=1}^\infty$ converges strongly to z, where $z:=P_\Upsilon(I-\mu S+tf)z$.*

Proof. Now,

$$\begin{aligned}
y_n - z &= \alpha_n t f(x_n) + (I - \alpha_n \mu S) T_{\lambda_n}(u_n) - z \\
&= ((I - \alpha_n \mu S)(T_{\lambda_n}(u_n) - (I - \alpha_n \mu S) T_{\lambda_n}(z)) \\
&\quad + ((I - \alpha_n \mu S) T_{\lambda_n}(z) - (I - \alpha_n \mu S) T(z)) \\
&\quad + \alpha_n t (f(x_n) - f(z)) + \alpha_n (t f(q) - \mu S(z)).
\end{aligned}$$

So, from (16) and (18), we derive

$$\begin{aligned}
\|y_n - z\|^2 &= \langle (I - \alpha_n \mu S)(T_{\lambda_n}(u_n) - (I - \alpha_n \mu S) T_{\lambda_n}(z), y_n - z \rangle \\
&\quad + \langle (I - \alpha_n \mu S) T_{\lambda_n}(z) - (I - \alpha_n \mu S) T(z), y_n - z \rangle \\
&\quad + \alpha_n t \langle f(x_n) - f(q), y_n - z \rangle + \alpha_n \langle -(\mu S - t f) z, y_n - z \rangle \\
&\leq (1 - \alpha_n \tau) \|u_n - z\| . \|y_n - z\| \\
&\quad + \lambda_n \gamma (1 + \alpha_n \mu k) \|z\| . \|z_n - z\| + \alpha_n t l \|x_n - z\| . \|y_n - z\| \\
&\quad + \langle -(\mu S - t f) z, y_n - z \rangle \\
&\leq (1 - \alpha_n (\tau - t l)) \|x_n - z\| . \|y_n - z\| \\
&\quad + \lambda_n \gamma (1 + \alpha_n \mu k) \|z\| . \|y_n - z\| + \alpha_n \langle -(\mu S - t f) z, y_n - z \rangle \\
&\leq (1 - \alpha_n (\tau - t l)) \frac{1}{2} (\|x_n - z\|^2 + \|y_n - z\|^2) \\
&\quad + \alpha_n \left[\langle -(\mu S - t f) z, y_n - z \rangle + \frac{\lambda_n}{\alpha_n} \gamma (1 + \alpha_n \mu S) \|z\| . \|y_n - z\| \right].
\end{aligned}$$

This implies that

$$\begin{aligned}
\|y_n - z\|^2 &\leq \frac{1 - \alpha_n (\tau - t l)}{1 + \alpha_n (\tau - t l)} \|x_n - z\|^2 \\
&\quad + \frac{2 \alpha_n}{1 + \alpha_n (\tau - t l)} \left[\langle -(\mu S - t f) z, y_n - z \rangle + \frac{\lambda_n}{\alpha_n} \gamma (1 + \alpha_n \mu k) \|z\| . \|y_n - z\| \right] \\
&\leq (1 - \alpha_n (\tau - t l)) \|x_n - z\|^2 + \frac{2 \alpha_n}{1 + \alpha_n (\tau - t l)} \Big[\langle -(\mu S - t f) z, y_n - z \rangle \\
&\quad + \frac{\lambda_n}{\alpha_n} \gamma (1 + \alpha_n \mu k) \|z\| . \|y_n - z\| \Big]. \qquad (49)
\end{aligned}$$

Using (14) in (49), we obtain

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \left\| \beta_n (x_n - z) + (1 - \beta_n) \left(\frac{1}{t_n} \int_0^{t_n} [G(u) y_n - G(u) p] du \right) \right\|^2 \\
&\leq \beta_n \|x_n - z\|^2 + (1 - \beta_n) \|y_n - z\|^2 \\
&\leq \beta_n \|x_n - z\|^2 + (1 - \beta_n) \Big((1 - \alpha_n (\tau - t l)) \|x_n - z\|^2 \\
&\quad + \frac{2 \alpha_n}{1 + \alpha_n (\tau - t l)} \Big[\langle -(\mu S - t f) z, y_n - z \rangle \\
&\quad + \frac{\lambda_n}{\alpha_n} \gamma (1 + \alpha_n \mu k) \|z\| . \|y_n - z\| \Big] \Big)
\end{aligned}$$

$$\begin{aligned}&\leq [1-(1-\beta_n)\alpha_n(\tau-tl)]\|x_n-z\|^2\\&+\frac{2\alpha_n(1-\beta_n)}{1+\alpha_n(\tau-tl)}\Big[\langle-(\mu S-tf)z,y_n-z\rangle\\&+\frac{\lambda_n}{\alpha_n}\gamma(1+\alpha_n\mu k)\|z\|.\|y_n-z\|\Big].\end{aligned}$$

Since $\{y_n\}$ is bounded by Lemma 8, there exists a constant $M>0$ such that

$$M\geq\|y_n-z\|,\quad n\geq 1.$$

Then, we have that

$$\|x_{n+1}-z\|^2 \leq (1-\delta_n)\|x_n-z\|^2+\alpha_n\sigma_n, \tag{50}$$

where $\delta_n:=(1-\beta_n)\alpha_n(\tau-tl)$ and
$\sigma_n:=\frac{2(1-\beta_n)}{1+\alpha_n(\tau-tl)}\left[\langle-(\mu S-tf)z,y_n-z\rangle+\frac{\lambda_n}{\alpha_n}\gamma(1+\alpha_n\mu k)\|z\|M\right]$.

