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On the solvability of a class of boundary value problems for systems of the integral equations with power nonlinearity on the whole axis

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Abstract. We investigate a class of boundary value problems for systems of convolution type integral equations on the whole axis with power nonlinearity. These problems have a direct application in the p-adic theory of open-closed strings. We prove the existence of odd rolling solutions to the problems. We also establish the integral asymptotic for the constructed solutions. The results are illustrated by examples of the equations under consideration.

Mathematics subject classification: 45G15, 45G05, 65R20.
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1 Introduction

We consider the following system of singular nonlinear integral equations on the whole axis

$$F_i^m(x) = (\mu_i(x) - 1)\, F_i^n(x) + \sum_{j=1}^{N} \int_{-\infty}^{+\infty} K_{ij}(x-t)\, F_j(t)\, dt,\ x \in \mathbb{R},\ i = 1, 2, \ldots, N \quad (1)$$

for real-valued measurable and odd function $F(x) = (F_1(x), F_2(x), \ldots, F_N(x))^T$ (T is transpose sign), assuming that

$$m,\, n \text{ are odd numbers and } \quad m > 2n; \quad (2)$$

$$\mu_i(0) = +\infty; \quad \mu_i(x) \geq 1, \quad x \in \mathbb{R}; \quad \lim_{x\to+\infty} \mu_i(x) = 1, \quad i = 1, 2, \ldots, N, \quad (3)$$

$$\mu_i(-x) = \mu_i(x),\ x \in \mathbb{R}^+ \equiv (0, +\infty); \quad \mu_i - 1 \in L_1(\mathbb{R}^+) \cap L_2(\mathbb{R}^+); \quad (4)$$

$$K_{ij}(x) > 0, \quad K_{ij}(-x) = K_{ij}(x),\ x \in \mathbb{R}; \quad K_{ij}(x) \downarrow \text{ for } x \text{ on } [0, +\infty), \quad (5)$$

$$K_{ij} \in L_1(\mathbb{R}) \cap C_M(\mathbb{R}), \quad i, j = 1, 2, \ldots, N, \quad (6)$$

$$a_{ij} \equiv \int_{-\infty}^{+\infty} K_{ij}(t)\, dt, \qquad A = (a_{ij})_{i,j=1}^{N\times N}, \quad r(A) = 1, \quad (7)$$

$$\upsilon_{ij} \equiv \int_{-\infty}^{+\infty} |x|\, K_{ij}(x)\, dx < +\infty, \quad i, j = 1, 2, \ldots, N, \quad (8)$$

where $C_M(\mathbb{R})$ is the space of continuous and essentially bounded functions on $\mathbb{R}$, $r(A)$ is the spectral radius of matrix A.

These equations are arising in studying system of interacting open, closed and open-closed strings. It should be noted that string theory is of considerable interest not only for p-adic mathematical physics, but also in other fields of natural science, for example, in cosmology. A significant number of articles (see [1]–[7]) are devoted to the study of the concrete one-dimensional case ($N = 1$) of (1). In particular case (see [6]–[7]) they describe the dynamics (rolling) of tachyon strings with a non-zero interaction constant λ when

$$m = p^2, \quad n = \frac{p(p-1)}{2} - 1 \text{ and } m, n, p \text{ are odd numbers,} \tag{9}$$

$$K(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}, \quad \mu(x) = \lambda^2 \frac{p-1}{2p} \left(\Phi^{p-1}(x) - 1\right) + 1, \; x \in \mathbb{R}, \quad \lambda \in [0, 1), \tag{10}$$

where

$$\begin{aligned} &\Phi(-0) = +\infty, \qquad \Phi(-x) = -\Phi(x), \; x > 0, \\ &\Phi(\pm\infty) = \mp 1, \qquad \Phi^{p-1} - 1 \in L_1(\mathbb{R}^+) \cap L_2(\mathbb{R}^+). \end{aligned} \tag{11}$$

The problem of the existence of nontrivial solutions of the one-dimensional equation (1) under the conditions (9)–(11) and $\lambda = 0$ was investigated in the paper [7] of V. S. Vladimirov. In [8]–[10] of one of the authors, a one-dimensional equation when $\lambda \equiv 1$ with a more general kernel is also investigated. The solutions of the boundary value problems for this case are assumed to be real continuous functions on the whole axis, which are different from the trivial solutions (vacua) ± 1, 0.

