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Transparency of Ore extensions over left σ-$(S,1)$ rings

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Abstract. Let R be a ring and σ be an endomorphism of R. Recall that a ring R is said to be a left σ-$(S,1)$ ring if for $a,b \in R$, $ab=0$ implies that $aRb=0$ and $\sigma(a)Rb=0$. In this paper we discuss a stronger type of primary decomposition (known as transparency) of a left σ-$(S,1)$ ring R, and Ore extension $R[x;\sigma]$.

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1 Introduction

A ring R always means an associative ring with identity $1 \neq 0$. The set of minimal prime ideals, prime radical, and the set of nilpotent elements of R are denoted by $Min.Spec(R)$, $P(R)$ and $N(R)$ respectively. The set of associated prime ideals of R (viewed as a right module over itself) is denoted by $Ass(R_R)$. Let R be a right Noetherian ring. For any uniform right R-module J, the assassinator of J is denoted by $Assas(J)$. Let M be a right R-module. Consider the set

$$\{Assas(J) \mid J \text{ is a uniform right } R\text{-submodule of } M\}.$$

We denote this set by $\mathbb{A}(M_R)$. The set of positive integers, the ring of integers, the field of rational numbers, the field of real numbers and the field of complex numbers are denoted by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ respectively unless otherwise stated.

Now let R be a ring and σ an endomorphism of R. Recall that the skew polynomial ring $R[x;\sigma]$ is the usual polynomial ring with coefficients in R, with usual addition, and multiplication subject to the relation $ax = x\sigma(a)$ for all $a \in R$. We define any $f(x) \in R[x;\sigma]$ to be of the form $f(x) = \sum_{i=0}^{n} x^i a_i$ as in McConnell and Robson [8]. We denote $R[x;\sigma]$ by $S(R)$. Skew-polynomial rings have been discussed by many authors, for example $[1,2,7,9]$.

By classical study of any commutative Noetherian ring we mean studying its primary decomposition. Further, existence of quotient rings, in particular, Artinian quotient rings etc. can be linked to primary decomposition of a Noetherian ring.

The notion of the quotient ring of a ring, the contractions and extensions of ideals arising thereby appear in Chapter 9 of [7].

It is also shown in [7] that if R is a commutative Noetherian ring and σ is an automorphism of R, then the skew-polynomial ring $R[x;\sigma]$ embeds in an Artinian ring. We would like to note that prime ideals (in particular minimal prime ideals and associated prime ideals) of skew polynomial rings have been the point of interest of many authors, for example Bhat [2, 7–9].

The above discussion leads us to discuss a stronger type of primary decomposition of a Noetherian ring. We call such a ring a transparent ring.

Definition 1 (see Definition 1.2 of [3])**.** A right Noetherian ring R is said to be a Transparent ring if there exists irreducible ideals I_j, $1 \leq j \leq n$ such that $\cap_{j=1}^{n} I_j = 0$ and each R/I_j has a right artinian quotient ring.

One can see that an integral domain is a transparent ring, a commutative Noetherian ring is a transparent ring and so is a right Noetherian ring having a right Artinian quotient ring. A fully bounded Noetherian ring is also an example of transparent ring.

We recall the following:

Theorem 1 (see Hilbert Basis Theorem, namely Theorem 1.14 of Goodearl and Warfield [7])**.** *Let R be a right/left Noetherian ring. Let σ be as above. Then the Ore extension $S(R) = R[x;\sigma]$ is right/left Noetherian.*

Definition 2. A ring R is called 2-primal if $P(R) = N(R)$, i. e. if the prime radical is a completely semiprime ideal. An ideal I of a ring R is called completely semiprime if $a^2 \in I$ implies that $a \in I$ for $a \in R$.

$\sigma(*)$-ring

Definition 3. Let R be a ring and σ an endomorphism of R. Then R is said to be a $\sigma(*)$-ring if $a\sigma(a) \in P(R)$ implies $a \in P(R)$ for $a \in R$.

Example 1. Let $R = \left\{ \begin{pmatrix} p & q \\ 0 & r \end{pmatrix};\ p,\ q,\ r \in F, \text{ a field} \right\}$. Now $P(R) = \left\{ \begin{pmatrix} p & 0 \\ 0 & r \end{pmatrix}; p, r \in F \right\}$.

