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Isohedral tilings by 8-, 10- and 12-gons for hyperbolic translation group of genus two

Elizaveta Zamorzaeva

Abstract. As is known there are 2 types of isoheral tilings of the Euclidean plane with disks for the translation group $p1$. In the hyperbolic plane there exist a countable series of translation groups, each group being characterized by its genus. For the hyperbolic translation group of genus two, isohedral tilings of the hyperbolic plane with disks are studied. The technique of adjacency symbols, developed by B. N. Delone for the Euclidean case, is used. In the present article we restrict the enumeration to tilings with 8-, 10- and 12-gons.

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1 Introduction

In the Euclidean plane the translation group $p1$ is one of 17 crystallographic plane groups, i. e. discrete isometry groups with compact fundamental domains. All the isohedral (tile-transitive) tilings of the Euclidean plane have been found. For the group $p1$ there are 2 types (as well as 2 Delone classes) of isohedral tilings of the Euclidean plane plane with disks, and these disks are parallelograms and center-symmetric hexagons (if convex). The orbifold symbol by Conway of the group $p1$ is $\circ$ (a circle).

The analog of $p1$ in the hyperbolic plane, i. e. a discrete group of translations with a compact (bounded) fundamental domain, is characterized by its genus, which is the genus of the quotient of the hyperbolic plane by the group. The Conway's orbifold symbol of such a group is $\circ\circ\cdots\circ$ where the number of circles is equal to its genus. So there exist a countable series of hyperbolic translation groups. The smallest genus of a hyperbolic translation group is 2, so the hyperbolic group of genus 2 with the orbifold symbol $\circ\circ$ is the simplest hyperbolic group of translations. Thus in the present paper we study isohedral tilings of the hyperbolic plane with disks for the group $\circ\circ$.

Isohedral tilings of the Euclidean plane with disks were obtained by several authors using different methods. Here we are going to apply methods analogous to those B. N. Delone used in [1] (see also [2]), and which the author described for the hyperbolic case in [3, 4]. Remark that the method developed by Z. Lučić and E. Molnár [5, 6] cannot be applied here because it gives only a small part of possible isohedral tilings for a hyperbolic translation group, although for some other groups it works perfectly. Also there exists one more method for obtaining k-isohedral tilings (that is with k transitivity classes of tiles) of all three 2-dimensional spaces of constant curvature. Some algorithms based on Delauney—Dress symbols were developed in [7]. Moreover, the algorithms were implemented on computer.

2 Basic notions

First we recall basic concepts which are needed here. Remark that we consider tilings with topological disks as in the classical monograph [8] it is done for the Euclidean plane.

A set W of closed topological disks in the plane is called a tiling of the plane with disks if every point of the plane belongs to at least one disk and no two disks have an inner point in common. The disks of a tiling are called tiles.

Definition 1. Let W be a tiling of the hyperbolic plane with disks, G be a discrete isometry group of the hyperbolic plane with a compact (bounded) fundamental domain. The tiling W is called isohedral with respect to the group G if the group G maps the tiling W onto itself and G acts transitively on the set of all disks of W.

Definition 2. Consider all possible pairs (W, G) where W is a tiling of the hyperbolic plane with disks which is isohedral with respect to a discrete hyperbolic isometry group G with a bounded fundamental domain. Two pairs (W, G) and (W', G') are said to belong to the same Delone class if there exists homeomorphic transformation φ of the plane which maps the tiling W onto the tiling W' and the relation $G = \varphi^{-1} G' \varphi$ holds.

In tilings under consideration the tiles are closed topological disks, so we need to define their vertices and edges. In our paper tilings are locally finite, i. e. any compact circular disk meets only a finite number of tiles.

A non-empty connected component of the intersection of two or more different tiles in a tiling is called a vertex of the tiling if it is a single point and is called an edge of the tiling otherwise. So a tile is a topological disk whose boundary is a simple closed curve. The boundary of a tile is divided by vertices of the tiling into curves that are edges of the tiling, so any tile may be considered as a curvilinear polygon.

A Delone class (W, G) is called fundamental if the group G acts one time transitively (simply transitively) on the set of tiles of W. In this case any tile is a fundamental domain of the group S, i. e. no two inner points of the tile are equivalent under the group G. Any translation group admits only fundamental Delone classes (or fundamental tilings).

Thus our task is to find all fundamental Delone classes of isohedral tilings in the hyperbolic plane with disks for the translation group of genus 2.