For the particular case of the system (1) in the paper [11] one boundary value problem is considered and the existence of a nonnegative (nontrivial) nondecreasing bounded and continuous solution of this problem is proved.

In this paper by using the results of [11] we investigate the properties of systems (1) with power nonlinearity, construct an iterative method for their solution and prove the convergence of iterations. We show that a class of boundary value problems for these systems of equations on the whole axis have the odd rolling solutions. At the end of the paper we present some examples.

2 On the solvability of one auxiliary system of nonlinear integral equations on the positive semi-axis

At first we consider one auxiliary system with the same kernels

$$\psi_i^m(x) = \sum_{j=1}^{N} \int_0^\infty \left(K_{ij}(x-t) - K_{ij}(x+t)\right) \psi_j(t)\, dt, \; x \in [0, +\infty), \qquad i = 1, 2, \ldots, N \tag{12}$$

with respect to continuous function $\psi(x) = (\psi_1(x), \psi_2(x), \ldots, \psi_N(x))^T$ on $[0, +\infty)$.

It is easy to see that $\psi_i(0) = 0, \; i = 1, 2, \ldots, N$. Below we show that this system has a nonnegative nontrivial nondecreasing and bounded solution on $[0, +\infty)$.

The existence of a nonnegative nondecreasing bounded and continuous solution of system (12)**.** We apply the Perron theorem to the matrix A, defined by formula (7). There exists a vector $\eta = (\eta_1, \eta_2, \dots, \eta_N)^T$ with positive components ($\eta_i > 0$) such that

$$\sum_{j=1}^{N} a_{ij}\eta_j = \eta_i, \quad i = 1, 2, \dots, N. \tag{13}$$

Denote

$$\eta_i^* \equiv \frac{\eta_i}{\min\limits_{1\le i\le N} \eta_i} \ge 1, \quad i = 1, 2, \dots, N. \tag{14}$$

Choosing as the initial approximation

$$\psi_i^{(0)}(x) \equiv \eta_i^*, \ \ x \in \mathbb{R}^+, \ \ i = 1, 2, \dots, N, \tag{15}$$

we introduce the following successive approximations for equation (12)

$$\left(\psi_i^{(k+1)}(x)\right)^m = \sum_{j=1}^{N} \int_0^\infty \left(K_{ij}(x-t) - K_{ij}(x+t)\right) \psi_j^{(k)}(t)\,dt, \qquad k = 0, 1, 2, \dots, \quad x \in \mathbb{R}^+, \ i = 1, 2, \dots, N. \tag{16}$$

First by induction we show that the iterative functions are monotone on $\mathbb{R}^+$.

$$\psi_i^{(k)}(x) \uparrow \text{ by } x \text{ on } \ \mathbb{R}^+, \ k = 0, 1, 2, \dots, \ i = 1, 2, \dots, N. \tag{17}$$

Let $x_1, x_2 \in \mathbb{R}^+$ be arbitrary numbers and $x_1 < x_2$. The assertion for $k = 0$ is obviously true (see (15)). Assuming $\psi_i^{(k)}(x_1) \le \psi_i^{(k)}(x_2)$ for certain $k \in \mathbb{N}$ and taking into account (5) in (16), we have

$$\left(\psi_i^{(k+1)}(x_1)\right)^m = \sum_{j=1}^{N} \left(\int_{-\infty}^{x_1} K_{ij}(t)\psi_j^{(k)}(x_1 - t)\,dt - \int_0^\infty K_{ij}(x_1+t)\psi_j^{(k)}(t)\,dt \right) \le$$

$$\le \sum_{j=1}^{N} \left(\int_{-\infty}^{x_2} K_{ij}(t)\psi_j^{(k)}(x_2 - t)\,dt - \int_0^\infty K_{ij}(x_2+t)\psi_j^{(k)}(t)\,dt \right) = \left(\psi_i^{(k+1)}(x_2)\right)^m.$$

Since $m\,(m \ge 3)$ is an odd number and the function $y = x^m$ is continuous and increasing on $\mathbb{R}$, from the obtained inequality for $k+1$ (17) follows, and consequently, for any natural k.

