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Properties of Generalized Semi-ideal-based Zero-divisor Graph of Posets

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Abstract. In this paper, we study properties of generalized semi-ideal-based zero-divisor graph structure of poset P, with respect to minimal elements of $P\backslash I$. We also investigate the interplay between the poset properties of P and the graph theoretic properties of $\widehat{G_I(P)}$

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1 Preliminaries

Throughout this paper, $(P, \leq)$ denotes a poset with zero element 0. For $M \subseteq P$, let $(M)^l := \{x \in P : x \leq m \text{ for all } m \in M\}$ be the lower cone of M in P and dually the upper cone. For $A, B \subseteq P$, we write $(A, B)^l$ instead of $(A \cup B)^l$. If $M = \{x_1, x_2, ..., x_n\}$ is finite, then we use the notation $(x_1, x_2, ..., x_n)^l$ instead of $(\{x_1, x_2, ..., x_n\})^l$.

Following [8], a nonempty subset I of P is called a semi-ideal of P if $b \in I$ and $a \leq b$, then $a \in I$. A proper semi-ideal I of P is called prime if for any $a, b \in P$, $(a, b)^l \subseteq I$ implies $a \in I$ or $b \in I$. It is easy to see that, for any subset A of P, $(A)^l$ is a semi-ideal of P. If $A = \{a\}$, for any $a \in P$, then $(a)^l$ is the smallest semi-ideal containing a.

In [8], R. Halaš and M. Jukl introduced the concept of a graph structure of a poset. Let $(P, \leq)$ be a poset with 0. Then the zero-divisor graph of P, denoted by $G(P)$, is an undirected graph whose vertices are just the elements of P with two distinct vertices x and y joined by an edge if and only if $(x, y)^l = \{0\}$. They proved some interesting results related with clique and chromatic number of this graph structure.

In [9], V. Joshi introduced the zero divisor graph of a poset P (with 0) with respect to an ideal I. Let I be an ideal of a poset P. Then the zero-divisor graph of poset P, denoted by $G_I(P)$, is an undirected graph whose vertices are the set $\{x \in P\backslash I \mid (x, y)^l \subseteq I \text{ for some } y \in P\backslash I\}$ and two distinct vertices x and y are adjacent if and only if $(x, y)^l \subseteq I$. V. Joshi characterized its diameter, cycles and complete bipartite graphs. The zero-divisor graph structures are studied by several authors [3, 6].

In 2009 P. Dheena and B. Elavarasan [5] introduced the notion of generalized zero divisor graphs of near-rings. Analogously, V. Joshi et al. [10] extended this structure to poset as follows. Let I be semi-ideal of P. Then the generalized semi-ideal based zero-divisor graph of P, denoted by $\widehat{G_I(P)}$, whose vertices are the set $\{x \in P\backslash I :$ there exists $y \in P\backslash I$ such that $(x_1, y_1)^l \subseteq I$ for some $x_1 \in (x)^l\backslash I$ and $y_1 \notin I\}$ with adjacent distinct vertices x and y if and only if $(x_1, y_1)^l \subseteq I$ for some $x_1 \in (x)^l\backslash I$ and $y_1 \in (y)^l\backslash I$. Clearly $G_I(P)$ is an induced subgraph of $\widehat{G_I(P)}$, and I is a prime semi-ideal of P if and only if $\widehat{G_I(P)} = \phi$. Also it is clear that $V(\widehat{G_I(P)}) \cup I = P$.

For distinct vertices x and y of a graph G, let $d(x, y)$ be the length of the shortest path from x to y. The diameter of a connected graph is the supremum of the distances between vertices. The core K of G is the union of all cycles of G. A cycle in a graph G is a path that begins and ends at the same vertex. The girth of G, denoted $gr(G)$, is the length of a shortest cycle in G and $gr(G) = \infty$ if G has no cycle. Following [7], let I be a semi-ideal of P. Then by the extension of I by $x \in P$ is meant the set $(I : x) = \{a \in P : (a, x)^l \subseteq I\}$.

In this paper the notations of graph theory are from [4], the notations of posets are from [7] and [9].

2 Properties of $\widehat{G_I(P)}$

In this section, we characterize diameter, cycles and girth of generalized zero-divisor graph $\widehat{G_I(P)}$ with respect to a minimal element of $P\backslash I$ for a semi-ideal I of P.

Theorem 1 (see [10])**.** *The following conditions on a simple graph $\widehat{G_I(P)}$ are equivalent:*

(i) $\widehat{G_I(P)}$ is the generalized zero divisor graph of a poset with respect to some non-prime semi-ideal.

(ii) $\widehat{G_I(P)}$ is the complete $r-$partite graph for $r > 1$.

(iii) $\widehat{G_I(P)}$ is the complement of the comparability graph for a poset P. (The comparability graph of P is the graph with vertex set P and two elements $x, y \in P$ are adjacent if and only if $x \leq y$ or $y \leq x$).

(iv) The complement of the $\widehat{G_I(P)}$ is a disjoint union of cliques.

(v) $\widehat{G_I(P)}$ satisfies the following property: If $x \in V(\widehat{G_I(P)})$ is adjacent to some vertex of an independent subset A of $V(\widehat{G_I(P)})$, then x is adjacent to every vertex in A.

