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Properties of annihilator graph of a commutative semigroup

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Abstract. Let S be a commutative semigroup with zero. Let $Z(S)$ be the set of all zero-divisors of S. We define the annihilator graph of S, denoted by $ANN_G(S)$, as the undirected graph whose set of vertices is $Z(S)^* = Z(S) - \{0\}$, and two distinct vertices x and y are adjacent if and only if $ann_S(xy) \neq ann_S(x) \cap ann_S(y)$. In this paper, we study some basic properties of $ANN_G(S)$ by means of $\Gamma(S)$. We also show that if $Z(S) \neq S$, then $ANN_G(S)$ is a subgraph of $\Gamma(S)$. Moreover, we investigate some properties of the annihilator graph $ANN_G(S)$ related to minimal prime ideals of S. We also study some connections between the domination numbers of annihilator graphs and zero-divisor graphs.

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1 Introduction

Let R be a commutative ring with nonzero identity, and let $Z(R)$ be its set of zero-divisors. There has been considerable attention in the literature to associating graphs with algebraic structures (see [1] and [4]). Probably the most attention has been to the zero-divisor graph $\Gamma(R)$ for a commutative ring R.

Throughout this paper S is a commutative semigroup with zero whose operation is written multiplicative and $S^* = S - \{0\}$. The set of all zero-divisors of S is denoted by $Z(S)$ and $Z(S)^* = Z(S) - \{0\}$. The concept of a zero divisor graph of S is a simple undirected graph $\Gamma(S)$ whose vertices are $Z(S)^*$ and for each two distinct elements x and y in $Z(S)^*$ x is adjacent to y in $\Gamma(S)$ if and only if $xy = 0$. In [7, 8] it is proved that $\Gamma(S)$ is connected and the diameter of $\Gamma(S)$ is less than or equal to three, if $\Gamma(S)$ contains a cycle, then its girth is less than or equal to four. In [2], Badawi introduced the concept of the annihilator graph for a commutative ring R, denoted by $AG(R)$, with vertices $Z(R)^*$ and two distinct vertices x and y adjacent if and only if $ann_R(xy) \neq ann_R(x) \cup ann_R(y)$. Khashyarmanesh and Afkhami in [9] studied the annihilator graph associated to a commutative semigroup S with zero.

In [3], P. P. Baruah, K. Patra defined and studied the annihilator graph $ANN_G(R)$ of a commutative ring R, where the set of vertices of $ANN_G(R)$ is $Z(R)^* = Z(R) - \{0\}$, and two distinct vertices x and y are adjacent if and only if $ann_R(xy) \neq ann_R(x) \cap ann_R(y)$. They proved that $ANN_G(R)$ is connected with diameter at most two. Also, if $ANN_G(R)$ contains a cycle, they obtained that girth of $ANN_G(R)$ is at most four.

In this paper, we give the definition of the annihilator graph in other way. In this paper, we define the annihilator graph for a commutative semigroup S, denoted by $ANN_G(S)$. The graph $ANN_G(S)$ is an undirected graph with vertex set $Z(S)^* = Z(S) - \{0\}$, and two distinct vertices x and y are adjacent if and only if $ann_S(xy) \neq ann_S(x) \cap ann_S(y)$, where $ann_S(x) = \{s \in S \mid xs = 0\}$. We investigate some basic properties of $ANN_G(S)$ by means of $\Gamma(S)$ and $AG(S)$. We also show that if $Z(S) \neq S$, then $ANN_G(S)$ is a subgraph of $\Gamma(S)$. Also, some relations between the domination numbers of $ANN_G(S)$ and $\Gamma(S)$ are studied.

In order to make this paper easier to follow, we recall various notions which will be used in the sequel. We use the standard terminology of graphs contained in [5]. Let G be an undirected graph. We denote the vertex set and the edge set of G by $V(G)$ and $E(G)$. We use the notation $x \sim y$ to denote that x is adjacent to y in G. The distance between two vertices x and y of G, denoted by $d(x, y)$, is the length of a shortest path connecting x and y, if such a path exists; otherwise, we use $d(x, y) = \infty$.