Now we turn to the proof of the convergence of iterations (15), (16). To do this, we first prove the monotonicity of iterations with respect to k.

$$\psi_i^{(k)}(x) \downarrow \ \text{ with respect to } k, \ x \in \mathbb{R}^+, \ i = 1, 2, \dots, N. \tag{18}$$

The properties of kernels ensure the fulfillment of the following inequalities

$$K_{ij}(x - t) \ge K_{ij}(x + t), \quad x, t \in \mathbb{R}^+, \ i, j = 1, 2, \dots, N. \tag{19}$$

Taking into consideration (15), (19), (7), (13) and (14) from (16) for $k=0$ we have

$$\left(\psi_i^{(1)}(x)\right)^m \leq \sum_{j=1}^{N} \int_0^{\infty} K_{ij}(x-t)\psi_j^{(0)}(t)\,dt \leq \sum_{j=1}^{N} \eta_j^* \int_{-\infty}^{+\infty} K_{ij}(t)dt =$$

$$= \sum_{j=1}^{N} a_{ij}\eta_j^* = \eta_i^* \leq (\eta_i^*)^m = \left(\psi_i^{(0)}(x)\right)^m, \quad x \in \mathbb{R}^+,\ i=1,2,\ldots,N.$$

Again using the monotonicity property of the function $y=x^m$ on $\mathbb{R}$, from the obtained inequality it follows $\psi_i^{(1)}(x) \leq \psi_i^{(0)}(x) \equiv \eta_i^*,\ x\in\mathbb{R}^+,\quad i=1,2,\ldots,N$. Let the inequalities $\psi_i^{(k)}(x) \leq \psi_i^{(k-1)}(x)$ hold for certain $k\in\mathbb{N}$. Then from (16) we have $\psi_i^{(k+1)}(x) \leq \psi_i^{(k)}(x),\ i=1,2,\ldots,N,\ x\in\mathbb{R}^+$. Consequently, the monotonicity of the iterations for any $k=0,1,2,\ldots$ is proved.

Using the results of the work [11] we prove that iterations (16) are bounded below. According to [11] the following system of equations

$$\begin{aligned} a_i\varphi_i^3(x)+(1-a_i)\varphi_i(x) &= \sum_{j=1}^{N}\int_0^{\infty}\left(K_{ij}(x-t)-K_{ij}(x+t)\right)\varphi_j(t)\,dt, \\ & x\in[0,\,+\infty), \quad i=1,2,\ldots,N \end{aligned} \tag{20}$$

for any $a_i\in(0,1]$ has a nonnegative (nontrivial) nondecreasing bounded and continuous solution $\varphi(x)=(\varphi_1(x),\varphi_2(x),\ldots,\varphi_N(x))^T$ and

$$\varphi_i(0)=0;\quad 0\leq\varphi_i(x)\leq\eta_i^* \quad x\in\mathbb{R}^+,\quad i=1,2,\ldots,N. \tag{21}$$

Let $\widetilde{\varphi}(x)=(\widetilde{\varphi}_1(x),\widetilde{\varphi}_2(x),\ldots,\widetilde{\varphi}_N(x))^T$ be the solution of system (20) for $a_i=1$, $i=1,2,\ldots,N$.

Now we show that the following two-sided inequalities hold:

$$\widetilde{\varphi}_i(x)\leq\psi_i^{(k)}(x)\leq\eta_i^*,\quad k=0,1,2,\ldots,\ x\in\mathbb{R}^+,\ i=1,2,\ldots,N. \tag{22}$$

