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Some properties of left-transitive quasigroups

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Abstract. Properties of left-transitive quasigroups (including their autotopisms and pseudoautomorphisms), their connections with some quasigroup classes are established. Left-transitive right GA-quasigroups are described.

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Basic concepts and definitions can be found in [1, 4, 11, 13].

1 Introduction

Definition 1. A quasigroup $(Q, \cdot)$ is said to be left-transitive if in this quasigroup the identity

$$xy \cdot xz = yz \tag{1}$$

holds [7, 14].

In the article [14] it is proved: if quasigroup $(Q, \cdot)$ satisfies identity of associativity $x \cdot yz = xy \cdot z$ (i.e., this quasigroup is a group), then (23)-parastrophe and (132)-parastrophe of this quasigroup satisfy left-transitive identity (1), (13)-parastrophe and (123)-parastrophe of this quasigroup satisfies right-transitive identity (i.e., the identity $yx \cdot zx = yz$).

In the articles [5, 6, 9, 12, 15] left-transitive quasigroups are called Ward quasigroups. In [9] important arguments for the need of the study of left-transitive quasigroups are presented. In this article it is noticed that Frobenius used group right division operation in his papers devoted to representation theory of groups [8].

From the results of mentioned above articles the following theorem follows.

Theorem 1. *Any left-transitive quasigroup $(G, \circ)$ can be obtained from a group $(G, +)$ (not necessary commutative) using the following construction*

$$x \circ y = -x + y = Ix + y, \tag{2}$$

where $x + Ix = 0$ for all $x, y \in G$ [14].

From Theorem 1 it follows that any left-transitive quasigroup $(Q, \circ)$ is unipotent, i. e., there exists a fixed element 0 of the set Q such that $x \circ x = 0$ for all $x \in Q$.

Remark 1. From Theorem 1 it follows that quasigroup $(Q, \circ)$ is an isotopic image of the group $(Q, +)$ with isotopy $(I, \varepsilon, \varepsilon)$.

Corollary 1. *Left-transitive quasigroup $(Q, \cdot)$ is commutative if and only if it is an abelian group any element of which has the order two.*

Proof. From commutativity, using equality (2), we obtain $-x + y = -y + x$ for all $x, y \in Q$. Therefore $-x = x$, $x + x = 0$ for all $x \in Q$.

Converse. It is clear that any commutative abelian group any element of which (with exception of 0) has the order two is left-transitive. □

The following corollary easily follows from Theorem 1, too.

Corollary 2. *Any left-transitive quasigroup $(Q, \cdot)$ has a left unit, i.e., there exists an element $f \in Q$ such that $f \cdot x = x$ for all $x \in Q$.*

Proof. If we put $x = y$ into the identity (1), then we have $xx \cdot xz = xz$. Therefore $f = x \cdot x$. □

If we use Theorem 1, then Lemma 2 takes the form:

Remark 2. Any left-transitive quasigroup $(Q, \circ)$ has a left unit, namely, the element 0 is its left unit.

Proof. If we put $x = 0$ in the equation (2), where 0 is unit of the group $(Q, +)$, then $0 \circ y = -0 + y = y$ for all $y \in Q$. □

Definition 2. A quasigroup $(Q, \cdot)$ is an LIP-quasigroup if in $(Q, \cdot)$ the following equation holds true:

$$I_l x(xy) = y \tag{3}$$

for all $x, y \in Q$, where I_l is a map of the set Q [1, 4].

Notice in fact the map I_l is a permutation of the set Q [4].

Remark 3. Any left-transitive quasigroup $(Q, \cdot)$ is an LIP-quasigroup [7].