As an immediate consequences of Theorem 1, we have the following:

Theorem 2 (see [10])**.** *Let I be a semi-ideal of P. Then $\widehat{G_I(P)}$ is connected and $diam(\widehat{G_I(P)}) \leq 2$.*

Theorem 3. *Let I be a semi-ideal of P and if $a - x - b$ is a path in $\widehat{G_I(P)}$, then either $I \cup \{x_1\}$ is a semi-ideal of P for some $x_1 \in (x)^l\backslash I$ or $a - x - b$ is contained in a cycle of length ≤ 4.*

Proof. Let $a-x-b$ be a path in $\widehat{G_I(P)}$. Then there exist $x_1, x_2 \in (x)^l \backslash I, a_1 \in (a)^l \backslash I$ and $b_1 \in (b)^l \backslash I$ such that $(a_1, x_1)^l \subseteq I$ and $(b_1, x_2)^l \subseteq I$. If $(a', b')^l \subseteq I$ for some $a' \in (a)^l \backslash I$ and $b' \in (b)^l \backslash I$, then $a - x - b - a$ is contained in a cycle of length ≤ 4. So assume that $(a_1, b_1)^l \nsubseteq I$ for all $a_1 \in (a)^l \backslash I$ and $b_1 \in (b)^l \backslash I$.

Case(i) Let $x_1 = x_2$. Then either $(I : a_1) \cap (I : b_1) = I \cup \{x_1\}$ or there exists $c \in (I : a_1) \cap (I : b_1)$ such that $c \notin I \cup \{x_1\}$. In the first case, $I \cup \{x_1\}$ is semi-ideal. In the second case, $a - x - b - c - a$ is contained in a cycle of length ≤ 4.

Case(ii) Let $x_1 \neq x_2$. If $(a_1, b_1)^l \subseteq I$, then $a - x - b - a$ is a cycle of length ≤ 4. Otherwise $(a_1, b_1)^l \nsubseteq I$. Then for each $z \in (a_1, b_1)^l \backslash I$, we have $(z, x_1)^l \subseteq (a_1, x_1)^l \subseteq I$ and $(z, x_2)^l \subseteq (b_1, x_2)^l \subseteq I$. Clearly either $x_1 \neq x$ or $x_2 \neq x$. Suppose $x_1 \neq x$. Then we have a path $a - x_1 - b$ and hence $a - x - b - x_1 - a$ is contained in a cycle of length ≤ 4. □

As an immediate consequence of Theorem 3, we have the following corollaries.

Corollary 1. *Let $|V(\widehat{G_I(P)})| \geq 3$ and if $I \cup \{x\}$ is not a semi-ideal of P for any $x \in P \backslash I$, then any edge in $\widehat{G_I(P)}$ is contained in a cycle of length ≤ 4, and therefore $\widehat{G_I(P)}$ is a union of triangles and squares.*

Corollary 2. *Let I be a semi-ideal of P. If $\widehat{G_I(P)}$ has cut vertex a, then $I \cup \{a_1\}$ is a semi-ideal of P for some $a_1 \in (a)^l \backslash I$.*

Corollary 3. *Let I be a semi-ideal of P. Then $\widehat{G_I(P)}$ can not be a pentagon.*

Theorem 4. *Let I be a semi-ideal of P. If $\widehat{G_I(P)}$ contains a cycle, then the core K of $\widehat{G_I(P)}$ is a union of triangles and rectangles. Moreover, any vertex in $\widehat{G_I(P)}$ is either a vertex of the core K of $\widehat{G_I(P)}$ or an end vertex of $\widehat{G_I(P)}$.*

Proof. It follows from Theorem 1. □

Corollary 4. *Let I be a semi-ideal of P. If P has a greatest element e with $|V(\widehat{G_I(P)})| \geq 2$, then $\widehat{G_I(P)} = K$, where K is the core of $\widehat{G_I(P)}$.*

For any $a, b \in P$, if $a \leq b$, then a and b are said to be comparable. Following [1], an element $x \in P \backslash I$ is called a minimal element of $P \backslash I$ if $y \in P \backslash I$ and $y \leq x$ implies that $y = x$. The set of all minimal elements of $P \backslash I$ is denoted by $Min(P \backslash I)$. It is easy to prove that for any semi-ideal I of P, I is prime if and only if $|Min(P \backslash I)| = 1$.

Lemma 1. *Let I be a semi-ideal of a poset P. Then we have the following*:

(*a*) *If $x, y \in Min(P \backslash I)$ with $x \neq y$, then $(x, y)^l \subseteq I$.*

(*b*) *Let $x \in Min(P \backslash I)$ and $y \in P \backslash I$. If $(y)^l \backslash ((x)^u \cup I) \neq \phi$, then $(x, y_i)^l \subseteq I$ for all $y_i \in (y)^l \backslash (x)^u \cup I$.*

(*c*) *For any $x \in P \backslash I$, $x \in V(\widehat{G_I(P)})$ if and only if there exists $x_1 \in (x)^l \backslash I$ such that $(I : x_1) \neq I$.*

(*d*) *If $|Min(P \backslash I)| \geq 2$, then $Min(P \backslash I) \subseteq V(G_I(P))$*

Lemma 2. *Let I be a semi-ideal of P and $V_1, V_2, ..., V_n$ be partitions of $\widehat{G_I(P)}$. If $x \in V_i \cap Min(P\backslash I)$, then $x \leq y$ for all $y \in V_i$.*

Proof. Let $x \in V_i \cap Min(P\backslash I)$. Suppose that $x \nleq y$ for some $y \in V_i$. Then there exists some $s \in (y)^l \cap Min(P\backslash I)$ such that x and s are adjacent which implies that x and y are adjacent, a contradiction to Theorem 1(ii). □