We prove the left-hand side of inequality of (22). It is easy to verify from (15) and (21) that the initial iteration is bounded $\psi_i^{(0)}(x)\equiv\eta_i^*\geq\widetilde{\varphi}_i(x)$. Let the $\psi_i^{(k)}(x)\geq\widetilde{\varphi}_i(x)$, $i=1,2,\ldots,N$, be valid for certain $k\in\mathbb{N}$. From (16) and (20) we have

$$\left(\psi_i^{(k+1)}(x)\right)^m \geq \sum_{j=1}^{N}\int_0^{\infty}\left(K_{ij}(x-t)-K_{ij}(x+t)\right)\widetilde{\varphi}_j^{(k)}(t)\,dt=\widetilde{\varphi}_i^3(x),\quad x\in\mathbb{R}^+.$$

Since $\psi_i^{(k)}(0)=0$ and in accordance to (17), all functions $\psi_i^{(k)}(x)$ are monotone on $\mathbb{R}^+$, therefore

$$\psi_i^{(k)}(x)>0,\quad x\in\mathbb{R}^+,\quad k=1,2,\ldots,\quad i=1,2,\ldots,N.$$

So, from the above obtained inequalities $\left(\psi_i^{(k)}(x)\right)^m \geq \widetilde{\varphi}_i^3(x)$ for odd number $m \geq 3$ it follows that

$$\psi_i^{(k)}(x) \geq \widetilde{\varphi}_i(x), \quad x \in \mathbb{R}^+, \quad i = 1, 2, \dots N.$$

Similarly, the right-hand side of inequality (22) is proved.

According to (15) and (16) from continuity of functions $\{K_{ij}\}_{i,j=1}^{N\times N}$ and x^m on $\mathbb{R}$ it follows

$$\psi_i^{(k)} \in C(\mathbb{R}^+), \quad k = 0, 1, 2, \dots, \ i = 1, 2, \dots, N. \tag{23}$$

Thus, we conclude that for every fixed $i \in \{1, 2, \dots, N\}$ the sequence of functions $\left\{\psi_i^{(k)}(x)\right\}_{k=0}^{\infty}$ has a pointwise limit as $k \to \infty$

$$\lim_{k\to\infty} \psi_i^{(k)}(x) = \psi_i(x), \quad i = 1, 2, \dots, N. \tag{24}$$

According to the Levi theorem [12] function $\psi(x)$ satisfies system (12). Moreover, due to (22), (17), (23) it follows that the following assertions hold

$$\widetilde{\varphi}_i(x) \leq \psi_i(x) \leq \eta_i^*, \quad x \in \mathbb{R}^+, \quad i = 1, 2, \dots, N, \tag{25}$$

$$\psi_i(x) \uparrow \text{ by } x \text{ on } \mathbb{R}^+, \quad i = 1, 2, \dots, N, \tag{26}$$

$$\psi_i \in C(\mathbb{R}^+), \quad i = 1, 2, \dots, N. \tag{27}$$

Consequently, taking into account (24), (26), (27) and Dini's theorem ([12]) we can state that the sequence of continuous functions $\{\psi_i^{(k)}(x)\}_{k=0}^{\infty}$ uniformly converges to continuous function $\psi_i(x)$ in each compact from $\mathbb{R}^+$.

The limit of the solution of the system (12) **at infinity.** From (25)–(27) it follows that there exits

$$\lim_{x\to+\infty} \psi_i(x) \equiv \lambda_i < +\infty, \quad i = 1, 2, \dots, N. \tag{28}$$

Using the continuity of the functions $\{\psi_i(t)\}_{i=1}^{N}$, the known limit relation for the convolution operation ([13])

$$\lim_{x\to+\infty} \int_0^{\infty} K_{ij}(x-t)\psi_j(t)dt = \lambda_j \int_{-\infty}^{+\infty} K_{ij}(t)dt = a_{ij}\lambda_j,$$

and also

$$0 \leq \left| \int_0^{\infty} K_{ij}(x+t)\psi_j(t)\,dt \right| \leq \sup_{t\geq 0} |\psi_j(t)| \int_x^{\infty} K_{ij}(\tau)\,d\tau \to 0,$$

$$\Rightarrow \lim_{x\to+\infty} \int_0^{\infty} K_{ij}(x+t)\psi_j(t)dt = 0$$

in (12) we get that

$$\lambda_i^m = \sum_{j=1}^{N} a_{ij}\lambda_j, \quad i = 1, 2, \ldots, N. \tag{29}$$