3 Solving Diophantine equations

We proceed following the scheme of work [1].

The first step is to solve Diophantine equations obtained from Euler theorem. For a tiling on the surface of genus g with N_0 vertices, N_1 edges and N_2 tiles, the Euler formula is: $N_0 - N_1 + N_2 = 2(1-g)$.

In an isohedral tiling each tile has the same cycle of valencies: $(\alpha_1, \alpha_2, \dots, \alpha_k)$ with each $\alpha_j \geq 3$, $j = 1, 2, \dots, k$. Denote by q_i the number of times the number i occurs in this cycle, then $N_0 = \sum \frac{q_i}{i}$, $N_1 = \sum \frac{q_i}{2}$, $N_2 = 1$, where $\sum q_i = k$ is the number of vertices, and for our group $g = 2$. Substituting in the Euler formula we obtain the Diophantine equation

$$\sum \frac{q_i}{i} = \frac{k}{2} - 3. \tag{1}$$

From $\sum \frac{q_i}{i} > 0$ and $i \geq 3$ we easily obtain additional inequalities $0 < \frac{k}{2} - 3 \leq \frac{\sum q_i}{3} = \frac{k}{3}$ which give us the restrictions $6 < k \leq 18$. Because a translation sends an edge of a tile onto another edge, it follows that the number k is even.

Thus we solve in integers the Diophantine equation (1) with conditions $i \geq 3$, $6 < k \leq 18$, k is even. Also we take in consideration that q_i is a multiple of i (from the geometrical sense of these numbers).

Now we list the obtained solutions of the equation (1):

$k = 18$ 333333333333333333
$k = 16$ 3333333333334444
$k = 14$ A : 33333333355555
B : 33333344444444
$k = 12$ A : 333333666666
B : 333444455555
C : 444444444444
$k = 10$ A : 3337777777
B : 4444666666
C : 5555555555
$k = 8$ 88888888

4 Adjacency symbols and adjacency diagrams

After the solutions have been obtained we form all possible ordered cycles of valencies $(\alpha_1, \alpha_2, \dots, \alpha_k)$ which may correspond to tiles of an isohedral tiling for a translation group. Here we take in consideration that all edges of a tile fall into pairs, two edges of a pair are mapped one onto another by translation. In fact we need all possible classes of ordered cycles of valencies up to equivalence.

To ordered cycles of valencies we apply adjacency symbols and adjacency diagrams developed by B. N. Delone in [1] (see also [2] and [3]). The method allows to obtain fundamental Delone classes of isohedral tilings.

Let a fundamental isohedral tiling (W, G) of the plane (either hyperbolic or Euclidean) with disks be given. Choose a tile and label consecutively all its edges with the letters a, b, ... Because the tiling is fundamental, applying the isometry group G yields a quite definite labelling of all the tiles of the tiling W. In a symbol the letter which labels the first chosen edge stands first, the letter which labels the adjacent edge of the neighbor tile stands next to it, then the lower index indicates the valency of the end vertex of the first edge, after that we pass to the second consecutive edge, and so on. If an isometry changes the orientation of a tile, this fact should be indicated with a bar over the second letter. In the present article we consider only translations, they preserve orientation, so no bars are needed. Such an adjacency symbol contains information both on generators and relations of the group, it fully determines the Delone class of the pair (W, G).

The method works as follows. We generate all possible adjacency symbols for each appropriate equivalence class of cycles. For each candidate in adjacency symbol we check if the condition of transition around a vertex is satisfied, for every vertex equivalence class. Further we must choose one representative among equivalent adjacency symbols.

Adjacency diagrams help us to determine (visually) whether two adjacency symbols correspond to the same Delone class. An adjacency diagram is a polygonal tile where the vertices are labelled with their valencies and the paired edges are connected with arcs.

5 Tilings with 8-gons

For $k = 8$ the only solution as well as the unique cycle of valencies is 8 : 88888888. So the tile is an 8-gon with all vertices of valency 8. It is convenient first to draw adjacency diagrams taking in consideration some restrictions. Because for our group all the isometries are translations, neighboring edges cannot be paired. Another forbidden situation is so-called 'parallel pairing': for a tile labelling $(\dots ab \dots b'a' \dots)$ two pairs aa' and bb' together cannot take place. As it was mentioned above we must check if the condition of transition around a vertex is satisfied. Here all the vertices are equivalent, so we examine only a vertex.