The diameter of G is $diam(G) = \sup\{d(x, y) \mid x$ and y are distinct vertices of $G\}$. The girth of G, denoted by $gr(G)$, is the length of a shortest cycle in G. (If G contains no cycle, then $gr(G) = \infty$). We say that G is connected if there exists a path between any two distinct vertices. A graph G is complete if any two distinct vertices are adjacent. The complete graph with n vertices will be denoted by K^n. Also, we say that G is totally disconnected if no two vertices of G are adjacent.

A complete bipartite graph is a graph G which may be partitioned into two disjoint nonempty vertex sets A and B such that two distinct vertices are adjacent if and only if they are in distinct vertex sets. If one of the vertex sets is singleton, we call G is a star graph. We denote the complete bipartite graph by $K^{m,n}$, where $|A| = m$ and $|B| = n$ (we allow m and n to be infinite cardinal); hence a star graph is a $K^{1,m}$.

Let G and H be two graphs. We use the notation $H \leq G$ (respectively, $H \cong G$) to denote that H is a subgraph of G (respectively, H is isomorphic to G). Also a subgraph H of G is called an induced subgraph if, for each two distinct vertices of H such that $x \sim y$ in G, we have $x \sim y$ in H.

Suppose that G and H are two graphs. H is called a refinement of G if $V(G) = V(H)$ and each edge in G is an edge in H.

A subset S of V is called a dominating set if every vertex in $V - S$ is adjacent to at least one vertex in S. The domination number $\gamma(G)$ of G is the minimum cardinality among the dominating sets of S. For any two graphs G and H, if G is identical to H, then we write $G = H$, otherwise, we write $G \neq H$. Throughout this paper, we consider commutative semigroups with more than one nonzero zero-divisor.

2 Basic properties of $ANN_G(S)$

In this section, we start by introducing some propositions for later applications in this paper and study some basic properties of the annihilator graph $ANN_G(S)$ in commutative semigroup.

The following proposition contains two cases of Lemma 2.1 of [2].

Proposition 1. *Let R be a commutative ring.*

(1) *Let x and y be distinct elements of $Z(R)^*$. Then $x \sim y$ is not an edge of $AG(R)$ if and only if $ann_R(xy) = ann_R(x)$ or $ann_R(xy) = ann_S(y)$.*

(2) *If $x \sim y$ is an edge of $\Gamma(R)$ for some distinct elements $x, y \in Z(R)^*$, then $x \sim y$ is an edge of $AG(R)$. In particular, if P is a path in $\Gamma(R)$, then P is a path in $AG(R)$.*

(3) *If $x \sim y$ is not an edge of $AG(R)$ for some distinct elements $x, y \in Z(R)^*$, then $ann_R(x) \subseteq ann_R(y)$ or $ann_R(y) \subseteq ann_R(x)$.*

Proposition 2 (see [6])**.** *If G is the graph of a semigroup then G satisfies all of the following conditions.*

(1) *G is connected.*

(2) *Any two vertices of G are connected by a path with ≤ 3 edges.*

(3) *If G contains a cycle then the core of G is a union of quadrilaterals and triangles, and any vertex not in the core of G is an end.*

(4) *For each pair x, y of nonadjacent vertices of G, there is a vertex z with $N(x) \cup N(y) \subset \overline{N(z)}$.*

Proposition 3 (see [10])**.** *Let R be a ring.*

(1) *diam $(\Gamma(R)) = 0$ if and only if R is (nonreduced and) isomorphic to either $\mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[y]}{y^2}$.*

(2) *diam $(\Gamma(R)) = 1$ if and only if $xy = 0$ for each distinct pair of zero divisors and R has at least two nonzero zero divisors.*

(3) *diam$(\Gamma(R)) = 2$ if and only if either (i) R is reduced with exactly two minimal primes and at least three nonzero zero divisors, or (ii) $Z(R)$ is an ideal whose square is not (0) and each pair of distinct zero divisors has a nonzero annihilator.*

(4) *diam$(\Gamma(R)) = 3$ if and only if there are zero divisors $a \neq b$ such that $(0 : (a, b)) = (0)$ and either (i) R is a reduced ring with more than two minimal primes, or (ii) R is nonreduced.*