For the system of nonlinear algebraic equations (29) we construct the following successive approximations

$$\begin{gathered}\left(\lambda_i^{(k+1)}\right)^m = \sum_{j=1}^{N} a_{ij}\lambda_j^{(k)}, \\ \lambda_i^{(0)} = \eta_i^*, \; k = 0, 1, 2, \ldots, \; i = 1, 2, \ldots, N.\end{gathered} \tag{30}$$

As before, by induction it is not difficult to verify the validity of the following facts

$$\lambda_i^{(k)} \downarrow \quad \text{with respect to } k; \quad \lambda_i^{(k)} \geq \frac{\eta_i^*}{\max\limits_{1\leq i\leq N} \eta_i^*}, \; k = 0, 1, 2, \ldots, \; i = 1, 2, \ldots, N.$$

Consequently, the sequence $\{\lambda_i^{(k)}\}_{k=0}^{\infty}$ has a limit as $k \to \infty$:

$$\lim_{k\to+\infty} \lambda_i^{(k)} = \lambda_i \quad i = 1, 2, \ldots, N,$$

and the numbers λ_i satisfy the system (29) and two-sided inequalities

$$\frac{\eta_i^*}{\max\limits_{1\leq i\leq N} \eta_i^*} \leq \lambda_i \leq \eta_i^*, \; i = 1, 2, \ldots, N.$$

Thus, the following lemma is true.

Lemma 1. *If for a matrix $A = (a_{ij})_{i,j=1}^{N\times N}$ with positive elements and with a spectral radius $r(A) = 1$ the following inequality holds*

$$\frac{\min\limits_{1\leq i,j\leq N} a_{ij}}{\max\limits_{1\leq i,j\leq N} a_{ij}} > \frac{1}{\sqrt[m-1]{m}}, \tag{31}$$

then for all odd numbers $m > 2$ the system (29) *has in the class*

$$\Lambda \equiv \left\{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)^T : \; \frac{\eta_i^*}{\max\limits_{1\leq i\leq N} \eta_i^*} \leq \lambda_i \leq \eta_i^*, \; i = 1, 2, \ldots, N\right\} \tag{32}$$

the unique solution being the limit of successive approximations (30)*.*

Proof. From (13), (14) and (31) it follows

$$\text{æ} \equiv \frac{\min\limits_{1\leq i\leq N} \eta_i^*}{\max\limits_{1\leq i\leq N} \eta_i^*} = \frac{\min\limits_{1\leq i\leq N} \sum\limits_{j=1}^{N} a_{ij}\eta_j^*}{\max\limits_{1\leq i\leq N} \sum\limits_{j=1}^{N} a_{ij}\eta_j^*} \geq \frac{\min\limits_{1\leq i,j\leq N} a_{ij}}{\max\limits_{1\leq i,j\leq N} a_{ij}} > \frac{1}{\sqrt[m-1]{m}}. \tag{33}$$

Then, assuming that the system (30) has in the class Λ two solutions λ and $\widetilde{\lambda}$, we have

$$(\lambda_i-\widetilde{\lambda}_i)(\lambda_i^{m-1}+\lambda_i^{m-2}\,\widetilde{\lambda}_i+\ldots+\lambda_i\,\widetilde{\lambda}_i^{m-2}+\widetilde{\lambda}_i^{m-1})=\sum_{j=1}^{N}a_{ij}(\lambda_j-\widetilde{\lambda}_j).$$

Since λ, $\widetilde{\lambda}\in\Lambda$, by (13), (14) it immediately follows that

$$|\lambda_i-\widetilde{\lambda}_i|\cdot m\left(\frac{\eta_i^*}{\max\limits_{1\le i\le N}\eta_i^*}\right)^{m-1}\le\sum_{j=1}^{N}a_{ij}|\lambda_j-\widetilde{\lambda}_j|\le\eta_i^*\cdot\max_{1\le j\le n}\frac{|\lambda_j-\widetilde{\lambda}_j|}{\eta_j^*}.$$