Thorough examination yields the following results. There are 4 Delone classes of isohedral tilings of the hyperbolic plane with 8-gonal disks for the translation group of genus 2, which are given by the following adjacency symbols: $8, 1$, $(ac_8bd_8ca_8db_8eg_8fh_8ge_8hf_8)$, which gives the well-known canonical identification, $8, 2$, $(ac_8be_8ca_8dg_8eb_8fh_8gd_8hf_8)$, $8, 3$, $(ac_8bf_8ca_8dg_8eh_8fb_8gd_8he_8)$, and $8, 4$, $(ae_8bf_8cg_8dh_8ea_8fb_8gc_8hd_8)$, which gives the most symmetric Riemann surface of genus 2 with the full automorphism group of order 96. The 2 less symmetric Delone classes are new. Using an adjacency symbol, one can reconstruct rather easy both the corresponding polygonal tile and the corresponding isohedral tiling of the hyperbolic plane. The corresponding adjacency diagrams are shown in Fig. 1. An adjacency symbol given above can be obtained from the adjacency diagram if the letter a labels the right bottom edge, going round counter-clockwise.

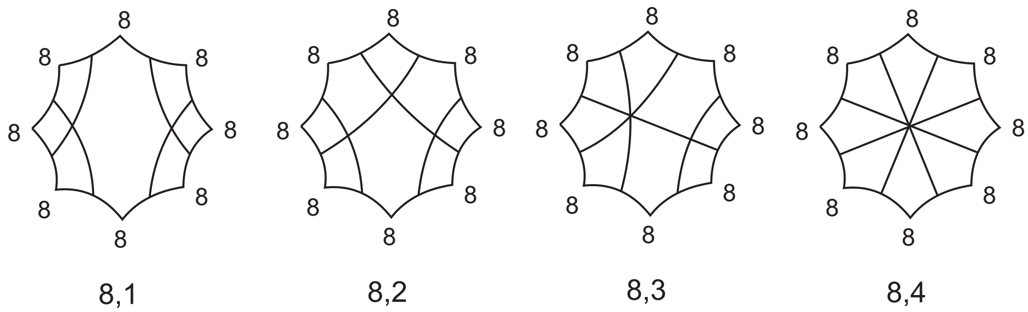


Figure 1. Adjacency diagrams for all 4 Delone classes of isohedral tilings of the hyperbolic plane by 8-gons

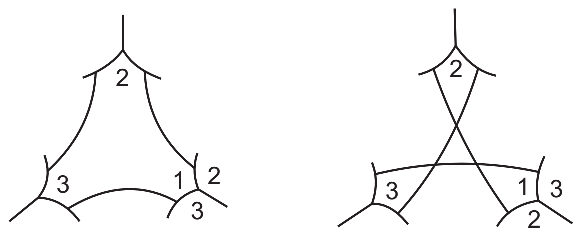


Figure 2. Two possible schemes as parts of adjacency diagrams for equivalent valencies 3

6 Tilings with 10-gons

For $k = 10$ there are 3 solutions of the Diophantine equations. We begin with the first solution $10A$: 3337777777.

For valencies 3 one more restriction appears: neighboring vertices cannot be equivalent because in this case the third equivalent vertex cannot be found. So this fact excludes pairs of edges separated by an edge with both ends of valency 3. Moreover, for valencies 3 only two schemes are possible as parts of adjacency diagrams, and they are depicted in Fig. 2.

Now we form possible classes of ordered cycles for the set $10A$. In order to avoid superfluous examination we exclude ordered cycles for which the edges cannot fall into pairs. In a pair two edges have ends of the same valency, in appropriate order. Thus we enumerate only eligible ordered cycles of valencies. In the set $10A$ the valencies 3 can be neither all consecutive nor two neighboring and one separate. So the valencies 3 must be all separate. To each eligible cycle of valencies we apply the technique of adjacency symbols and check the condition of transition around a vertex, to a vertex 3 and a vertex 7.