Proposition 4 (see [11])**.** *Suppose for a fixed integer $n \geq 2$, that $R = R_1 \times R_2 \times \ldots \times R_n$, where R_i is an integral domain for each $i = 1, 2, \ldots, n$. Then*

(1) *$\gamma(\Gamma(R)) = n = w(\Gamma(R))$ if $n \geq 3$.*

(2) *$\gamma(\Gamma(R)) = 2 = w(\Gamma(R))$ if $n = 2$ and $min\{|R_1|, |R_2|\} \geq 3$.*

(3) *$\gamma(\Gamma(R)) = 1 < w(\Gamma(R))$ if $n = 2$ and $min\{|R_1|, |R_2|\} = 2$.*

Lemma 1. *Let S be a commutative semigroup.*

(1) *If x and y are distinct elements of $Z(S)^*$, then $x \sim y$ is not an edge of $ANN_G(S)$ if and only if $ann_S(x) = ann_S(xy) = ann_S(y)$.*

(2) *If* $x \sim y$ *is an edge of* $\Gamma(S)$ *for some distinct elements* $x, y \in Z(S)^*$*, then* $x \sim y$ *is an edge of* $ANN_G(S)$*.*

(3) *If* $d_{\Gamma(S)}(x, y) = 3$ *for some distinct elements* x, y $Z(S)^*$*, then* $x \sim y$ *is an edge of* $ANN_G(S)$*.*

(4) *If* $x \sim y$ *is not an edge of* $ANN_G(S)$ *for some distinct elements* $x, y \in Z(S)^*$*, then there is a* $w \in Z(S)^* - \{x, y\}$ *such that* $x \sim w \sim y$ *is a path in* $\Gamma(S)$*, and hence* $x \sim w \sim y$ *is also a path* $ANN_G(S)$*.*

(5) *If* $x \sim y$ *is an edge of* $AG(S)$ *for some distinct elements* $x, y \in Z(S)^*$*, then* $x \sim y$ *is an edge of* $ANN_G(S)$*.*

(6) *If* $ANN_G(S) = \Gamma(S)$*, then* $ANN_G(S) = AG(S)$*.*

Proof. (1) Suppose that $x \sim y$ is not an edge of $ANN_G(S)$. Then $ann_S(xy) = ann_S(x) \cap ann_S(y)$ by definition. Thus $ann_S(xy) \subseteq ann_S(x)$ and $ann_S(xy) \subseteq ann_S(y)$. But $ann_S(x) \subseteq ann_S(xy)$ and $ann_S(y) \subseteq ann_S(xy)$. Hence $ann_S(x) = ann_S(xy) = ann_S(y)$.

Conversely, suppose that $ann_S(x) = ann_S(xy) = ann_S(y)$. Then $ann_S(xy) = ann_S(x) \cap ann_S(y)$. Hence $x \sim y$ is not an edge of $ANN_G(S)$ by definition.

(2) Suppose that $x \sim y$ is an edge of $\Gamma(S)$ for some distinct elements $x, y \in Z(S)^*$. Then $xy = 0$ and $ann_S(xy) = ann_S(0) = S$. Since $x \neq 0$, $y \neq 0$, then $ann_S(x) \neq S$ and $ann_S(y) \neq S$. Therefore $ann_S(xy) \neq ann_S(x)$ and $ann_S(xy) \neq ann_S(y)$. Hence $x \sim y$ is an edge of $ANN_G(S)$ by (1).

(3) Suppose that $d_{\Gamma(S)}(x, y) = 3$ for some distinct elements $x, y \in Z(S)^*$. So assume $x \sim a \sim b \sim y$ is a shortest path connecting x and y in $\Gamma(S)$, where $a, b \in Z(S)^*$ and $a \neq b$. This implies $xa = 0$, $ab = 0$, $by = 0$, $xb \neq 0$ and $ay \neq 0$. Now $xa = 0 \Rightarrow xya = 0 \Rightarrow a \in ann_S(xy)$ and $by = 0 \Rightarrow xyb = 0 \Rightarrow b \in ann_S(xy)$. Thus $\{a, b\} \subseteq ann(xy)$ such that $a \notin ann_S(y)$ and $b \notin ann_S(x)$. Therefore $ann_S(xy) \neq ann_S(x)$ and $ann_S(xy) \neq ann_S(y)$. Hence $x \sim y$ is an edge of $ANN_G(S)$ by (1).