Define a map $\sigma : R \to R$ by $\sigma\begin{pmatrix} p & q \\ 0 & r \end{pmatrix} = \begin{pmatrix} p & 0 \\ 0 & r \end{pmatrix}$. Clearly, σ is an endomorphism of R, and one can be easily see that R is a $\sigma(*)$-ring.

Remark 1. A $\sigma(*)$-ring R is 2-primal, for let $a \in R$ be such that $a^2 \in P(R)$. Then

$$a\sigma(a)\sigma(a\sigma(a)) = a\sigma(a)\sigma(a)\sigma^2(a) = a\sigma(a^2)\sigma^2(a) \in \sigma(P(R)) = P(R).$$

Therefore, $a\sigma(a) \in P(R)$ and hence $a \in P(R)$. So $P(R)$ is completely semiprime and hence R is 2-primal.

Weak σ-rigid ring

Definition 4 (see Ouyang [10])**.** Let R be a ring and σ an endomorphism of R satisfying $a\sigma(a) \in N(R)$ if and only if $a \in N(R)$ for $a \in R$. Then R is called a weak σ-rigid ring.

Example 2. Let $W_1[F]$ be the first Weyl algebra over a field F with characteristic zero. Then $W_1[F] = F[\mu, \lambda]$, the polynomial ring with indeterminates μ and λ with $\lambda\mu = \mu\lambda + 1$.
Now let R be the ring
$$\begin{pmatrix} W_1[F] & W_1[F] \\ 0 & 0 \end{pmatrix}.$$
Now the prime radical $P(R)$ of R is
$$\begin{pmatrix} 0 & W_1[F] \\ 0 & 0 \end{pmatrix}.$$
Define an endomorphism $\sigma : R \to R$ by
$$\sigma\Big(\begin{pmatrix} \mu & \lambda \\ 0 & 0 \end{pmatrix}\Big) = \begin{pmatrix} \mu & 0 \\ 0 & 0 \end{pmatrix}.$$
Then R is a weak σ-rigid ring.

$(S, 1)$-rings

Definition 5 (see Kwak [9])**.** Let R be a ring. Then R is called an $(S, 1)$-ring if for a, $b \in R$, $ab = 0$ implies $aRb = 0$.

Example 3. 1. Let $R = \mathbb{Z}\times\{0\}\times\{0\}\times\mathbb{Z}$. The only elements $a, b \in R$ satisfying $ab = 0$ are of the type $a = (p, 0, 0, 0)$ and $b = (0, 0, 0, s)$; $p, s \in \mathbb{Z}$. Now for all $k = (q, 0, 0, r) \in R$,
$$akb = (p, 0, 0, 0)(q, 0, 0, r)(0, 0, 0, s) = (0, 0, 0, 0).$$
This implies that R is an $(S, 1)$-ring.

2. Let $R = \Big\{\begin{pmatrix} p & q \\ 0 & r \end{pmatrix}; p,\ q,\ r \in \mathbb{Z}\Big\}$. Let $A = \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix}$; $0 \neq p$, $0 \neq q \in \mathbb{Z}$. Then $AB = 0$. Now let $K = \begin{pmatrix} r & s \\ 0 & 0 \end{pmatrix} \in R$, with $s \neq 0$. Then

$$AKB = \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} r & s \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix}$$
$$= \begin{pmatrix} 0 & psq \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus R is not an $(S,1)$-ring.

Left σ-$(S,1)$-ring

We generalize the notion of an $(S,1)$-ring R by involving an endomorphism σ of R as follows:

Definition 6. Let R be a ring and σ an endomorphism of R. We call R a left σ-$(S,1)$ ring if for $a, b \in R$, $ab = 0$ implies that $aRb = 0$ and $\sigma(a)Rb = 0$.