In the notation introduced on the left side of (33), we get

$$m\,æ^{m-1}\frac{|\lambda_i-\widetilde{\lambda}_i|}{\eta_i^*}\le\max_{1\le j\le N}\frac{|\lambda_j-\widetilde{\lambda}_j|}{\eta_j^*},\quad\forall i=1,2,\ldots,N.$$

From the previous inequality it follows that

$$\big(m\,æ^{m-1}-1\big)\max_{1\le j\le N}\frac{|\lambda_j-\widetilde{\lambda}_j|}{\eta_j^*}\le 0. \tag{34}$$

Since $æ^{m-1}>\frac{1}{m}$ (see (33)), then from (34) we get $\lambda_i=\widetilde{\lambda}_i,\ \ i=1,2,\ldots,N.$ □

Thus, if the condition (31) is fulfilled, the solution $\psi(x)$ of the system (12) has in the class Λ (see (32)) a unique limit $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_N)^T$. Consequently, the below theorem is valid.

Theorem 1. *Suppose kernels* $\{K_{ij}\}_{i,j=1}^{N\times N}$ *possess the properties* (5)–(7). *Then for any odd number* $m>2$ *the system* (12) *has a nonnegative* (*non-trivial*) *continuous nondecreasing and bounded solution on* $[0,+\infty)$ *and the estimates* (25) *are valid.*

Moreover, if kernels satisfy the additional condition (31), *then in the class* Λ, *defined by formula* (32), *the solution of the boundary problem* (12), (28) *is unique.*

Asymptotic behavior of the solution of the system (12)**.** Below we show that the solution of the system has one more important property:

$$\lambda_i-\psi_i\in L_1(\mathbb{R}^+),\quad i=1,2,\ldots,N. \tag{35}$$

We consider for the boundary value problem (12) and (28) the following successive approximations

$$\begin{gathered}\Big(\psi_i^{(k+1)}(x)\Big)^m=\sum_{j=1}^{N}\int_0^\infty\left(K_{ij}(x-t)-K_{ij}(x+t)\right)\psi_j^{(k)}(t)\,dt,\\ \psi_i^{(0)}(x)\equiv\lambda_i,\quad k=0,1,2,\ldots,\quad x\in\mathbb{R}^+,\quad i=1,2,\ldots,N.\end{gathered} \tag{36}$$

Obviously, for these iterations for any $i \in \{1, 2, \dots, N\}$ the assertions are valid

$$\psi_i^{(k)}(x) \downarrow \text{ with respect to } k, \quad x \in \mathbb{R}^+; \tag{37}$$

$$\widetilde{\varphi}_i(x) \leq \psi_i^{(k)}(x) \leq \lambda_i, \quad k = 0, 1, 2, \dots, \quad x \in \mathbb{R}^+; \tag{38}$$

$$\psi_i^{(k)}(x) \uparrow \text{ by } x \text{ on, } R, \ k = 0, 1, 2, \dots; \tag{39}$$

$$\psi_i^{(k)} \in C(\mathbb{R}^+), \quad k = 0, 1, 2, \dots \tag{40}$$

By induction on k we prove the validity of the inclusions (35). Indeed, for $k = 0$ they are obvious. Suppose that (35) hold for certain $k \in \mathbb{N}$. Then according to (29), (36) and (7) we have

$$\lambda_i^m - \left(\psi_i^{(k+1)}(x)\right)^m =$$
$$= \left(\lambda_i - \psi_i^{(k+1)}(x)\right) \Big[\lambda_i^{m-1} + \lambda_i^{m-2}\, \psi_i^{(k+1)}(x) + \dots$$
$$\dots + \lambda_i \left(\psi_i^{(k+1)}(x)\right)^{m-2} + \left(\psi_i^{(k+1)}(x)\right)^{m-1}\Big] =$$
$$= \sum_{j=1}^N a_{ij}\lambda_j - \sum_{j=1}^N \int_0^\infty K_{ij}(x-t)\psi_j^{(k)}(t)dt + \sum_{j=1}^N \int_0^\infty K_{ij}(x+t)\psi_j^{(k)}(t)dt =$$
$$= \sum_{j=1}^N \int_0^\infty K_{ij}(x-t)(\lambda_j - \psi_j^{(k)}(t))dt + \sum_{j=1}^N \lambda_j \int_x^\infty K_{ij}(t)dt + \sum_{j=1}^N \int_0^\infty K_{ij}(x+t)\psi_j^{(k)}(t)dt.$$