Proof. In identity (1) we substitute $y = f$, where $fx = x$ for all $x \in Q$, and obtain the following equality: $xf \cdot xz = fz$. Further we have $xf \cdot xz = z$. From the last equality we have that $R_f = I_l$. □

Corollary 3. *In left-transitive quasigroup $(Q, \cdot)$ the following equality holds : $R_f^{-1} = R_f$.*

Proof. From Remark 3 we have $R_f = I_l$. Then $R_f^{-1} = I_l^{-1}$. But in LIP-quasigroups $I_l = I_l^{-1}$ [1, 13]. □

Theorem 2. *Any loop which is an isotope of left-transitive quasigroup $(Q, \cdot)$ is a group [7].*

Proof. First method. From Theorem 1 it follows that any left-transitive quasigroup $(Q, \cdot)$ is isotope of group $(Q, +)$ of the form $x \circ y = Ix + y$, where $x + Ix = 0$ for all $x \in Q$. Therefore among loop isotopes of quasigroup $(Q, \cdot)$ there exists a group.

Further from Albert theorem [1, 13] it follows that any loop which is an isotope of left-transitive quasigroup $(Q, \cdot)$ is a group.

Second method. Consider loop isotope of quasigroup $(Q, \cdot)$ of the form

$$x + y = R_f^{-1}x \cdot L_f^{-1}y = xf \cdot y. \tag{4}$$

We have used Lemma 2 and Corollary 3. From Theorem 1 it follows that this isotope is a group.

Further from Albert theorem it follows that any loop which is an isotope of left-transitive quasigroup $(Q, \cdot)$ is a group. □

Corollary 4. *In left transitive quasigroup* $(Q, \cdot)$ *and its isotopic group* $(Q, +)$ *(Theorem 1) we have* $R^{\cdot}_f = I^+$.

Proof. The proof follows from the equalities (2) and (4). □

Definition 3. A quasigroup $(Q, \cdot)$ is said to be left Bol quasigroup if the identity

$$x(y \cdot xz) = R_{e_x}^{-1}(x \cdot yx) \cdot z \tag{5}$$

holds true, where $x \cdot e_x = x$ for any $x \in Q$ [7].

A quasigroup $(Q, \cdot)$ is said to be right Bol quasigroup if the identity

$$(yx \cdot z)x = yL_{f_x}^{-1}(xz \cdot x) \tag{6}$$

holds true, where $f_x \cdot x = x$ for any $x \in Q$ [7].

In the left-transitive quasigroups the identity (6) is transformed into the following right Bol loop identity

$$(yx \cdot z)x = y(xz \cdot x), \tag{7}$$

since any left-transitive quasigroup has the left unit f (Lemma 2).

Example 1. We give an example of left transitive quasigroup. This is (23)-parastrophe of the group S_3.

*	0	1	2	3	4	5
0	0	1	2	3	4	5
1	2	0	1	4	5	3
2	1	2	0	5	3	4
3	3	4	5	0	1	2
4	4	5	3	2	0	1
5	5	3	4	1	2	0

2 Results

2.1 Left-transitive quasigroups and some other quasigroup classes

In other way the following theorem is proved in [7].

Theorem 3. *Any left-transitive quasigroup $(Q,\cdot)$ is a left Bol quasigroup.*

Proof. Consider loop isotope of quasigroup $(Q,\cdot)$ of the form

$$x\circ y = R_f^{-1}x\cdot L_f^{-1}y = xf\cdot y. \tag{8}$$

From Theorem 2 it follows that this isotope is a group.

We write the left side of the equality (1) in the following form:

$$x(y\cdot xz) = R_f x\circ R_f y\circ R_f x\circ z. \tag{9}$$

The right side can be rewritten in the form: $R_{e_x}^{-1}(x\cdot yx)\cdot z = R_f t\circ z$, where $t = R_{e_x}^{-1}(x\cdot yx)$. Then $x\cdot yx = R_{e_x}t = t\cdot e_x$. Passing in the last equality to the operation "$\circ$" we have $R_f x\circ R_f y\circ x = R_f t\circ e_x$,

$$R_f x\circ R_f y\circ x\circ (e_x)^{-1} = R_f t. \tag{10}$$

From the equality $x\cdot e_x = x$ we have $R_f x\circ e_x = x$, $(e_x)^{-1} = x^{-1}\circ R_f x$. If in the equality (10) we substitute the last equality, then we have $R_f x\circ R_f y\circ x\circ x^{-1}\circ R_f x = R_f t$, $R_f x\circ R_f y\circ R_f x = R_f t$. Therefore,

$$R_f x\circ R_f y\circ R_f x\circ z = R_f t\circ z. \tag{11}$$

The right side of the equation (9) coincides with the left side of the equation (11) that proves the lemma. □