(4) Suppose that $x \sim y$ is not an edge of $ANN_G(S)$ for some distinct elements $x, y \in Z(S)^*$. Then $ann_S(x) = ann_S(y) = ann_S(xy)$ by (1). Also $x \sim y$ is not an edge of $\Gamma(S)$ by (2) and hence $xy \neq 0$. Therefore $w \in \{x, y\}$, then $xy = 0$, a contradiction. Thus $w \in Z(S)^* - \{x, y\}$ such that $x \sim w \sim y$ is a path in $\Gamma(S)$. Hence $x \sim w \sim y$ is a path in $ANN_G(S)$ by (2).

(5) Suppose that $x \sim y$ is an edge of $AG(S)$ for some distinct elements $x, y \in Z(S)^*$. Then $ann_S(xy) \neq ann_S(x)$ and $ann_S(xy) \neq ann_S(y)$ by (1) of Proposition 1. Hence $x \sim y$ is an edge of $ANN_G(S)$ by (1).

(6) Let $ANN_G(S) = \Gamma(S)$. If possible, suppose that $ANN_G(S) \neq AG(S)$. Then there are some distinct elements $x, y \in Z(S)^*$ such that $x \sim y$ is an edge of $ANN_G(S)$ that is not an edge of $AG(S)$. So $x \sim y$ is not an edge of $\Gamma(S)$ by (2) of Proposition 1, and hence $ANN_G(S) \neq \Gamma(S)$, a contradiction. Thus $ANN_G(S) = AG(S)$. □

In view of Lemma 1 (4), we have Theorem 1.

Theorem 1. *Let* S *be a commutative semigroup with* $|Z(S)^*| \geq 2$*. Then* $ANN_G(S)$ *is connected and diam* $(ANN_G(S)) \leq 2$*.*

Proof. Let x and y be two distinct elements of $Z(S)^*$. If $x \sim y$ is an edge of $ANN_G(S)$, then $d(x, y) = 1$.

Suppose that $x \sim y$ is not an edge of $ANN_G(S)$. Then there is a $w \in Z(S)^* - \{x, y\}$ such that $x \sim w \sim y$ is a path in $\Gamma(S)$, and hence $x \sim w \sim y$ is a path in $ANN_G(S)$ by Lemma 1 (4). Thus $d(x, y) = 2$. Hence $ANN_G(S)$ is connected and diam $(ANN_G(S)) \leq 2$. □

3 Some properties of $ANN_G(S)$ by means of $AG(S)$ and $\Gamma(S)$

In this section, we express some basic properties of $ANN_G(S)$ by $\Gamma(S)$.

Theorem 2. *If $Z(S) \neq S$, then $\Gamma(S) \leq ANN_G(S)$.*

Proof. Since $Z(S) \neq S$, there exists an element $a \in S - Z(S)$. Then, for each two distinct elements $x, y \in S^*$, we have $ax \neq 0$ and $ay \neq 0$. Hence $a \notin ann_S(x)$ and $a \notin ann_S(y)$. Thus $a \notin ann_S(x) \cap ann_S(y)$. Now let x and y be adjacent in $\Gamma(S)$.

Therefore $xy = 0$. Hence $a(xy) = 0 \Rightarrow a \in ann_S(xy)$. So, $ann_S(xy) \neq ann_S(x) \cap ann_S(y)$, which implies that x is adjacent to y in $ANN_G(S)$. We also have $V(\Gamma(S)) = V(ANN_G(S))$, and so $\Gamma(S)$ is a subgraph of $ANN_G(S)$. □

Recall that a monoid is a semigroup with an identity.