For the set $10A$ there are 4 eligible ordered cycles of valencies. For cycle $10A1$: 3737377777 the only Delone class $10A1$ of isohedral tilings of the hyperbolic plane exists and it is given by the adjacency symbol $(ad_3be_7cf_3da_7eb_3fc_7gi_7hj_7ig_7jh_7)$. For cycle $10A2$: 3737737777 also the only Delone class $10A2$ exists and it has the

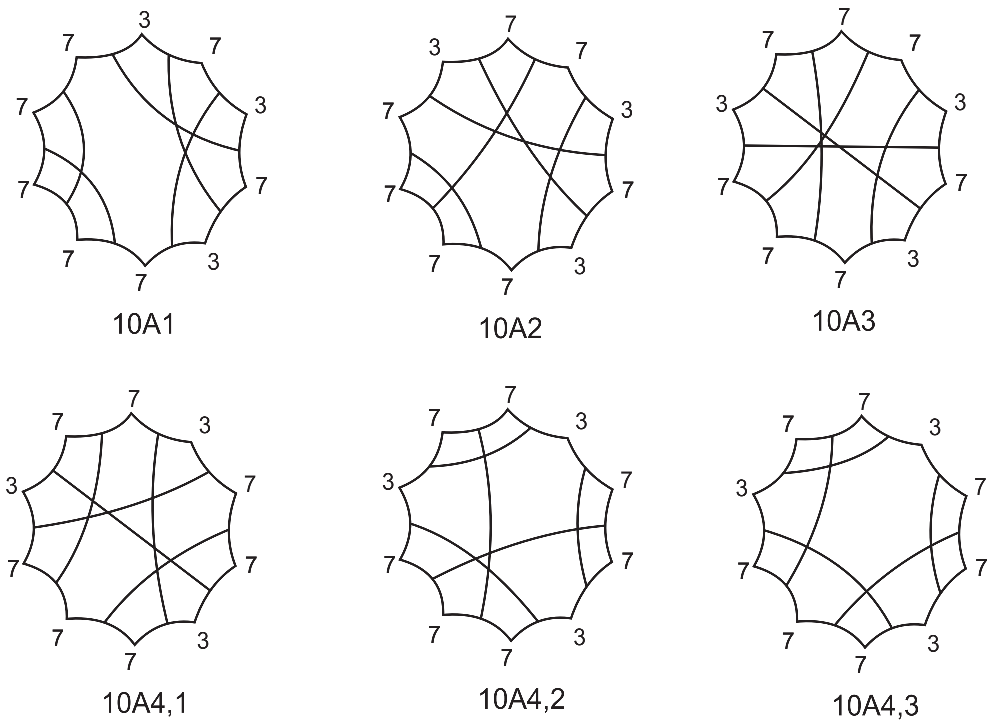


Figure 3. Adjacency diagrams for all 6 Delone classes of isohedral tilings for the set of valencies $10A$

adjacency symbol $(ad_3bf_7cg_3da_7ei_7fb_3gc_7hj_7ie_7jh_7)$. For cycle $10A3$: 3737773777 the Delone class $10A3$ with the adjacency symbol $(ad_3bg_7ch_3da_7ei_7fj_7gb_3hc_7ie_7jf_7)$ is the only possible as well. For cycle $10A4$: 3773773777 there exist 3 Delone classes of isohedral tilings of the hyperbolic plane with the following symbols: $10A4, 1$, $(ae_3bg_7cj_7dh_3ea_7fi_7gb_3hd_7if_7jc_7)$, $10A4, 2$, $(ah_3bd_7ci_7db_3eg_7fj_7ge_3ha_7ic_7jf_7)$, and $10A4, 3$, $(ah_3bd_7cj_7db_3eg_7fi_7ge_3ha_7if_7jc_7)$. The corresponding adjacency diagrams are shown in Fig. 3.

Turn to the second solution of the equation $10B$: 4444666666. Form possible classes of ordered cycles for this set. We enumerate only eligible ordered cycles of valencies. If all the vertices of valency 4 are consecutive, it yields 3 edges 44 which cannot be paired. If 3 vertices of valency 4 are consecutive and one vertex 4 is separate, it yields 2 neighboring edges 44 which also cannot be paired. So examine the situation when there are two pairs of neighboring vertices 4. To each eligible cycle of valencies we apply adjacency symbols and check the condition of transition around a vertex, to a vertex 4 and a vertex 6.

For cycle $10B1$: 4464466666 the only Delone class $10B1$ of isohedral tilings of the hyperbolic plane exists and it has the symbol $(ac_4be_4ca_6df_4eb_4fd_6gi_6hj_6ig_6jh_6)$. For cycle $10B2$: 4466446666 also the only Delone class $10B2$ exists and it is given by the adjacency symbol $(ac_4bf_4ca_6di_6eg_4fb_4ge_6hj_6id_6jh_6)$. For cycle $10B3$: 4466644666 the Delone class $10B3$ with the adjacency symbol $(ac_4bg_4ca_6di_6ej_6fh_4gb_4hf_6id_6je_6)$

is the only possible as well.