Using the estimates of (38) and taking into account that the functions $\widetilde{\varphi}_i$, as a solution of the system (20), are nonnegative nondecreasing bounded and continuous, for any $i \in \{1, 2, \dots, N\}$ we obtain

$$\left(\lambda_i - \psi_i^{(k+1)}(x)\right) \left(\lambda_i^{m-1} + \lambda_i^{m-2}\, \widetilde{\varphi}_i(x)\right) \leq$$
$$\leq \sum_{j=1}^N \int_0^\infty K_{ij}(x-t)(\lambda_j - \psi_j^{(k+1)}(t))dt + 2\sum_{j=1}^N \lambda_j \int_x^\infty K_{ij}(t)dt$$

or

$$\begin{aligned} &\lambda_i^m \left(1 + \frac{\widetilde{\varphi}_i(x)}{\lambda_i}\right) \frac{\lambda_i - \psi_i^{(k+1)}(x)}{\lambda_i} \leq \\ &\leq \sum_{j=1}^N \int_0^\infty K_{ij}(x-t)(\lambda_j - \psi_j^{(k)}(t))dt + 2\sum_{j=1}^N \lambda_j \int_x^\infty K_{ij}(t)dt, \quad x \in \mathbb{R}^+. \end{aligned} \tag{41}$$

Using Fubini's theorem ([12]) to kernels with the property (8) it is easy to check

$$\int_x^\infty K_{ij}(t)dt \in L_1(\mathbb{R}^+), \quad i, j = 1, 2, \dots, N. \tag{42}$$

Taking into account the inclusion (42), the condition (6) and the inductive assumption in (41) we conclude that $\lambda_i - \psi_i^{(k+1)} \in L_1(\mathbb{R}^+)$, $i,j = 1,2,\dots,N$. Then, integrating both sides of (41) in x from 0 to $+\infty$ and taking into account (5)–(8), we have

$$\lambda_i^m \int_0^\infty \left(1 + \frac{\widetilde{\varphi}_i(x)}{\lambda_i}\right) \frac{\lambda_i - \psi_i^{(k+1)}(x)}{\lambda_i}\, dx \leq$$
$$\leq \sum_{j=1}^N \int_0^\infty \int_0^\infty K_{ij}(x-t)\left(\lambda_j - \psi_j^{(k+1)}(t)\right) dt\, dx + 2\sum_{j=1}^N \lambda_j \int_0^\infty \int_x^\infty K_{ij}(t)\, dt\, dx \leq$$
$$\leq \sum_{j=1}^N \int_0^\infty \left(\lambda_j - \psi_j^{(k+1)}(t)\right) \int_{-\infty}^t K_{ij}(u)\, du\, dt + \sum_{j=1}^N \lambda_j \upsilon_{ij}. \quad (43)$$

Taking into account the property (7) of the even kernel and the formula (29), we estimate the first sum on the right-hand side of the last inequality (43)