Right σ-$(S,1)$ ring can be defined in a similar way; i.e. for $a, b \in R$, $ab = 0$ implies that $aRb = 0$ and $aR\sigma(b) = 0$

Example 4. Let $R = \mathbb{Z} \times \{0\} \times \{0\} \times \mathbb{Z}$. Define a map $\sigma : R \to R$ by $\sigma(p,0,0,q) = (0,0,0,q)$. Then σ is an endomorphism of R. Now the only elements $a \in R$ and $b \in R$ satisfying $ab = 0$ are of the type $a = (p,0,0,0)$ and $b = (0,0,0,q); p, q \in \mathbb{Z}$. Now for all $k = (r,0,0,s) \in R$,

$$akb = (p,0,0,0)(r,0,0,s)(0,0,0,q) = (0,0,0,0).$$

Also

$$\sigma(a)kb = (0,0,0,0)(r,0,0,s)(0,0,0,q) = (0,0,0,0).$$

Therefore, R is a left σ-$(S,1)$ ring. In fact R is also a right σ-$(S,1)$ ring.

From the definition it is clear that a left or right σ-$(S,1)$ ring is an $(S,1)$-ring, but the converse is not true.

Remark 2. We note that a right σ-$(S,1)$ ring need not be a left σ-$(S,1)$ ring, and vice versa.

Example 5. Let $R = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix};\ a,\ b \in \mathbb{Z} \right\}$. Define a map $\sigma : R \to R$ by $\sigma\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}$. Then σ is an endomorphism of R.

Let $A = \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix}$; $0 \neq p,\ 0 \neq q \in \mathbb{Z}$.

Then $AB = 0$. Now let $K = \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} \in R$, with $s \neq 0$. Then

$$\begin{aligned} AKB &= \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix} \\ &= \begin{pmatrix} pr & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Therefore, R is an $(s, 1)$-ring.

Now,

$$\begin{aligned} AK\sigma(B) &= \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} pr & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Thus, R is also a right σ-$(S, 1)$ ring.

However,

$$\begin{aligned} \sigma(A)KB &= \begin{pmatrix} 0 & 0 \\ 0 & p \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & ps \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & psq \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Thus, R is not a left σ-$(S, 1)$ ring.

Now onwards, we consider left σ-$(S, 1)$ rings.

Proposition 1. *Let R be a ring, σ an automorphism of R such that R is a left σ-$(S, 1)$ ring. Then R is 2-primal.*

Proof. Let R be a left σ-$(S, 1)$ ring. Then by Theorem 1.5 of [11] R is 2-primal, which implies that $P(R)$ is completely semiprime. We give a sketch of proof.

Let $a \in N(R)$, say $a^k = 0$. If $a \notin P$ for some prime ideal P, then $ax_1a \notin P$ for some element $x_1 \in R$. Continuing this procedure we get elements $x_i \in R$ such that P does not contain $b = ax_1a \ \ldots\ x_{k-1}a$. But, R is an $(S, 1)$-ring, so $a^k = 0$ implies $b = 0$, hence $b \in P$, a contradiction. □

Proposition 2. *Let R be a ring, σ an automorphism of R such that R is a left σ-$(S, 1)$ ring. Then R is a $\sigma(*)$-ring.*

Proof. Since R is a left σ-$(S, 1)$ ring, by Proposition 1 R is 2-primal and $P(R)$ is completely semiprime.

We will show that R is a weak σ-rigid ring. Let $b \in R$ be such that $b\sigma(b) \in N(R)$. Now $b\sigma(b)\sigma^{-1}(b\sigma(b)) \in N(R)$ implies that $b^2 \in N(R)$, and so $b \in N(R)$. Therefore, R is a weak σ-rigid ring, and is also a $\sigma(*)$- ring. □

Remark 3. However, the converse of Proposition 1 and Proposition 2 is not true. In 1 R is a $\sigma(*)$-ring and, therefore, is also a 2-primal ring, but it is not a left σ-$(S, 1)$ ring (even not an $(S, 1)$-ring).