The situation when 2 vertices of valency 4 are consecutive and the other 2 vertices 4 are separate yields the only edge 44 and is not admissible. So examine the situation when all the vertices of valency 4 are separate. Then 6 vertices of valency 6 must fall into 4 groups. In the cycle 4646464666 2 edges 66 are neighboring which is not admissible. For cycle $10B4 : 4646466466$ there exist 2 Delone classes of isohedral tilings of the hyperbolic plane with the following adjacency symbols: $10B4, 1$, $(ad_4be_6ci_4da_6eb_4fh_6gj_6hf_4ic_6jg_6)$ and $10B4, 2$, $(ad_4bh_6cf_4da_6ei_4fc_6gj_6hb_4ie_6jg_6)$. For cycle $10B5 : 4646646466$ the only Delone class $10B5$ with the adjacency symbol $(ag_4bh_6ci_4df_6ej_6fd_4ga_6hb_4ic_6je_6)$ is possible.

At last turn to the third solution of the equation $10C : 5555555555$. It is the only ordered cycle, too. Here there are less restrictions for pairing. Now we begin with pairs with more near edges, then take pairs with less near edges. We apply the technique of adjacency symbols and check the condition of transition around a vertex, to two non-equivalent vertices. For the set $10C$ there are 6 Delone classes of isohedral tilings of the hyperbolic plane given by the following adjacency symbols: $10C, 1$, $(ac_5bd_5ca_5db_5ej_5fh_5gi_5hf_5ig_5je_5)$, $10C, 2$, $(ac_5be_5ca_5di_5eb_5fh_5gj_5hf_5id_5jg_5)$, $10C, 3$, $(ac_5bf_5ca_5dh_5ej_5fb_5gi_5hd_5ig_5je_5)$, $10C, 4$, $(ad_5be_5ch_5da_5eb_5fi_5gj_5hc_5if_5jg_5)$, $10C, 5$, $(ad_5bi_5cf_5da_5eh_5fc_5gj_5he_5ib_5jg_5)$, and $10C, 6$, $(af_5bg_5ch_5di_5ej_5fa_5gb_5hc_5id_5je_5)$. The two last Delone classes are known, $10C, 6$ gives the regular map of order 20 (the Riemann surface of genus 2 with the full automorphism group of order 20). The first four Delone classes are new and less symmetric.

7 Tilings with 12-gons

For $k = 12$ there are 3 solutions of the Diophantine equation. We begin with the first solution $12A : 333333666666$.

We form possible classes of ordered cycles for the set $12A$. As was mentioned above we enumerate only eligible ordered cycles of valencies. We analyze which groups of vertices of the same valency are admissible in a cycle, first for valency 3. If all the valencies 3 are consecutive, it yields 5 edges 33 and is not admissible. Falling into 5 consecutive and a separate valency 3: 33333 and 3 yields 4 neighboring edges 33 which cannot be divided into pairs because neighboring vertices 3 cannot be equivalent. Falling into 4 and 2 consecutive valencies 3: 3333 and 33 yields 3 neighboring and a separate edge 33 which cannot be divided into two pairs, too, for the same reason. Falling into 4 consecutive and 2 separate valencies 3: 3333, 3 and 3 yields three edges 33 and therefore is not admissible.

Falling into two groups 333 and 333 suits us and it allows 3 classes of eligible ordered cycles of valencies. To each eligible cycle of valencies we apply the technique of adjacency symbols and check the condition of transition around a vertex, for all three classes of equivalent vertices: two vertices of valency 3 and a vertex 6.

For cycle $12A1 : 333633366666$ the only Delone class $12A1$ of isohedral tilings of the hyperbolic plane exists and it is given by the adjacency symbol

$(ad_3bf_3cg_3da_6eh_3fb_3gc_3he_6ik_6jl_6ki_6lj_6)$. For cycle $12A2$: 333663336666 also the only Delone class $12A2$ with the symbol $(ad_3bg_3ch_3da_6ek_6fi_3gb_3hc_3if_6jl_6ke_6lj_6)$ exists. For cycle $12A3$: 333666333666 the Delone class $12A3$ with the adjacency symbol $(ad_3bh_3ci_3da_6ek_6fl_6gj_3hb_3ic_3jg_6ke_6lf_6)$ is the only possible as well.