Proposition 1. *Left-transitive quasigroup $(Q,\cdot)$ is isotopic to an abelian group if and only if translation R_f is an automorphism of quasigroup $(Q,\cdot)$.*

Proof. It is easy to check. □

In [9] conditions when left-transitive quasigroup $(Q,\cdot)$ is isotopic to an abelian group are given using Theorem 1 in the following form:

Proposition 2. *Left-transitive quasigroup $(Q,\cdot)$ is isotopic to an abelian group if and only if permutation I, $x + Ix = 0$ for all $x\in Q$, is an automorphism of group $(Q,+)$.*

Proof. It is easy to check. □

Lemma 1. *Left-transitive quasigroup $(Q,\cdot)$ satisfies the following identity (right Bol loop identity)*

$$(zx\cdot y)x = z(xy\cdot x) \tag{12}$$

if and only if permutation R_f is an automorphism of quasigroup $(Q,\cdot)$.

Proof. The proof follows from equality (8) using direct calculations. □

Remark 4. Using the equality (2) we can rewrite identity (12) in the following form $-z - y = -y - z$. From this equality is follows that group $(Q, +)$ is commutative and that permutation $Ix = -x$ for all $x \in Q$ is an automorphism of group $(Q, +)$.

Definition 4. A quasigroup $(Q, \cdot)$ is said to be Moufang quasigroup if the identity

$$x(y \cdot xz) = (x \cdot yf_x)x \cdot z \tag{13}$$

holds true, where $f_x x = x$ for any $x \in Q$ [3].

Proposition 3. *Any left-transitive quasigroup $(Q, \cdot)$ is a Moufang quasigroup if and only if translation R_f is an automorphism of quasigroup $(Q, \cdot)$.*

Proof. It is known that any left and right Bol quasigroup is a Moufang quasigroup. See, for example [2, p. 80]. Result follows from Theorem 3 and Lemma 1. □

Definition 5. A quasigroup $(Q, \cdot)$ with the identity $x \cdot yz = xy \cdot e_x z$, where $x \cdot e_x = x$ for all $x \in Q$, is called left F-quasigroup.

Theorem 4. *Any left-transitive quasigroup $(Q, \cdot)$ is a left F-quasigroup if and only if translation R_f is an automorphism of the quasigroup $(Q, \cdot)$.*

Proof. It is clear that by proving this lemma we can use any of the approaches to left-transitive quasigroups, namely the approach from Theorem 1 or Theorem 2.

Here we use the approach from Theorem 1. In this case from the equality $x \cdot e_x = x$ we have that $e_x = 2x$ for any $x \in Q$. Suppose that left-transitive quasigroup $(Q, \cdot)$ is a left F-quasigroup. Then we can rewrite the equality (13) in the form

$$-x + (-y + z) = -(-x + y) + (-2x + z). \tag{14}$$

Using group properties, further we have

$$-x - y + z = -y + x - 2x + z, -x - y = -y - x. \tag{15}$$

Therefore, the group $(Q, +)$ is commutative and in order to finish the proof we can apply Proposition 1.

Converse. Suppose that in left-transitive quasigroup $(Q, \cdot)$ the translation R_f is an automorphism of the quasigroup $(Q, \cdot)$. By Proposition 1 the group $(Q, +)$ is commutative and the identity (14) is true. □

2.2 Nuclei of left-transitive quasigroups

Definition 6. A set N_l of a quasigroup $(Q, \cdot)$ which consists of all elements $a \in Q$ such that

$$a \cdot xy = ax \cdot y \tag{16}$$

for all $x, y \in Q$ is called left nucleus of quasigroup $(Q, \cdot)$ [11].