2 Prime ideals of left $\boldsymbol{\sigma}$-$(\boldsymbol{S}, 1)$ rings and their extensions

Proposition 3. *Let R be a right Noetherian ring which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R such that R is a left σ-$(S, 1)$ ring. Then $\sigma(U) = U$ for all $U \in MinSpec(R)$.*

Proof. By Proposition 1 R is 2-primal and by Proposition 2 it is a $\sigma(*)$-ring. Now the result follows by Proposition 2.1 of [5]. □

Proposition 4. *Let R be a right Noetherian ring and σ an automorphism of R such that R is a left σ-$(S, 1)$ ring. Then $U \in Min.Spec(R)$ implies that U is a completely prime ideal.*

Proof. Suppose that U is not completely prime. Then there exist $a, b \in R \backslash U$ with $ab \in U$. Consider U_i as in 3. Let c be any element of $b(U_2 \cap U_3 \cap ... \cap U_n)a$. Then $c^2 \in \cap_{i=1}^{n} U_i = P(R)$. So $c \in P(R)$ and, thus $b(U_2 \cap U_3 \cap ... \cap U_n)a \subseteq U$. Therefore, $bR(U_2 \cap U_3 \cap ... \cap U_n)Ra \subseteq U$ and, as U is prime, $a \in U$, $U_i \subseteq U$ for some $i \neq 1$ or $b \in U$. None of these can occur, so U is completely prime. □

Lemma 1. *Let R be a right Noetherian ring which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R such that R is a left σ-$(S, 1)$ ring. Then*

1. *If U is a minimal prime ideal of R, then $S(U)$ is a minimal prime ideal of $S(R)$ and $S(U) \cap R = U$.*
2. *If P is a minimal prime ideal of $S(R)$, then $P \cap R$ is a minimal prime ideal of R.*

Proof. R is a left σ-$(S, 1)$ ring implies that R is $\sigma(*)$-ring by Proposition 2. Now the proof follows on the same lines as in Lemma 2.2 of [5]. □

Proposition 5. *Let R be a right Noetherian ring which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R such that R is a left σ-$(S, 1)$ ring. Then $U \in Min.Spec(R)$ implies that $S(U)$ is a completely prime ideal of $S(R)$.*

Proof. Let $U \in Min.Spec(R)$. Proposition 3 implies that $\sigma(U) = U$. Also Proposition 4 implies that U is a completely prime ideal of R. Now the result follows by Theorem 2.4 of [6]. □

Theorem 2. *Let R be a semiprime right Noetherian ring. Let σ be an automorphism of R such that R is a left σ-$(S, 1)$ ring. Then $P \in Ass(S(R)_{S(R)})$ if and only if there exists $U \in Ass(R_R)$ such that $S(P \cap R) = P$ and $P \cap R = U$.*

Proof. R being right Noetherian implies that $Ass(R_R) = \mathbb{A}(R_R)$ (Remark 2). R a left σ-$(S,1)$ ring implies that $\sigma(U) = U$ for all $U \in Min.Spec(R)$ by Proposition 3. Now the result follows on the same lines as in Theorem A of [5]. $\square$

Corollary 1. *Let R be a right Noetherian $(S,1)$-ring, and σ an automorphism of R. Then $P \in Ass(S(R)_{S(R)})$ if and only if there exists $U \in Ass(R_R)$ such that $S(P \cap R) = P$ and $P \cap R = U$.*

3 Transparency of a left σ-$(S,1)$ ring

We are now in a position to state and prove the main result in the form of the following theorem:

Theorem 3. *Let R be a semiprime right Noetherian ring and σ an automorphism of R such that R is a left σ-$(S,1)$ ring. Then R is a transparent ring, and $S(R) = R[x;\sigma]$ is a transparent ring.*

Proof. R is a right Noetherian ring, therefore, there exist finitely many minimal prime ideals U_j, $1 \leq j \leq n$ of R by Theorem 3.4 of [7]. Also R is semiprime, therefore, $\cap_{j=1}^{n} U_j = 0$. Now R/U_j has a right Artinian quotient ring by Theorem 6.15 of Goodearl and Warfield [7]. Hence R is a *transparent ring*.

Now $R[x;\sigma]$ is right Noetherian by Hilbert Basis Theorem, namely Theorem 1.14 of Goodearl and Warfield [7]. Now R is left σ-$(S,1)$ ring, therefore, Proposition 3 implies that $\sigma(U_j) = U_j$, for all j, $1 \leq j \leq n$. Therefore, $S(U_j)$ is a minimal prime ideal of $S(R)$ by Lemma 1, and $\cap_{j=1}^{n} S(U_j) = 0$.

Now $S(R)/S(U_j)$ has also an Artinian quotient ring by Theorem (2.11) of Bhat [1]. Hence $S(R) = R[x;\sigma]$ is a *transparent ring*. $\square$

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