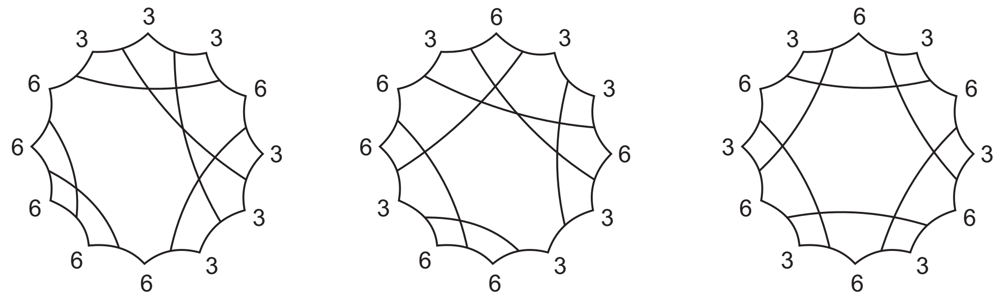
Falling into 3 consecutive, 2 consecutive and a separate valency 3: 333, 33 and 3 yields 3 edges 33 and it is not admissible. Falling into 3 consecutive and 3 separate valencies 3: 333, 3, 3, and 3 yields 2 neighboring edges 33 which cannot be paired and it is not admissible. Falling into 3 pairs of consecutive valencies 3: 33, 33 and 33 yields 3 edges 33 and it is not admissible. Falling into two pairs of cosecutive and 2 separate valencies 3 suits us and it allows 6 classes of eligible ordered cycles of valencies.

For cycle $12A4$: 336336366366 the only Delone class $12A4$ of isohedral tilings of the hyperbolic plane exists and it is given by the adjacency symbol $(ak_3be_3cg_6dh_3eb_3fj_6gc_3hd_6il_6jf_3ka_6li_6)$. For cycle $12A5$: 336336636366 also the only Delone class $12A5$ with the symbol $(ai_3be_3cj_6dk_3eb_3fh_6gl_6hf_3ia_6jc_3kd_6lg_6)$ exists. For cycle $12A6$: 336363366366 the Delone class $12A6$ with the adjacency symbol $(ae_3bg_3cj_6dh_3ea_6fk_3gb_3hd_6il_6jc_3kf_6li_6)$ is the only possible as well. For cycle $12A7$: 336363663366 the Delone class $12A7$ with the adjacency symbol $(ae_3bj_3cf_6dk_3ea_6fc_3gi_6hl_6ig_3jb_3kd_6lh_6)$ is the only possible. For cycle 336366336366 there is no Delone class of isohedral tilings of the hyperbolic plane. For cycle $12A8$: 336366336636 the only Delone class $12A8$ exists and it has the adjacency symbol $(ae_3bh_3ck_6di_3ea_6fj_6gl_3hb_3id_6jf_6kc_3lg_6)$. For cycle 336366363366 there is no Delone class.

Falling into 2 consecutive and 4 separate valencies 3 yields one edge 33 and it is not admissible. The last possibility is that all the valencies alternate. For cycle $12A9$: 363636363636 there exist 4 Delone classes of isohedral tilings of the hyperbolic plane: $12A9,1$ with the adjacency symbol $(ad_3be_6cf_3da_6eb_3fc_6gj_3hk_6il_3jg_6kh_3li_6)$, $12A9,2$ with the adjacency symbol $(ad_3bg_6ch_3da_6ej_3fk_6gb_3hc_6il_3je_6kf_3li_6)$, $12A9,3$ with the adjacency symbol $(aj_3be_6ch_3dk_6eb_3fi_6gl_3hc_6if_3ja_6kd_3lg_6)$, and $12A9,4$ with the adjacency symbol $(aj_3be_6cl_3dg_6eb_3fi_6gd_3hk_6if_3ja_6kh_3lc_6)$. The last Delone class $12A9,4$ is known, it gives the second rich Riemann surface of genus 2 with the full automorphism group of order 48. The other Delone classes are new. The adjacency diagrams for the Delone classes of isohedral tilings $12A1$, $12A4$ and $12A9,4$ are shown in Fig. 4.

Turn to the second solution of the equation $12B$: 333444455555. Form possible classes of ordered cycles for this set, enumerating only eligible ordered cycles of valencies. First we analyze valencies 3, then valencies 4 and at last valencies 5. If all the vertices of valency 3 are consecutive, it yields 2 neighboring edges 33 which cannot be paired. If 2 vertices 3 are consecutive and a vertex 3 is separate, it yields the only edge 33 which is not admissible. Thus all the vertices 3 must be separate. For each eligible cycle of valencies we apply the technique of adjacency symbols and check the condition of transition around a vertex, for all three classes of equivalent vertices: of valency 3, of valency 4 and of valency 5.