Lemma 3. *Let I be a semi-ideal of P and $V_1, V_2, \ldots, V_n$ be partitions of $V(\widehat{G_I(P)})$. If $|(x)^l \cap Min(P\backslash I)| = 1$ and for any $t \in (x)^l \cap Min(P\backslash I)$, then t and x are in the same partition of $V(\widehat{G_I(P)})$.*

Proof. Suppose that t and x are in different partitions of $V(\widehat{G_I(P)})$. Then $(t, x')^l \subseteq I$ for some $x' \in (x)^l\backslash I$. Clearly $x' \neq t$ and $x' \notin Min(P\backslash I)$. Thus $t \in (t, x')^l \subseteq I$, a contradiction. □

Lemma 4. *Let $V_1, V_2, ..., V_n$ be partitions of $V(\widehat{G_I(P)})$.*

(*i*) *If $|(x)^l \cap Min(P\backslash I)| = 1$ for $x \in V_i\backslash Min(P\backslash I)$, then $|V_i| > 1$.*

(*ii*) *If $|(x)^l \cap Min(P\backslash I)| \geq 2$ for all $x \in V_i$, then $|V_i| = 1$.*

(*iii*) *If $|V_i| > 1$, then $|(x)^l \cap Min(P\backslash I)| = 1$ for any $x \in V_i$.*

Proof. (*i*) Let $t \in (x)^l \cap Min(P\backslash I)$. Then by Lemma 3, t and x are in the same partition of $V(\widehat{G_I(P)})$ and hence $|V_i| > 1$.

(*ii*) Suppose that $|V_i| > 1$. Then there exist distinct elements x and y in V_i such that $x_1, x_2 \in (x)^l \cap Min(P\backslash I)$ and $y_1, y_2 \in (y)^l \cap Min(P\backslash I)$. If $\{y_1, y_2\} \cap ((y)^l \cup \{x_1, x_2\}) \neq \phi$, then x and y are adjacent. Otherwise $\{y_1, y_2\} \cap ((y)^l \cup \{x_1, x_2\}) = \phi$. Then again x and y are adjacent, a contradiction to $x, y \in V_i$.

(*iii*) Let x and y be distinct vertices in V_i and suppose that $|(x)^l \cap Min(P\backslash I)| > 1$ for some $x \in V_i$. Then there exist distinct elements s and t in $(x)^l \cap Min(P\backslash I)$ such that s and t are adjacent which imply that x and y are adjacent, a contradiction. □

The main result of this section is given as follows.

Theorem 5. *Let I be a semi-ideal of P. Then we have the following*:

(*i*) *$diam(\widehat{G_I(P)}) = 1$ if and only if $|(x)^l \cap Min(P\backslash I)| \geq 2$ for every $x \in V(\widehat{G_I(P)})\backslash Min(P\backslash I)$.*

(*ii*) *$diam(\widehat{G_I(P)}) = 2$ if and only if $|(x)^l \cap Min(P\backslash I)| \leq 1$ for some $x \in V(\widehat{G_I(P)})\backslash Min(P\backslash I)$.*

Proof. (*i*) Assume that $diam(\widehat{G_I(P)}) = 1$. Clearly $Min(P\backslash I) \subseteq V(\widehat{G_I(P)})$ by Lemma 1. Suppose that $|(x)^l \cap Min(P\backslash I)| = 1$ for some $x \in V(\widehat{G_I(P)})\backslash Min(P\backslash I)$. Then for any $t \in (x)^l \cap Min(P\backslash I)$ by Lemma 3, t and x are in same partitions of $\widehat{G_I(P)}$, a contradiction to $diam(\widehat{G_I(P)}) = 1$.

Conversely, assume that $|(x)^l \cap Min(P\backslash I)| \geq 2$ for every $x \in V(\widehat{G_I(P)})\backslash Min(P\backslash I)$. Then there exist at least two distinct elements s and t in $(x)^l \cap Min(P\backslash I)$ such that $(s, t)^l \subseteq I$ which implies $x - s$ and $x - t$ are edges in $\widehat{G_I(P)}$.

Suppose there exists $y(\neq x) \in V(\widehat{G_I(P)})$ such that x is not adjacent to y. If $y \in Min(P \backslash I) \cup \{s,t\}$, then $x - y$ is an edge in $\widehat{G_I(P)}$, a contradiction. Otherwise $y \notin Min(P \backslash I) \cup \{s,t\}$. Then by assumption, there exist atleast two distinct elements s_1 and t_1 in $(y)^l \cap Min(P \backslash I)$ such that $(s_1,t_1)^l \subseteq I$. If $s_1 \in \{s,t\}$, then $x-y$ is an edge in $\widehat{G_I(P)}$. Otherwise $s_1 \notin \{s,t\}$. Then $(s_1,t)^l \subseteq I$ which implies $x-y$ is an edge in $\widehat{G_I(P)}$, again a contradiction. Thus $x-y$ is an edge in $\widehat{G_I(P)}$ and hence $diam(\widehat{G_I(P)}) = 1$.

(*ii*) It follows from (i) and Theorem 2. □

As an immediate consequence, we have the following corollaries.

Corollary 5. *If $|(x)^l \cap Min(P \backslash I)| \geq 2$ for every $x \in V(\widehat{G_I(P)}) \backslash Min(P \backslash I)$, then $\widehat{G_I(P)}$ is a complete graph.*

Corollary 6. *Let I be a semi-ideal of a poset P. For any two distinct $x, y \in Min(P \backslash I)$, we have the following:*

(i) If $(x,y)^u = V(\widehat{G_I(P)}) \backslash Min(P \backslash I)$, then $diam(\widehat{G_I(P)}) = 1$.