Corollary 1. *Let S be a monoid such that $1 \neq 0$. Then $\Gamma(S) \leq ANN_G(S)$.*

Proposition 5. *Let $Z(S) \neq S$. Then $ANN_G(S)$ is connected with diameter less than four and $gr(ANN_G(S)) \leq 4$ or ∞.*

Proof. Since $\Gamma(S)$ is a subgraph of $ANN_G(S)$, it follows, from (1) of Proposition 2. □

Proposition 6. *Suppose that for each two distinct elements x and y of S, we have $xy = 0$, $xy = y$ or $xy = x$, then $ANN_G(S) \leq \Gamma(S)$.*

Proof. Let x and y be adjacent vertices in $ANN_G(S)$. Hence $ann_S(x) \neq ann_S(xy) \neq ann_S(y)$ by Lemma 1 (1). Then $xy \neq x$ and $xy \neq y$. So, by hypothesis, $xy = 0$. Thus $x \sim y$ is an edge in $\Gamma(S)$. Because $V(\Gamma(S)) = V(ANN_G(S))$, we have $ANN_G(S) \leq \Gamma(S)$. □

The next corollary immediately follows from Theorem 2 and Proposition 6.

Corollary 2. *Suppose that $Z(S) \neq S$ and, for each two distinct elements x and y of S, we have $xy = 0$, $xy = y$ or $xy = x$, then $ANN_G(S) \cong \Gamma(S)$.*

The next theorem shows that there exists a path in $ANN_G(S)$ which is not a path in $\Gamma(S)$.

Theorem 3. *Let x and y be distinct nonzero adjacent vertices in $ANN_G(S)$ and $xy \neq 0$. If there is a $w \in ann_S(xy) - \{x, y\}$ such that $wx \neq 0$ or $wy \neq 0$, then $x \sim w \sim y$ is a path in $ANN_G(S)$ which is not a path in $\Gamma(S)$ and $ANN_G(S)$ contains a cycle C of length 3 such that at least two edges of C are not the edges of $\Gamma(S)$.*

Proof. Suppose that $x \sim y$ is an edge of $ANN_G(S)$ with $xy \neq 0$. Assume that there is a $w \in ann_S(xy) - \{x, y\}$ such that $wx \neq 0$ or $wy \neq 0$. Hence $x \notin ann_S(w)$ or $y \notin ann_S(w)$. Then $x \notin ann_S(w) \cap ann_S(y)$. Since $w \in ann_S(xy)$, we have $0 = w(xy) = x(wy)$, and hence $x \in ann_S(wy)$. Thus $ann_S(wy) \neq ann_S(w) \cap ann_S(y)$. Therefore w is adjacent to y in $ANN_G(S)$. Similarly, $y \in ann_S(xw) - (ann_S(x) \cap ann_S(w))$. Hence w is adjacent to x in $ANN_G(S)$. Thus $x \sim w \sim y$ is a path in $ANN_G(S)$. Since $wx \neq 0$ or $wy \neq 0$, we have $x \sim w \sim y$ is not a path in $\Gamma(S)$. Therefore, $C : x \sim w \sim y \sim x$ is a cycle of length 3 in $ANN_G(S)$ and at least two edges C are not the edges of $\Gamma(S)$. □

Theorem 4. *Let x and y be distinct adjacent vertices in $ANN_G(S)$ such that $xy \neq 0$, for some $x, y \in Z(S)^*$. If $x^2y \neq 0$ and $xy^2 \neq 0$, then there exists $w \in Z(S)^* - \{x, y\}$ such that $x \sim w \sim y$ is a path in $ANN_G(S)$ which is not a path in $\Gamma(S)$. In this case $gr(ANN_G(S)) = 3$.*

Proof. Suppose that x is adjacent to y in $ANN_G(S)$ and $xy \neq 0$. Then there exists $w \in ann_S(xy) - (ann_S(x) \cap ann_S(y))$ such that $w \neq 0$. If $w = x$, then $wxy = x^2y = 0$, which is impossible. If $w = y$, then $wxy = xy^2 = 0$, which is impossible. Therefore $w \in ann_S(xy) - \{x, y\}$ such that $wx \neq 0$ or $wy \neq 0$. Thus $x \sim w \sim y$ is a path in $ANN_G(S)$ which is not a path in $\Gamma(S)$. Now $C : x \sim w \sim y \sim x$ is a cycle in $ANN_G(S)$ of length three, and so $gr(ANN_G(S)) = 3$. □

Recall that the semigroup S is called reduced if, for each $x \in S$, $x^n = 0$ implies that $x = 0$.