By (48) and $\lambda_n=o(\alpha_n)$, we get $\limsup_{n\to\infty}\sigma_n\leq 0$. Now applying Lemma 4 to (50) we conclude that $x_n\to z$ as $n\to\infty$. This completes the proof. □

Remark 1. Examples of sequences $\{\alpha_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$, $\{t_n\}_{n=1}^\infty$, $\{r_n\}_{n=1}^\infty$ and $\{\lambda_n\}_{n=1}^\infty$ in Theorem 2 are

$$\alpha_n=\frac{1}{n^{\frac14}},\quad \beta_n=\frac{1}{(n+1)^{\frac14}},\quad t_n=n,\quad r_n=\frac{n}{n+1},\quad \lambda_n=\frac{1}{n+1},\quad n\geq 1.$$

Corollary 1. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $F:C\times C\to\mathbb{R}$ be a bifunction satisfying (A1)-(A4), $\psi:C\to H$ be a monotone mapping and let $g:C\to\mathbb{R}$ be a real-valued convex function, and assume that the gradient ∇g is $1/L$-ism with a constant $L>0$. Let $f:H\to H$ be a contraction with the constant $0<l<1$. Let $\mathcal{T}:=\{G(u):0\leq u<\infty\}$ be a one-parameter nonexpansive semigroup on H such that $\Upsilon:=F(\mathcal{T})\cap\Gamma\cap GEP(F,\psi)\neq\emptyset$, and $\{t_n\}\subset(0,\infty)$ be a sequence such that $\lim_{n\to\infty}t_n=\infty$. Suppose $\{x_n\}_{n=1}^\infty$ and $\{u_n\}_{n=1}^\infty$ are generated by $x_1\in H$ as follows:*

$$\begin{cases}F(u_n,y)+\langle\psi x_n,y-u_n\rangle+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq 0,\forall\ y\in C;\\ T_{\lambda_n}(u_n)=P_C(I-\gamma\nabla g_{\lambda_n})u_n;\\ x_{n+1}=\beta_nx_n+(1-\beta_n)\Big(\frac{1}{t_n}\int_0^{t_n}G(u)[\alpha_nf(x_n)+(1-\alpha_n)T_{\lambda_n}(u_n)]du\Big),\end{cases} \tag{51}$$

where $u_n=Q_{r_n}(x_n)$, $\nabla g_{\lambda_n}=\nabla g+\lambda_nI$, $T_{\lambda_n}=P_C(I-\gamma\nabla g_{\lambda_n})$, $\gamma\in(0,2/L)$. Let $\{\beta_n\}$, $\{r_n\}$, $\{\alpha_n\}$, $\{\lambda_n\}$ satisfy the following conditions:

(i) $\lim_{n\to\infty}\beta_n=0$, $\sum_{n=1}^\infty|\beta_{n+1}-\beta_n|<\infty$;

(ii) $\alpha_n\subset(0,1)$ $\lim_{n\to\infty}\alpha_n=0$, $\sum_{n=0}^\infty\alpha_n=\infty$, $\sum_{n=1}^\infty|\alpha_{n+1}-\alpha_n|<\infty$;

(iii) $\{\lambda_n\} \subset (0, 2/\gamma - L)$, $\lambda_n = o(\alpha_n)$, $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$;

(iv) $\{r_n\} \subset (0, \infty)$, $\liminf_{n\to\infty} r_n > 0$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;

(v) $\lim_{n\to\infty} \frac{t_n - t_{n-1}}{t_n} \frac{1}{\alpha_n(1-\beta_n)} = 0$.

Then $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* z, *where* $z := P_\Upsilon f(z)$.

4 Application

Consider the problem of finding a zero of maximal monotone operator in a Hilbert space H. It is well known (see [3]) that the initial value problem

$$\frac{du(t)}{dt} + Au(t) \ni 0 \text{ for every } t \geq 0, \ u(0) = x,$$

for any $x \in \overline{D(A)}$ has a unique solution $u : [0, \infty) \to H$ and $\overline{D(A)}$ is closed and convex. Putting $G(t)x = u(t)$, we have that the family of mappings $\mathcal{T} = \{G(t) : 0 \leq t < \infty\}$ of $\overline{D(A)}$ onto itself is a one-parameter nonexpansive semigroup on $\overline{D(A)}$. Moreover, we know from [3] that $A^{-1}0 = F(\mathcal{T})$. So, we can apply our Theorem 2 to find zero of A with $H = \overline{D(A)}$. Then the method (14) has the form $x_1 \in H$,

$$\begin{cases} y_n = (1 - \alpha_n)x_n, \\ x_{n+1} = (1 - \beta_n)x_n + \beta_n \left(\frac{1}{t_n} \int_0^{t_n} G(s) y_n ds \right), \ n \geq 1. \end{cases}$$

Acknowledgement: The first author acknowledges with thanks the bursary and financial support from Department of Science and Technology and National Research Foundation, Republic of South Africa Center of Excellence in Mathematical and Statistical Sciences (DST-NRF COE-MaSS) Doctoral Bursary. Opinions expressed and conclusions arrived are those of the authors and are not necessarily to be attributed to the CoE-MaSS.

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Received June 16, 2017