A set N_r of a quasigroup $(Q, \cdot)$ which consists of all elements $a \in Q$ such that

$$x \cdot ya = xy \cdot a \tag{17}$$

for all $x, y \in Q$ is called right nucleus of quasigroup $(Q, \cdot)$ [11].

Theorem 5. *Left nucleus $(N_l, \cdot)$ of left transitive quasigroup $(Q, \cdot)$ is a normal subgroup of quasigroup $(Q, \cdot)$ which consists of elements of order two that lie in the centre of group $(Q, +)$.*

Proof. It is well known [1,11] that the sets N_l and N_r form subgroups of quasigroup $(Q, \cdot)$.

Using representation (2) we can rewrite the equality (16) in the following form $-a - x + y = -(-a + x) + y$, $-a - x = -x + a$. If $x = 0$, then we have $-a = a$, $a + a = 0$.

The last means that the element a has order two, element a lies in the centre $(Z, +)$ of the group $(Q, +)$, left nucleus $(N_l, +)$ is an abelian normal subgroup of the group $(Q, +)$.

From Remark 1 and results on normal congruences of isotopic quasigroups [13] it follows that $(N_l, \cdot) \trianglelefteq (Q, \cdot)$. □

Theorem 6. *If left transitive quasigroup $(Q, \cdot)$ has non-empty right nucleus, then it is a commutative 2-group.*

Proof. Using the representation (2) we can rewrite the equality (17) in the following form $-x+(-y+a) = -(-x+y)+a$, $-x-y = -y+x$, $-y = x-y+x$, $-x = y+x-y$, $Ix = y + x - y$ for all $x, y \in Q$.

The last means that the permutation I is an automorphism of the group $(Q, +)$, the group $(Q, +)$ is commutative. From the equality $-x = y + x - y$ we have $-x = x, x + x = 0$. The last means that in the group $(Q, +)$ all non-zero elements have order two. □

Corollary 5. *If left transitive quasigroup $(Q, \cdot)$ is associative, then $(Q, \cdot)$ is an abelian group any element of which (with the exception of its unit) has order two.*

Proof. The proof follows from Theorem 5. In this case left nucleus coincides with quasigroup $(Q, \cdot)$. □

2.3 Some morphisms of left transitive quasigroups

First next theorem was proved in [7]. Our proof differs from the proof given in [7].

Theorem 7. *Any autotopy (α, β, γ) of left-transitive quasigroup $(Q, \cdot)$ has the form*

$$(\alpha, \beta, \gamma) = (L^{\cdot}_a, R^{\cdot}_b, R^{\cdot}_f L^{\cdot}_a R^{\cdot}_b) R^{\cdot}_f \theta, \tag{18}$$

where a, b are fixed elements of the set Q, θ is an automorphism of $(Q, \cdot)$ and $(Q, +)$.

Proof. We shall use representation (2) of left-transitive quasigroup. From Remark 1 it follows that left-transitive quasigroup $(Q, \cdot)$ and group $(Q, +)$ are isotopic with isotopy $(I, \varepsilon, \varepsilon)$. Therefore autotopy groups of these quasigroups are isomorphic [1, 13], and forms of autotopies are conjugate with isotopy $(I, \varepsilon, \varepsilon)$.

It is well known [1,13] that any autotopy of a group $(Q, +)$ has the following form $(L^+_a, R^+_b, L^+_a R^+_b)\theta$, where a, b are fixed elements of the set Q, θ is an automorphism of group $(Q, +)$. Therefore any autotopy of quasigroup $(Q, \cdot)$ has the form

$$(IL^+_a I, R^+_b, L^+_a R^+_b)\theta. \tag{19}$$

It is known, if permutation θ is an automorphism of a quasigroup $(Q, \cdot)$, then θ is automorphism of any parastrophe of this quasigroup [13]. From representation (2) we have that $L^{\cdot}_x = IL^+_x$, $IL^{\cdot}_x = L^+_x$, $R^{\cdot}_x = R^+_x I$, $R^{\cdot}_x I = R^+_x$ for all $x \in Q$. Using the facts that $I^2 = \varepsilon$ and $I = R^{\cdot}_f$ we can rewrite (19) in the following form