(ii) For any $t \in V(\widehat{G_I(P)}) \backslash Min(P \backslash I)$. If $t \notin (x,y)^u$ and $|Min(P \backslash I)| = 2$, then $diam(\widehat{G_I(P)}) = 2$.

Proof. (i) Let $s \in (x,y)^u$. Then $|(s)^l \cap Min(P \backslash I)| \geq 2$. By Theorem 5(i), we have $diam(\widehat{G_I(P)}) = 1$.

(ii) Let $t \in V(\widehat{G_I(P)}) \backslash (x,y)^u$. Then either $x \leq t$ or $y \leq t$ and thus $diam(\widehat{G_I(P)}) = 2$, by Theorem 5 □

Theorem 6 (see Theorem 4.2 [1])**.** *Let P be a poset. Then we have $girth(\widehat{G_I(P)}) \in \{3, 4, \infty\}$.*

Proof. Suppose that $girth(G_I(P)) \neq \infty$. Then there exists a cycle of minimal length n in $G_I(P)$, say $x_1 - x_2 - ... - x_n - x_1$. Let $n \geq 5$. So there exists $t \in (x'_2, x'_4)^l \backslash I$ for all $x'_2 \in (x_2)^l \backslash I$ and $x'_4 \in (x_4)^l \backslash I$ such that $(t, x'_1) \subseteq I$ and $(t, x'_5) \subseteq I$ for some $x'_1 \in (x_1)^l \backslash I$ and $x'_5 \in (x_5)^l \backslash I$ which implies $x_1 - t - x_5 - ... - x_n - x_1$ is a cycle of length $n-2$, a contradiction. Thus $n = 3$ or 4, so $girth(G_I(P)) = 3$ or 4. Thus $girth(\widehat{G_I(P)}) \in \{3, 4, \infty\}$. □

Theorem 7. *Let P be a poset. Then we have the following:*

(i) Either $|Min(P \backslash I)| \geq 3$ or $|(x)^l \cap Min(P \backslash I)| = 2$ for some $x \in V(\widehat{G_I(P)}) \backslash Min(P \backslash I)$ if and only if $girth(\widehat{G_I(P)}) = 3$.

(ii) $|Min(P \backslash I)| = 2$ with $|(t)^l \cap Min(P \backslash I)| = 1$ for all $t \in (P \backslash I)$ and there exist two elements $x, y \in V(\widehat{G_I(P)}) \backslash Min(P \backslash I)$ such that $(x)^l \cap Min(P \backslash I) \neq (y)^l \cap Min(P \backslash I)$ if and only if $girth(\widehat{G_I(P)}) = 4$.

(iii) $|Min(P \backslash I)| = 2$ and $(x)^l \cap Min(P \backslash I) = (y)^l \cap Min(P \backslash I)$ with $|(x)^l \cap Min(P \backslash I)| = 1$ for all $x, y \in V(\widehat{G_I(P)}) \backslash Min(P \backslash I)$ if and only if $girth(\widehat{G_I(P)}) = \infty$.

Proof. (*i*) If $|Min(P\backslash I)| \geq 3$, then by Lemma 1(a), $girth(\widehat{G_I(P)}) = 3$. So we may assume that $|Min(P\backslash I)| < 3$.

Suppose that $|(x)^l \cap Min(P\backslash I)| = 2$ for some $x \in V(\widehat{G_I(P)})\backslash Min(P\backslash I)$. Then $|Min(P\backslash I)| = 2$. Let $s, t \in (x)^l \cap Min(P\backslash I)$ for $x \in V(\widehat{G_I(P)})\backslash Min(P\backslash I)$. Then by Lemma 1, s and t are adjacent, and so $s - t - x - s$ is a cycle of length 3 and hence the $girth(\widehat{G_I(P)}) = 3$.

Conversely, assume that $girth(\widehat{G_I(P)}) = 3$. Then $\widehat{G_I(P)}$ has a cycle $a - b - c - a$ of length 3. Suppose $|Min(P\backslash I)| < 3$ and $|(x)^l \cap Min(P\backslash I)| \neq 2$ for all $x \in V(\widehat{G_I(P)})\backslash Min(P\backslash I)$. If $|Min(P\backslash I)| = 1$, then I is a prime semi-ideal of P and hence $\widehat{G_I(P)} = \phi$, a contradiction. Otherwise $|Min(P\backslash I)| = 2$. Then we must have $|(x)^l \cap Min(P\backslash I)| = 1$ and at least one of the element in $\{a, b, c\}$ is not minimal say, c. Let s and t be distinct elements in $Min(P\backslash I)$. Then either $s \in (c)^l$ or $t \in (c)^l$. If $s \in (c)^l$, then $t \in (a)^l \cap (b)^l$. Indeed, if $t \notin (a)^l \cap (b)^l$, then $s \in (a)^l \cap (b)^l$ which implies a and b are not adjacent, a contradiction. If $t \in (c)^l$, then $s \in (a)^l \cap (b)^l$. Indeed, if $s \notin (a)^l \cap (b)^l$, then $t \in (a)^l \cap (b)^l$ which implies a and b are not adjacent, a contradiction.