Corollary 3. *Let S be a reduced commutative semigroup. Suppose that there exist distinct elements x, y in $Z(S)^*$ such that x is adjacent to y in $ANN_G(S)$ and $xy \neq 0$. Then there exists $w \in ann_S(xy) - \{x, y\}$ such that $x \sim w \sim y$ is a path in $ANN_G(S)$ which is not a path in $\Gamma(S)$.*

Proof. Since S is reduced and $xy \neq 0$, we have $x^2y^2 = (xy)^2 \neq 0$. This implies $x^2y \neq 0$ and $xy^2 \neq 0$. Now the claim clearly follows from Theorem 4. □

Theorem 5. *Let S be a reduced semigroup, where $\Gamma(S) \leq ANN_G(S)$ and $\Gamma(S) \neq ANN_G(S)$. Then $gr(ANN_G(S)) = 3$. Moreover, there is a cycle of length three in $ANN_G(S)$ such that at least two edges of C are not the edges of $\Gamma(S)$.*

Proof. Since $ANN_G(S) \neq \Gamma(S)$, there are some distinct elements $x, y \in Z(S)^*$ such that $x \sim y$ is an edge of $ANN_G(S)$ that is not an edge of $\Gamma(S)$. As S is reduced, we have $(xy)^2 \neq 0$. This implies $x^2y \neq 0$ and $xy^2 \neq 0$. Now the claim follows from Theorem 4. □

Corollary 4. *Let S be a reduced semigroup, $Z(S) \neq S$ and $\Gamma(S) \neq ANN_G(S)$. Then $gr(ANN_G(S)) = 3$.*

The following example shows that the condition that S being reduced in Corollary 4 is necessary.

Example 1. Let $S = \mathbb{Z}_8$ be a multiplicative semigroup. Then S is not a reduced semigroup and $\Gamma(S) \cong K^{1,2}$ and $ANN_G(S) \cong K^3$. Also $\bar{2} \sim \bar{6}$ is an edge in $ANN_G(S)$ which is not an edge in $\Gamma(S)$ and $\bar{4}$ is the only element belonging to $ann_S(\bar{2}.\bar{6}) - \{\bar{2}, \bar{6}\}$. But $\bar{2} \sim \bar{4} \sim \bar{6}$ is a path in $ANN_G(S)$ which is a path in $\Gamma(S)$.

Theorem 6. *Let x and y be distinct adjacent vertices in $ANN_G(S)$ such that x is not adjacent to y in $AG(S)$. Then there is a $w \in Z(S)^* - \{x, y\}$ such that $x \sim w \sim y$ is a path in $ANN_G(S)$ and $ANN_G(S)$ contains a cycle $C : x \sim w \sim y \sim x$ of length three such that exactly one edge of C is not an edge of $AG(S)$.*

Proof. Suppose that $x \sim y$ is an edge of $ANN_G(S)$ that is not an edge of $AG(S)$. Then $ann_S(x) \subseteq ann_S(y)$ or $ann_S(y) \subseteq ann_S(x)$ by (3) of Proposition 1, and there is a $w \in Z(S)^* - \{x, y\}$ such that $x \sim w \sim y$ is a path in $\Gamma(S)$. Thus $x \sim w \sim y$ is a path in $ANN_G(S)$ by Lemma 1 (2). Hence $C : x \sim w \sim y \sim x$ is a cycle of length three in $ANN_G(S)$. We have $x \sim w \sim y$ is a path in $AG(S)$ and thus exactly one edge of C is not an edge of $AG(S)$. □

Corollary 5. *Suppose that $\Gamma(S) \leq ANN_G(S)$ and $ANN_G(S) \neq \Gamma(S)$, then $gr(ANN_G(S)) = 3$. Moreover, there is a cycle C in $ANN_G(S)$ of length three, such that exactly one edge of C is not an edge of $AG(S)$.*