$$(L^{\cdot}_a I, R^{\cdot}_b I, IL^{\cdot}_a R^{\cdot}_b I)\theta = (L^{\cdot}_a, R^{\cdot}_b, R^{\cdot}_f L^{\cdot}_a R^{\cdot}_b)R^{\cdot}_f\theta, \tag{20}$$

where θ is an automorphism of quasigroup $(Q, \cdot)$. □

Definition 7. The last component of an autotopy of a quasigroup is called a quasi-automorphism [1].

Lemma 2. *The groups of second and third components of the group of all autotopisms of an LIP-quasigroup coincide [13, Lemma 2.40. 1.].*

Lemma 3. *Any quasiautomorphism of left-transitive quasigroup* $(Q, \cdot)$ *has the form* $R^{\cdot}_b R^{\cdot}_f \theta$.

Proof. From Remark 3 it follows that any left-transitive quasigroup is an LIP-quasigroup. The rest follows from Lemma 2 and Theorem 7. □

Right and left pseudoautomorphisms of a quasigroup are autotopies of a special kind.

Definition 8. A bijection θ of a set Q is called a right pseudoautomorphism of a quasigroup $(Q, \cdot)$ if there exists at least one element $c \in Q$ such that $\theta x \cdot (\theta y \cdot c) = \theta(x \cdot y) \cdot c$ for all $x, y \in Q$, i.e.,

$$(\theta, R_c\theta, R_c\theta) \tag{21}$$

is an autotopy of a quasigroup $(Q, \cdot)$. The element c is called a companion of θ.

A bijection θ of a set Q is called a left pseudoautomorphism of a quasigroup $(Q, \cdot)$ if there exists at least one element $c \in Q$ such that $(c \cdot \theta x) \cdot \theta y = c \cdot \theta(x \cdot y)$ for all $x, y \in Q$, i.e.,

$$(L_c\theta, \theta, L_c\theta) \tag{22}$$

is an autotopy of a quasigroup $(Q, \cdot)$. The element c is called a companion of θ [1].

It is well known, if a quasigroup has non-trivial left and right pseudoautomorphism, then it is a loop [1, 11].

2.4 G-property of left-transitive quasigroups

Definition 9. G-loop is a loop which is isomorphic to all its loop isotopes (LP-isotopes) [4, 11].

The importance of the study of pseudoautomorphisms follows from the following theorem of V. D. Belousov.

Theorem 8. *A loop $(L, \cdot)$ is a G-loop if and only if every element $x \in L$ is a companion of some right and some left pseudoautomorphism of $(L, \cdot)$ [1, Theorem 3.8].*

V. D. Belousov's result opens the way for the study of G-property, i.e., for the study of G-loops and left (right) G-quasigroups. There exist connections between quasigroup nuclei and pseudoautomorphisms of a quasigroup [1, p. 47].

We start from the following definition.

Definition 10. A bijection α of a set Q is called a right A-pseudoautomorphism of a quasigroup $(Q, \cdot)$ if there exists a bijection β of the set Q such that the triple (α, β, β) is an autotopy of quasigroup $(Q, \cdot)$.

A bijection β of a set Q is called a left A-pseudoautomorphism of a quasigroup $(Q, \cdot)$ if there exists a bijection α of the set Q such that the triple (α, β, α) is an autotopy of quasigroup $(Q, \cdot)$ [13, Definition 1.159].

Notice sets of all the first, second, and third components of right (left) A-pseudo-automorphisms of a quasigroup $(Q, \cdot)$, sets of right (left) A-pseudo-automorphisms of a quasigroup $(Q, \cdot)$ form groups relative to operation of multiplication of these A-pseudoautomorphisms as autotopisms of the quasigroup $(Q, \cdot)$ [13, Theorem 1.161.].