(*ii*) Assume that $|Min(P\backslash I)| = 2$ with $|(t)^l \cap Min(P\backslash I)| = 1$ for all $t \in P\backslash I$ and there exist two elements $x, y \in V(\widehat{G_I(P)})\backslash Min(P\backslash I)$ such that $(x)^l \cap Min(P\backslash I) \neq (y)^l \cap Min(P\backslash I)$. By (i), $girth(\widehat{G_I(P)}) \neq 3$. Let s and t be distinct elements in $Min(P\backslash I)$. Then $s \in (x)^l \cap Min(P\backslash I)$ and $t \in (y)^l \cap Min(P\backslash I)$ which implies $s - t - x - y - s$ is a cycle of length 4 and so $girth(\widehat{G_I(P)}) = 4$.

Conversely, assume that $girth(\widehat{G_I(P)}) = 4$. Then $\widehat{G_I(P)}$ has a cycle $a-b-c-d-a$ of length 4. Then by (i), $|Min(P\backslash I)| < 3$ and $|(x)^l \cap Min(P\backslash I)| \neq 2$ for all $x \in V(\widehat{G_I(P)})\backslash Min(P\backslash I)$. Clearly $|Min(P\backslash I)| \neq 1$. So $|Min(P\backslash I)| = 2$. Since $|(x)^l \cap Min(P\backslash I)| \neq 2$, we must have $|(x)^l \cap Min(P\backslash I)| = 1$ for all $x \in V(\widehat{G_I(P)})\backslash Min(P\backslash I)$. Let s and t be distinct elements in $Min(P\backslash I)$. If $(x)^l \cap Min(P\backslash I) = (y)^l \cap Min(P\backslash I)$ for all $x, y \in V(\widehat{G_I(P)})\backslash Min(P\backslash I)$, then x and y are in the same partition of $V(\widehat{G_I(P)})$ and hence x and y are not adjacent. Thus for every $x \in V(\widehat{G_I(P)})\backslash Min(P\backslash I)$, x is adjacent to either s or t but not both and hence $\widehat{G_I(P)}$ is a star graph, a contradiction to $girth(\widehat{G_I(P)}) = 4$.

(*iii*) It follows from (i), (ii) and Theorem 6. □

Theorem 8. *Let P be a poset. If the $girth(\widehat{G_I(P)}) = 4$, then $G_I(P)$ and $\widehat{G_I(P)}$ are isomorphic.*

Proof. Suppose that $girth(\widehat{G_I(P)}) = 4$. Then by Theorem 7, $|Min(P\backslash I)| = 2$ with $|(t)^l \cap Min(P\backslash I)| = 1$ for all $t \in (P\backslash I)$ and there exist two elements $x, y \in V(\widehat{G_I(P)})\backslash Min(P\backslash I)$ such that $(x)^l \cap Min(P\backslash I) \neq (y)^l \cap Min(P\backslash I)$. Suppose that there exists $x \in V(\widehat{G_I(P)})\backslash V(G_I(P))$. Then $x \notin Min(P\backslash I)$. Let a and b be distinct elements in $Min(P\backslash I)$. Then a and b are adjacent in $V(G_I(P))$. Since $x \notin V(G_I(P))$, we have $(a, x)^l \nsubseteq I$ and $(b, x)^l \nsubseteq I$. By minimality of a and b, we have $a < x$ and $b < x$ which imply $a - b - x - a$ is a cycle of length 3 in $\widehat{G_I(P)}$, a contradiction to $girth(\widehat{G_I(P)} = 4$. Thus $V(G_I(P)) = V(\widehat{G_I(P)})$.

Suppose that $x - y$ is an edge in $\widehat{G_I(P)}$, but not in $G_I(P)$. Then atleast one is not minimal in $\widehat{G_I(P)}$. Suppose that $y \notin Min(P\backslash I)$. Since $x - y$ is not an edge in $G_I(P)$, there exist $s, t \in V(G_I(P))$ such that $x - s - y$ or $x - s - t - y$. If $x - s - t - y$ is a path in $V(G_I(P))$, then there exist $r \in (x, y)^l \backslash I$ such that $r - s - t - r$ is a cycle of length 3 in $G_I(P)$. So the $girth(\widehat{G_I(P)} = 3$, a contradiction. Thus we have a path $x - s - y$ in $G_I(P)$ which implies $x - s - y - x$ is a cycle of length 3 in $\widehat{G_I(P)}$, a contradiction. □

Theorem 9. *Let P be a poset. If $\widehat{G_I(P)}$ is a star graph, then $G_I(P)$ and $\widehat{G_I(P)}$ are isomorphic.*

Proof. Suppose that $\widehat{G_I(P)}$ is a star graph. Then by Theorem 7 (iii), $|Min(P\backslash I)| = 2$ and $|(x)^l \cap Min(P\backslash I) = (y)^l \cap Min(P\backslash I)| = 1$ for every $x, y \in V(\widehat{G_I(P)}) \backslash Min(P\backslash I)$. Suppose that $V(\widehat{G_I(P)}) \nsubseteq V(G_I(P))$. Then there exists $x \in V(\widehat{G_I(P)}) \backslash V(G_I(P))$. Clearly $x \notin Min(P\backslash I)$. Let a and b be distinct elements in $Min(P\backslash I)$. Then a and b are adjacent in $V(G_I(P))$. Since $x \notin V(G_I(P))$, we have $(a, x)^l \nsubseteq I$ and $(b, x)^l \nsubseteq I$ which imply $a < x$ and $b < x$. Then $b - x$ and $a - x$ in $\widehat{G_I(P)}$ and hence $a - b - x - a$ is a cycle of length 3 in $\widehat{G_I(P)}$, a contradiction. Thus $V(G_I(P)) = V(\widehat{G_I(P)})$.