For cycle $12B1$: 343434555554 the only Delone class $12B1$ of isohedral tilings of



12A1 12A4 12A9,4

Figure 4. Adjacency diagrams for the Delone classes of isohedral tilings $12A1$, $12A4$ and $12A9,4$

hyperbolic plane with the adjacency symbol $(ad_3be_4cf_3da_4eb_3fc_4gl_5hj_5ik_5jh_5ki_5lg_4)$ exists. Here there are 6 edges with ends of valencies 3 and 4. Now diminish the number of edges 34 to 4. Grouping 3434, 43 and a separate 4 requires that one of two pairs of edges should consist of neighboring edges which is not admissible. Grouping 34, 43, 34, and 4 as well as grouping 34, 34, 43, and 4 yield three edges 34 which cannot be paired, and they are not admissible. Grouping 434, 34 and 43 as well as grouping 434, 43 and 34 yield 4 edges 34 and will be examined. Both groups require that five valencies 5 should fall into 3 groups, too. Grouping 555 and two separate 5 yields two neighboring edges 55 which cannot be paired, it is not admissible. Grouping 55, 55 and a separate 5 yields two separate edges 55 and will be examined. By combining these groupings we obtain 6 classes of eligible ordered cycles of valencies.

For cycle $12B2$: 345345543554 the only Delone class $12B2$ exists and it is given by the adjacency symbol $(ae_3bi_4cl_5dj_3ea_4fh_5gk_5hf_4ib_3jd_5kg_5lc_4)$. For cycle $12B3$: 345435534554 the Delone class $12B3$ with the adjacency symbol $(ai_3be_4cl_5dj_4eb_3fh_5gk_5hf_3ia_4jd_5kg_5lc_4)$ is also the only possible. For cycles 345534543554 and 345543534554 there are no isohedral tilings. For cycle $12B4$: 345345543554 the Delone class $12B4$ with the adjacency symbol $(ae_3bi_4cl_5dj_5ea_4fh_5gk_5hf_4ib_3jd_5kg_5lc_4)$ is the only possible. For cycle $12B5$: 345434554355 the Delone class $12B5$ is the only possible as well and it has the adjacency symbol $(ak_3be_4ci_5dg_4eb_3fj_4gd_5hl_5ic_4jf_3ka_5lh_5)$.

Now diminish the number of edges 34 to 2. Grouping 34, 43, 3, 4, and 4, we combine with 5 separate valencies 5 and obtain 8 classes of eligible ordered cycles of valencies. For cycle $12B6$: 345354354545 the only Delone class $12B6$ with the adjacency symbol $(ae_3bg_4ck_5dh_3ea_5fj_4gb_3hd_5il_4jf_5kc_4li_5)$ exists. For cycle 345435354545 there are no isohedal tilings. For cycle $12B7$: 345354543545 the Delone class $12B7$ with the adjacency symbol $(ae_3bi_4cf_5dj_3ea_5fc_4gk_5hl_4ib_3jd_5kg_4lh_5)$ is the only possible. For cycle $12B8$: 345435453545 there exist two Delone classes: $12B8,1$ with the adjacency symbol $(aj_3be_4cg_5dl_4eb_3fi_5gc_4hk_5if_3ja_5kh_4ld_5)$ and $12B8,2$ with the symbol $(aj_3be_4ck_5dh_4eb_3fi_5gl_4hd_5if_3ja_5kc_4lg_5)$. For cycle $12B9$: 345354545435 the

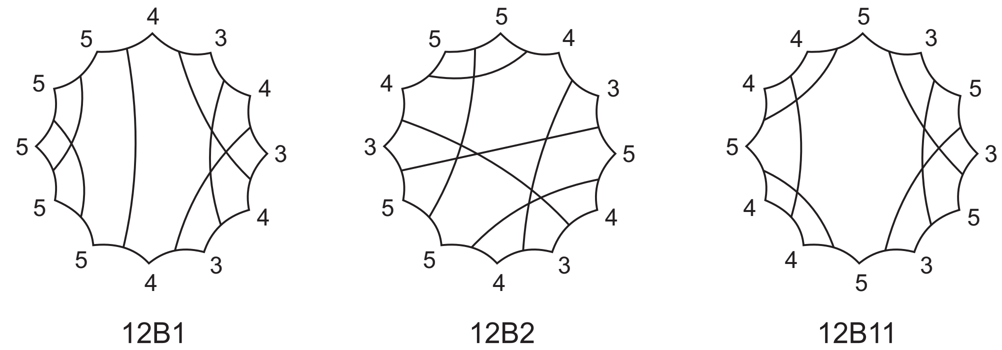


Figure 5. Adjacency diagrams for the Delone classes of isohedral tilings $12B1$, $12B2$ and $12B11$