Proof. Since $ANN_G(S) \neq AG(S)$, there are x and y in $Z(S)^*$ such that x is adjacent to y in $ANN_G(S)$ that is not an edge in $AG(S)$. Now the claim follows from Theorem 6. □

Lemma 2. *Let S be a reduced commutative semigroup, and let $z \in Z(S)^*$. Then $ann_S(z) = ann_S(z^n)$ for each positive integer $n \geq 2$.*

Proof. Let $x \in ann_S(z^n)$. So $xz^n = 0$, which implies that $0 = (xz^n).x^{n-1} = x^n z^n = (xz)^n$. Since S is reduced, we have $xz = 0$ and $x \in ann_S(z)$. Also it is clear that $ann_S(z) \subseteq ann_S(z^n)$. Thus $ann_S(z^n) = ann_S(z)$. □

Lemma 3. *Let S be reduced, and $ANN_G(S)$ be a complete graph. Then, for each $x \in Z(S)^*$, we have $x^2 = x$.*

Proof. Let $x \in Z(S)^*$. If possible, suppose that $x^2 \neq x$. Since S is reduced, we have $x^3 \neq 0$. Now $ann_S(x) = ann_S(x^2)$ and $ann_S(x) = ann_S(x^3)$ by Lemma 2. Therefore $ann_S(x) = ann_S(x^3) = ann_S(x^2)$ and hence x is not adjacent to x^2 in $ANN_G(S)$, a contradiction. □

Theorem 7. *Let S be reduced and $ANN_G(S)$ be complete. Then $\Gamma(S)$ is complete.*

Proof. Suppose that x is not adjacent to y in $\Gamma(S)$. Since $ANN_G(S)$ is a complete graph, therefore x is adjacent to y in $ANN_G(S)$ and $xy \neq 0$. We have $xy \neq x$, $xy \neq y$ and $x^2 = x$ by Lemma 3. Then $x(xy) = x^2 \cdot y = xy$ and hence x is not adjacent to y in $ANN_G(S)$. It is a contradiction. Thus x is adjacent to y in $\Gamma(S)$. Therefore, $\Gamma(S)$ is complete. □

If $Z(S) \neq S$, then by Theorem 2, $\Gamma(S) \leq ANN_G(S)$. Now the next corollary immediately follows from Theorem 7.

Corollary 6. *Suppose that S is reduced and $Z(S) \neq S$. Then $ANN_G(S)$ is complete if and only if is $\Gamma(S)$ complete.*

Definition 1. (1) A nonempty subset I of S is called an ideal if $xS \subseteq I$, for each $x \in I$.

(2) An ideal P in S is called a prime ideal of S if $xSy \subseteq P$ implies $x \in P$ or $y \in P$.

(3) A prime ideal P is said to be a minimal prime ideal belonging to I if $I \subseteq P$ and there is no prime ideal Q in S such that $I \subseteq Q \subseteq P$.

Recall that if S is a reduced commutative semigroup, then it has at least two minimal prime ideal. So for a reduced commutative semigroup S, we have $|\min(S)| \geq 2$ where $min(S)$ is the set of all minimal prime ideals of S. If $Z(S)$ is an ideal of S, then $|\min(S)|$ may be infinite, as $Z(S) = \cup\{I \mid I \in \min(S)\}$ [6].

Theorem 8. *Let S be a reduced commutative semigroup and suppose that $Z(S)$ is an ideal of S. Then $\Gamma(S) \neq ANN_G(S) \neq AG(S)$ and $gr(ANN_G(S)) = 3$.*

Proof. Let $z \in Z(S)^*$ and $c \in ann_S(z) - \{0\}$. We have $c \neq z$, as S is reduced. Since $Z(S)$ is an ideal of S, we have $c+z \in Z(S)^* - \{c, z\}$. Since $(c+z)z = cz+z^2 = z^2 \neq 0$, we have $(c+z) \sim z$ is not an edge of $\Gamma(S)$.