Suppose that $x - y$ is an edge $\widehat{G_I(P)}$ but not in $G_I(P)$. Then at least one is not minimal in $\widehat{G_I(P)}$. Suppose that $y \notin Min(P\backslash I)$. Since $x - y$ is not an edge in $G_I(P)$, there exists $s \in V(G_I(P))$ such that $x - s - y$ which implies $x - s - y - x$ is a cycle of length 3 in $\widehat{G_I(P)}$, a contradiction. □

Following [2], $\widehat{G_I(P)}$ is called a generalized tree if for every $x \in P$ and for every $x_1, x_2 \in (x)^l \backslash I$, either $x_1 \leq x_2$ or $x_2 \leq x_1$.

In general, zero-divisor graphs $G_I(P)$ and $\widehat{G_I(P)}$ are not isomorphic. The following theorem shows that under certain condition both graphs $G_I(P)$ and $\widehat{G_I(P)}$ are isomorphic.

Theorem 10. *Let P be a poset. If $\widehat{G_I(P)}$ is a generalized tree, then $G_I(P)$ and $\widehat{G_I(P)}$ are isomorphic.*

Proof. If $\widehat{G_I(P)}$ is a generalized tree of a poset P, then $|(x)^l \cap Min(P\backslash I)| = 1$ for every $x \in V(\widehat{G_I(P)})$. By Theorem 7 (ii) and (iii), $girth(\widehat{G_I(P)})$ is 4 or ∞. Thus $G_I(P)$ and $\widehat{G_I(P)}$ are isomorphic by Theorems 8 and 9. □

Theorem 11. *If $|Min(P\backslash I)| = n$, then there are at most n-partitions $\{V_i\}_{i=1}^n$ with* $|V_i| \geq 1$.

Proof. Let $Min(P\backslash I) = \{a_1, a_2, ..., a_n\}$ and $V_1, V_2, ..., V_r$ be r-partitions of $\widehat{G_I(P)}$ where $r \geq n$. Take $a_i \in V_i$ for $i \in \{1, 2, ..., n\}$. Suppose that there are at least

$(n+1)-$partitions $\{V_i\}_{i=1}^{n+1}$ of $V(\widehat{G_I(P)})$ with $|V_i| > 1$. Then there exist distinct s and t in V_{n+1} such that $s, t \notin Min(P\backslash I)$. By Lemma 4 (iii), $|(s)^l \cap Min(P\backslash I)| = 1$ and $|(t)^l \cap Min(P\backslash I)| = 1$. Let us assume that $a_1 \in (s,t)^l \cap Min(P\backslash I)$. Then a_1, s and t are in same partitions of $\widehat{G_I(P)}$, a contradiction. □

Definition 1. For any $x \in V(\widehat{G_I(P)})$, $\widehat{N(x)} = \{y \in P\backslash I \ : \ (x_1, y_1)^l \subseteq I \ for \ some \ x_1 \in (x)^l\backslash I \ and \ y_1 \in (y)^l\backslash I\}$.

It is clear that if $a \leq b$, then $\widehat{N(a)} \subseteq \widehat{N(b)} \cup \{b\}$, and if $a \leq b$ and $|(a)^l \cap Min(P\backslash I) = (b)^l \cap Min(P\backslash I)| = 1$, then $\widehat{N(a)} = \widehat{N(b)}$.

The following example shows that $\widehat{N(x)} \cup I$ need not be a semi-ideal of P for $x \in Min(P\backslash I)$.

Example 1. Let $P = \{1, 2, 3, 4, 12\}$. Then P is a poset under the relation divisiblity and $I = \{1\}$ is a semi-ideal of P. Here $2 \in Min(P\backslash I)$ and $\widehat{N(2)} \cup I = \{1, 3, 12\}$ is not a semi-ideal of P.

The following theorem illustrates that under certain condition $\widehat{N(x)} \cup I$ is a semi-ideal of P.

Theorem 12. *Let $V_1, V_2, ..., V_n$ are n-partitions of $\widehat{G_I(P)}$. If $x \in V_i$ is minimal and $(x)^u \subseteq V_i$, then $\widehat{N(x)} \cup I$ is a prime semi-ideal of P.*

Proof. Let $x \in V_i$ be a minimal element and $(x)^u \subseteq V_i$. Let $a, b \in P$ with $a \leq b$ and $b \in \widehat{N(x)} \cup I$. If $b \in \widehat{N(x)}$, then $b \notin V_i$, and so $a \notin V_i$ which implies $a \in \widehat{N(x)} \subseteq \widehat{N(x)} \cup I$.

Now we claim that $\widehat{N(x)} \cup I$ is a prime semi-ideal of P. Suppose that $(a,b)^l \subseteq \widehat{N(x)} \cup I$ and $a, b \notin \widehat{N(x)} \cup I$. Then a, b, x are in same partition of $\widehat{G_I(P)}$ which implies a and b are comparable. Thus either $a \in \widehat{N(x)}$ or $b \in \widehat{N(x)}$, a contradiction. □

Example 1 shows that one can not drop the condition $(x)^u \subseteq V_i$ in Theorem 12.