only Delone class $12B9$ with the symbol $(ae_3bk_4ch_5dl_3ea_5fi_4gj_5hc_4if_5jg_4kb_3ld_5)$ exists. For cycle 345454353545 there are no isohedral tilings. For cycle $12B10$: 345453545435 the Delone class $12B10$ is the only possible and it has the symbol $(ag_3bk_4ch_5di_4ej_5fl_3ga_5hc_4id_5je_4kb_3lf_5)$. For cycle 345454543535 there are no isohedral tilings.

Now consider the situation where no edge 34 exists, so vertices 3 and 4 never are neighbors. The set of 3 valencies 3 and 4 valencies 4 (altogether 7) should be divided into 5 groups. As we know all the vertices 3 must be separate so 4 vertices 4 should be divided into 2 groups. Grouping 444 and 4 yields 2 neighboring edges 44 which cannot be paired, it is not admissible. Therefore grouping 44 and 44 remains. Thus we combine groups 3, 3, 3, 44, and 44 with 5 separate 5 and obtain two classes of eligible ordered cycles of valencies. For cycle $12B11$: 353535445445 the Delone class $12B11$ with the adjacency symbol $(ad_3be_5cf_3da_5eb_3fc_5gi_4hk_4ig_5jl_4kh_4lj_5)$ is the only possible. For cycle $12B12$: 353544535445 the Delone class $12B12$ with the symbol $(ad_3bh_5ci_3da_5eg_4fk_4ge_5hb_3ic_5jl_4kf_4lj_5)$ is the only possible too. The adjacency diagrams for the Delone classes of isohedral tilings $12B1$, $12B2$ and $12B11$ are shown in Fig. 5.

Turn to the third solution of the equation $12C$: 444444444444. It determines the only ordered cycle of valencies. So we apply adjacency symbols and check the condition of transition around a vertex, for all three classes of equivalent vertices of valency 4. Here the number of possible adjacency symbols is rather large. For convenience, we begin with symbols which have the element ac and thoroughly examine all the possibilities, then form symbols which have no pairing for edges separated by one edge, and so on.

There are 6 Delone classes of isohedral tilings of the hyperbolic plane: $12C,1$, given by the adjacency symbol $(ac_4be_4ca_4df_4eb_4fd_4gi_4hk_4ig_4jl_4kh_4lj_4)$, $12C,2$, given by the adjacency symbol $(ac_4bf_4ca_4dj_4eg_4fb_4ge_4hk_4il_4jd_4kh_4li_4)$, $12C,3$, given by the adjacency symbol $(ac_4bh_4ca_4df_4ek_4fd_4gi_4hb_4ig_4jl_4ke_4lj_4)$, $12C,4$, given by the adjacency symbol $(ac_4bh_4ca_4dj_4ek_4fl_4gi_4hb_4ig_4jd_4ke_4lf_4)$, $12C,5$, given by the adjacency symbol $(ad_4bf_4ck_4da_4ei_4fb_4gj_4hl_4ie_4jg_4kc_4lh_4)$, and $12C,6$, given by the adjacency symbol $(ad_4bi_4cf_4da_4ej_4fc_4gk_4hl_4ib_4je_4kg_4lh_4)$.

8 Conclusions

We can list intermediate results. For the translation group of genus two the number of Delone classes of isohedral tilings of the hyperbolic plane is as follows: 4 Delone classes of tilings with 8-gons, $6 + 6 + 6 = 18$ Delone classes of tilings with 10-gons, and $12 + 13 + 6 = 31$ Delone classes of tilings with 12-gons. The 16 Delone classes of tilings are given both by adjacency symbols and adjacency diagrams, for the other 37 Delone classes of tilings only adjacency symbols are given. The enumeration of tilings with 14-gons, 16-gons and 18-gons will be given in the next article.

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Elizaveta Zamorzaeva
Moldova State University,
60 Mateevici str., Chisinau, MD-2009
Republic of Moldova
E-mail: *zamorzaeva@yahoo.com*

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