We shall denote the above listed groups using the letter Π with various indexes as follows: ${}_1\Pi_l^A$, ${}_2\Pi_l^A$, ${}_3\Pi_l^A$, ${}_1\Pi_r^A$, ${}_2\Pi_r^A$, and ${}_3\Pi_r^A$. The letter A in the right upper corner means that this is an autotopical pseudoautomorphism. For example, ${}_2\Pi_r^A$ denotes the group of second components of right A-pseudo-automorphisms of a quasigroup $(Q, \cdot)$.

The following lemma shows that in "loop" case right and left A-pseudo-automorphisms are transformed into standard pseudoautomorphisms.

Proposition 4. *In a right loop $(Q, \cdot)$ with the right identity element e, any right A-pseudoautomorphism is a right pseudoautomorphism.*

In a left loop $(Q, \cdot)$ with the left identity element f, any left A-pseudoautomorphism is a left pseudoautomorphism [13, Lemma 1.165.].

Definition 11. A quasigroup $(Q, \cdot)$ is called a right GA-quasigroup if the group ${}_2\Pi_r^A$ (or the group ${}_3\Pi_r^A$) is transitive on the set Q.

A quasigroup $(Q, \cdot)$ is called a left GA-quasigroup if the group ${}_1\Pi_l^A$ (or the group ${}_3\Pi_l^A$) is transitive on the set Q.

Right and left GA-quasigroups are called GA-quasigroups.

Theorem 9. *Case 1. Autotopy of left-transitive quasigroup* $(Q, \cdot)$ *is a right A-pseudoautomorphism if and only if there exists an element* $a \in Q$ *such that*

$$R^{\cdot}_f = L^{\cdot}_a. \tag{23}$$

Case 2. Autotopy of left-transitive quasigroup $(Q, \cdot)$ *is a left A-pseudoautomorphism if and only if the following equality is true:*

$$L^{\cdot}_a = R^{\cdot}_f L^{\cdot}_a R^{\cdot}_b \tag{24}$$

for some fixed $a, b \in Q$.

Proof. The proof follows from Corollary 3 and Theorem 7. □

Theorem 10. *Left-transitive quasigroup* $(Q, \cdot)$ *is a right GA-quasigroup if and only if* $(Q, \cdot)$ *is an abelian 2-group.*

Proof. ⇒ Here we use equality (2). We can rewrite equality (23) $R_f x = L_a x$, $x \cdot f = a \cdot x$ for all $x \in Q$ in the form:

$$-x + f = -a + x. \tag{25}$$

If we put $x = 0$ in the equation (25), then we have $f = -a$, $-f = f = a$. Therefore $a = f = 0$ (Corollary 2, Remark 2).

Then the equality (25) can be rewritten in the form $-x = x$, $x + x = 0$ for all $x \in Q$. Thus $x \cdot y = -x + y = x + y = y + x = -y + x = y \cdot x$. We have used the well known fact [10] that any group in which all non-zero elements have the order two, is commutative. Properties of commutative left-transitive quasigroups are given in Corollary 1.

⇐ It is easy to see. □

Corollary 6. *A left-transitive quasigroup* $(Q, \cdot)$ *is a GA-quasigroup if and only if* $(Q, \cdot)$ *is an abelian 2-group.*

Proof. Any GA-quasigroup is a right GA-quasigroup. □

Given in Example 1 quasigroup does not satisfy equality (24). Then there exist left-transitive quasigroups that are not left or right GA-quasigroups. Commutative left-transitive quasigroup is a right GA-quasigroup. It is easy to see that this quasigroup is also a G-loop.

2.5 Simple left-transitive quasigroups

Proposition 5. Left-transitive quasigroup $(Q, \cdot)$ is simple if and only if the group $(Q, +)$ is simple.

Proof. From the results given in [13, Remark 1.308] and the form of isotopy (Remark 1) it follows that sets (lattices) of normal congruences of left-transitive quasigroup $(Q, \cdot)$ and of corresponding group $(Q, +)$ are equal. □

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