Now $ann_S((c+z)z) = ann_S(z^2) = ann_S(z)$. But $ann_S(c+z) \subset ann_S(z) = ann_S((c+z)z)$. Since $ann_S((c+z)z) = ann_S(z)$, we have $(c+z) \sim z$ is not an edge of $AG(S)$ by (1) of Proposition 1. Also since $ann_S((c+z)z) \neq ann_S(c+z)$, we have $(c+z) \sim z$ is an edge of $ANN_G(S)$ by Lemma 1 (1). Thus $\Gamma(S) \neq ANN_G(S) \neq AG(S)$ and hence $gr(ANN_G(S)) = 3$ by Theorem 5. □

Theorem 9. *Suppose that S is a reduced commutative semigroup and $|\min(S)| \geq 3$. Then $ANN_G(S) \neq \Gamma(S)$ and $gr(ANN_G(S)) = 3$.*

Proof. If $Z(S)$ is an ideal of S, then $ANN_G(S) \neq \Gamma(S)$ by Theorem 8. Now assume that $Z(S)$ is not an ideal of S. Since $|\min(S)| \geq 3$, we have $diam(\Gamma(S)) = 3$ by Proposition 3. Thus $ANN_G(S) \neq \Gamma(S)$ by Theorem 1. As S is reduced and $ANN_G(S) \neq \Gamma(S)$, we have $gr(ANN_G(S)) = 3$ by Theorem 5. □

Theorem 10. *Let S be a reduced commutative semigroup. Then $ANN_G(S) = \Gamma(S)$ if and only if $|\min(S)| = 2$.*

Proof. Suppose that $ANN_G(S) = \Gamma(S)$. Then we have $|\min(S)| = 2$ by Theorem 9. Conversely, suppose that $|\min(S)| = 2$. Let P and Q be the two minimal prime ideals of S. Since S is reduced, we have $Z(S) = P \cup Q$ and $P \cap Q = \{0\}$. Let $x, y \in Z(S)^*$. Suppose that $x, y \in P$. So neither $x \in Q$ nor $y \in P$ and thus $xy \neq 0$. Since $PQ \subseteq P \cap Q = \{0\}$, we have $ann_S(xy) = ann_S(x) = ann_S(y) = Q$. Hence $x \sim y$ is not an edge of $ANN_G(S)$ by Lemma 1 (1).

Similarly, if $x, y \in Q$, then $x \sim y$ is not an edge of $ANN_G(S)$. If $x \in P$ and $y \in Q$, then $xy = 0$, hence $x \sim y$ is an edge of $\Gamma(S)$. Therefore by Lemma 1 (2), $x \sim y$ is an edge of $ANN_G(S)$. Thus each edge of $ANN_G(S)$ is an edge of $\Gamma(S)$. Hence $ANN_G(S) = \Gamma(S)$. □

Corollary 7. *Let S be a reduced commutative semigroup. Then $ANN_G(S) = AG(S) = \Gamma(S)$ if and only if $|\min(S)| = 2$.*

Theorem 11. *Let S be a reduced commutative semigroup. Then the following statements are equivalent:*

(1) *$|\min(S)| = 2$ and at least one minimal prime ideal of S has exactly two distinct elements;*

(2) *$AG(S) = \Gamma(S) = K^{1,n}$ for some $n \geq 1$;*

(3) *$ANN_G(S) = K^{1,n}$ for some $n \geq 1$.*

Proof. Since $|\min(S)| = 2$, we have $ANN_G(S) = AG(S) = \Gamma(S)$ by Corollary 7. But $AG(S) = \Gamma(S) = K^{1,n}$ for some $n \geq 1$. Hence $ANN_G(S) = K^{1,n}$. □

Theorem 12. *Let S be a reduced commutative semigroup. Then we have the following:*

(1) *If $\gamma(ANN_G(S)) = 1$, then $\gamma(\Gamma(S)) \in \{1, 2\}$.*

(2) *If $|\min(S)| = 2$, $\gamma(\Gamma(S)) = 1$, then $\gamma(ANN_G(S)) = 1$.*

Proof. The result follows from Theorem 11 and Proposition 4. Since $\gamma(ANN_G(S)) = 1$ for a star graph $K^{1,n}$. □

Corollary 8. *Let S be a reduced commutative semigroup and $|\min(S)| < \infty$. If $\gamma(ANN_G(S)) > 1$, then $\gamma(ANN_G(S)) = \gamma(\Gamma(S))$.*

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