Theorem 13. *If $x \in Min(P\backslash I)$, then $(V(\widehat{G_I(P)})\backslash \widehat{N(x)}) \cup I$ is a semi-ideal of P. Moreover if $|Min(P\backslash I)| = 2$, then $(V(\widehat{G_I(P)})\backslash \widehat{N(x)}) \cup I$ is a prime semi-ideal of P.*

Proof. Let $a, b \in P$ with $a \leq b$ and $b \in (V(\widehat{G_I(P)})\backslash \widehat{N(x)}) \cup I$. If $b \in V(\widehat{G_I(P)})\backslash \widehat{N(x)}$, then b and x are in same partition of $\widehat{G_I(P)}$ which implies a and x are in same partition of $\widehat{G_I(P)}$, and so $a \in V(\widehat{G_I(P)})\backslash \widehat{N(x)}$. Thus $(V(\widehat{G_I(P)})\backslash \widehat{N(x)}) \cup I$ is a semi-ideal of P.

Let $y \in Min(P\backslash I)$. Suppose that $(a,b)^l \subseteq (V(\widehat{G_I(P)})\backslash \widehat{N(x)}) \cup I$ and $a, b \notin (V(\widehat{G_I(P)})\backslash \widehat{N(x)}) \cup I$. If a and b are in same partition of $\widehat{G_I(P)}$, then a and b are comparable. So either $a \in (V(\widehat{G_I(P)})\backslash \widehat{N(x)}) \cup I$ or $b \in (V(\widehat{G_I(P)})\backslash \widehat{N(x)}) \cup I$, a contradiction.

If a and b are in different partition of $\widehat{G_I(P)}$, then $y \in (a)^l \cap (b)^l$ which implies $y \in V(\widehat{G_I(P)}) \backslash \widehat{N(x)}$, a contradiction. Thus either $a \in (V(\widehat{G_I(P)}) \backslash \widehat{N(x)}) \cup I$ or $b \in (V(\widehat{G_I(P)}) \backslash \widehat{N(x)}) \cup I$, and hence $(V(\widehat{G_I(P)}) \backslash \widehat{N(x)}) \cup I$ is a prime semi-ideal of P. □

As an immediate consequence, we have the following corollary.

Corollary 7. *If $x \in Min(P \backslash I)$ and $u \in (V(\widehat{G_I(P)}) \backslash \widehat{N(x)}) \cup I$, then $(u,v)^l \subseteq (V(\widehat{G_I(P)}) \backslash \widehat{N(x)}) \cup I$ for $v \in P$.*

Proof. It follows from Theorem 13. □

The following example shows that one can not drop the condition $|Min(P \backslash I)| = 2$ in Theorem 13.

Example 2. Let $P = \{1,2,3,5,6,12\}$ be a poset under the relation divisiblity. Then $I = \{1\}$ is a semi-ideal of P. Take $x = 2$. Here $(3,5)^l \subseteq I$, but $3,5 \notin (V(\widehat{G_I(P)}) \backslash \widehat{N(x)}) \cup I$ and hence $(V(\widehat{G_I(P)}) \backslash \widehat{N(x)}) \cup I$ is not a prime semi-ideal of P.

We use $Spec(P)$ for the spectrum of prime semi-ideals of P. For any semi-ideal J of P and $a \in P$, we define $V(a) = \{I \in Spec(P) : a \in I\}$ and $D(I) = Spec(P) \backslash V(I)$. Let $V(J) = \bigcap_{a \in J} V(a)$. Then $F = \{V(J) : J \text{ is a semi-ideal of } P\}$ is closed under finite union and arbitrary intersections, so that there is a topology on $Spec(P)$ for which F is the family of closed sets. This is called the Zariski topology [6]. Let $\mathbb{P}$ be the intersection of all prime semi-ideals of P. Then we set $Supp(a) = \bigcap_{x \in (\mathbb{P}:a)} V(x)$. Also $B = \{D(a) : a \in P\}$ form a basis for a topology on $Spec(P)$, and $V(a) = V((a)^l)$ and $D(a) \subseteq Supp(a)$.

Lemma 5. *Let P be a poset. Then $a \in V(\widehat{G_{\mathbb{P}}(P)})$ if and only if $Supp(a_1)$ is non-empty and $Supp(a_1) \neq Spec(P)$ for some $a_1 \in (a)^l \backslash \mathbb{P}$.*

Proof. Let $a \in V(\widehat{G_{\mathbb{P}}(P)})$. Then there exists $b_1 \in P \backslash \mathbb{P}$ such that $(a_1,b_1)^l \subseteq \mathbb{P}$ for some $a_1 \in (a)^l \backslash \mathbb{P}$, so $D(a_1) \cap D(b_1) = \phi$. Suppose that $Supp(a_1) = \phi$. Then $x \notin P_1$ for some $x \in (\mathbb{P} : a_1)$ which implies $a_1 \in \mathbb{P}$, a contradiction. Suppose that $Supp(a_1) = Spec(P)$ for all $a_1 \in (a)^l \backslash \mathbb{P}$. Then for any prime ideal $P_1 \in Supp(a_1) = V((\mathbb{P} : a_1))$, we have $b_1 \in \mathbb{P}$, a contradiction.

Conversely assume that $Supp(a_1)$ is non-empty and $Supp(a_1) \neq Spec(P)$ for some $a_1 \in (a)^l \backslash \mathbb{P}$. Then there exists $x \in P \backslash \mathbb{P}$ such that $(a_1,x)^l \subseteq \mathbb{P}$ which implies $a \in V(\widehat{G_{\mathbb{P}}(P)})$. □

Lemma 6. *Let P be a poset and $a,b,c \in \widehat{G_{\mathbb{P}}(P)}$ be distinct vertices. Then the following statements are equivalent.*

(i) c is adjacent to both a and b.

(ii) There exist $a_1 \in (a)^l \backslash \mathbb{P},\ b_1 \in (b)^l \backslash \mathbb{P}$ *and* $c_1, c_2 \in (c)^l \backslash \mathbb{P}$ *with* $D(a_1) \cap D(c_1) = \phi$ *and* $D(b_1) \cap D(c_2) = \phi$

(iii) There exist $a_1 \in (a)^l \backslash \mathbb{P},\ b_1 \in (b)^l \backslash \mathbb{P}$ *and* $c_1, c_2 \in (c)^l \backslash \mathbb{P}$ *with* $D(a_1) \cup D(b_1) \subseteq V(c_1)$

(iv) There exist $a_1 \in (a)^l \backslash \mathbb{P},\ b_1 \in (b)^l \backslash \mathbb{P}$ *and* $c_1, c_2 \in (c)^l \backslash \mathbb{P}$ *with* $Supp(a_1) \cup supp(b_1) \subseteq V(c_1)$.

Proof. It is easy to prove. □

Theorem 14. *Let* P *be a poset. Then we have the following:*

(i) $d(a, b) = 1$ *if and only if* $D(a_1) \cap D(b_1) = \phi$ *for some* $a_1 \in (a)^l \backslash \mathbb{P}$ *and* $b_1 \in (b)^l \backslash \mathbb{P}$.

(ii) $d(a, b) = 2$ *if and only if* $D(a_1) \cap D(b_1) \neq \phi$ *and* $Supp(a_1) \cup Supp(b_1) \neq Spec(P)$ *for all* $a_1 \in (a)^l \backslash \mathbb{P}$ *and* $b_1 \in (b)^l \backslash \mathbb{P}$.

Proof. (i) The proof follows from Lemma 6.

(ii) Assume that $d(a, b) = 2$. Then there exists $c \in P \backslash \mathbb{P}$ such that $a - c - b$. Then $(a_1, c_1)^l \subseteq \mathbb{P}$ and $(b', c')^l \subseteq \mathbb{P}$ for some $a_1 \in (a)^l \backslash \mathbb{P},\ b' \in (b)^l \backslash \mathbb{P}$ and $c_1, c' \in (c)^l \backslash \mathbb{P}$. By (i), we have $D(a_1) \cap D(b_1) \neq \phi$.

Case(i) $\mathbf{c_1 = c'}$. Then $c_1 \in (\mathbb{P} : a_1) \cap (\mathbb{P} : b')$. Since $c \in P \backslash \mathbb{P}$, there exists a prime ideal P_1 such that $c_1 \notin P_1$ which implies $(\mathbb{P} : a_1) \nsubseteq P_1$ and $(\mathbb{P} : b') \nsubseteq P_1$. Thus $P_1 \notin Supp(a_1) \cup Supp(b')$ and hence $Supp(a_1) \cup Supp(b') \neq Spec(P)$ for all $a_1 \in (a)^l \backslash \mathbb{P}$ and $b' \in (b)^l \backslash \mathbb{P}$.

Case(ii) $\mathbf{c_1 \neq c'}$. If c_1 is adjacent to c', then either $a - c'$ or $b - c_1$, by Lemma 2. Suppose that $a - c'$. Then we have a path $a - c' - b$, and by Case (i), $Supp(a_1) \cup Supp(b_1) \neq Spec(P)$ for all $a_1 \in (a)^l \backslash \mathbb{P}$ and $b_1 \in (b)^l \backslash \mathbb{P}$. Suppose that $b - c_1$. Then we have a path $b - c_1 - a$, and by Case (i), $Supp(a_1) \cup Supp(b_1) \neq Spec(P)$ for all $a_1 \in (a)^l \backslash \mathbb{P}$ and $b_1 \in (b)^l \backslash \mathbb{P}$.

Suppose that c_1 is not adjacent to c'. Then there exists $t \in (c_1', c'')^l$ such that $t \notin \mathbb{P}$ for all $c_1' \in (c_1)^l \backslash \mathbb{P}$ and $c'' \in (c')^l \backslash \mathbb{P}$. Thus $t \in (\mathbb{P} : a_1) \cap (\mathbb{P} : b')$. Since $t \in P \backslash \mathbb{P}$, there exists a prime ideal P_2 such that $t \notin P_2$ which implies $(\mathbb{P} : a_1) \nsubseteq P_2$ and $(\mathbb{P} : b') \nsubseteq P_2$. Thus $P_2 \notin Supp(a_1) \cup Supp(b')$ and hence $Supp(a_1) \cup Supp(b') \neq Spec(P)$ for all $a_1 \in (a)^l \backslash \mathbb{P}$ and $b' \in (b)^l \backslash \mathbb{P}$.

Conversely assume that $D(a_1) \cap D(b_1) \neq \phi$ and $Supp(a_1) \cup Supp(b_1) \neq Spec(P)$ for all $a_1 \in (a)^l \backslash \mathbb{P}$ and $b_1 \in (b)^l \backslash \mathbb{P}$. Then there exists a prime ideal P_1 such that $P_1 \notin Supp(a_1) \cup Supp(b_1)$ for all $a_1 \in (a)^l \backslash \mathbb{P}$ and $b_1 \in (b)^l \backslash \mathbb{P}$. Hence there exists $c \in (\mathbb{P} : a_1)$ and $d \in (\mathbb{P} : b_1)$ such that $c, d \notin P_1$. Clearly $(c, d)^l \nsubseteq \mathbb{P}$. Then there exists $t \in (c, d)^l \backslash \mathbb{P}$ such that $a - t - b$ is a path of length 2 and hence $d(a, b) = 2$. □

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