$$\sum_{j=1}^N \int_0^\infty \left(\lambda_j - \psi_j^{(k+1)}(t)\right) \int_{-\infty}^t K_{ij}(u)\, du\, dt \leq$$
$$\leq \sum_{j=1}^N \int_0^1 \left(\lambda_j - \psi_j^{(k+1)}(t)\right) \int_{-\infty}^t K_{ij}(u)\, du\, dt + \sum_{j=1}^N a_{ij} \int_1^\infty \left(\lambda_j - \psi_j^{(k+1)}(t)\right) dt \leq$$
$$\leq \max_{1\leq j\leq N} \int_0^1 \frac{\lambda_j - \psi_j^{(k+1)}(t)}{\lambda_j}\, dt \cdot \sum_{j=1}^N \lambda_j \int_{-\infty}^1 K_{ij}(u) du +$$
$$+ \max_{1\leq j\leq N} \int_1^\infty \frac{\lambda_j - \psi_j^{(k+1)}(t)}{\lambda_j}\, dt \cdot \sum_{j=1}^N a_{ij}\lambda_j \leq \lambda_i^m \cdot \max_{1\leq j\leq N} \int_1^\infty \frac{\lambda_j - \psi_j^{(k+1)}(t)}{\lambda_j}\, dt +$$
$$+ \max_{1\leq j\leq N} \int_0^1 \frac{\lambda_j - \psi_j^{(k+1)}(t)}{\lambda_j}\, dt \cdot \left(\sum_{j=1}^N a_{ij}\lambda_j - \sum_{j=1}^N \lambda_j \int_1^\infty K_{ij}(u) du\right) =$$
$$= \lambda_i^m \max_{1\leq j\leq N} \int_0^1 \frac{\lambda_j - \psi_j^{(k+1)}(t)}{\lambda_j}\, dt \cdot \left(1 - \frac{1}{\lambda_i^m} \sum_{j=1}^N \lambda_j \int_1^\infty K_{ij}(u) du\right) +$$
$$+ \lambda_i^m \max_{1\leq j\leq N} \int_1^\infty \frac{\lambda_j - \psi_j^{(k+1)}(t)}{\lambda_j}\, dt.$$

Substituting this inequality into (43), dividing both sides by λ_i^m, then transforming its left-hand side according to the properties of the function $\widetilde{\varphi}_i(x)$, we get the following estimate

$$\int_0^1 \frac{\lambda_i - \psi_i^{(k+1)}(x)}{\lambda_i}\,dx + \left(1 + \frac{\widetilde{\varphi}_i(1)}{\lambda_i}\right)\int_1^\infty \frac{\lambda_i - \psi_i^{(k+1)}(x)}{\lambda_i}\,dx \le$$

$$\le \left(1 - \min_{1\le i\le N} \frac{1}{\lambda_i^m}\sum_{j=1}^N \lambda_j \int_1^\infty K_{ij}(u)du\right)\cdot \max_{1\le j\le N}\int_0^1 \frac{\lambda_j - \psi_j^{(k+1)}(t)}{\lambda_j}\,dt +$$

$$+ \max_{1\le j\le N}\int_1^\infty \frac{\lambda_j - \psi_j^{(k+1)}(t)}{\lambda_j}\,dt + \max_{1\le i\le N}\frac{1}{\lambda_i^m}\sum_{j=1}^N \lambda_j \upsilon_{ij}, \quad i = 1,2,\dots,N.$$

For $a_i,\ b_i \ge 0,\ i = 1,2,\dots,N$, we have the easily verifiable identity $\max\limits_{1\le i\le N}(a_i + b_i) = \max\limits_{1\le i\le N} a_i + \max\limits_{1\le i\le N} b_i$. Applying this identity to the left-hand side of the last inequality and denoting

$$C_1 = \min_{1\le i\le N}\frac{1}{\lambda_i^m}\sum_{j=1}^N \lambda_j \int_1^\infty K_{ij}(u)du \ >\ 0, \quad C_2 = \min_{1\le i\le N}\frac{\widetilde{\varphi}_i(1)}{\lambda_i} \ >\ 0,$$

$$C = \frac{\max\limits_{1\le i\le N}\frac{1}{\lambda_i^m}\sum\limits_{j=1}^N \lambda_j\,\upsilon_{ij}}{\min\{C_1,\ C_2\}},$$

we obtain the uniform estimate

$$\max_{1\le i\le N}\int_0^\infty \frac{\lambda_i - \psi_i^{(k+1)}(x)}{\lambda_i}\,dx \ \le\ C,\ \forall k = 0,1,2,\dots. \tag{44}$$

By (39), (40), (44) according to the Levi theorem, pointwise convergence of successive approximations (36) it follows that $\lim\limits_{k\to\infty}\psi_i^{(k)}(x) = \psi_i(x),\ i = 1,2,\dots,N$ and limit function $\psi(x) = (\psi_1(x),\psi_2(x),\dots,\psi_N(x))^T$ satisfies system (12). Furthermore, the inclusions $\lambda_i - \psi_i \in L_1(\mathbb{R}^+)$ and the inequalities

$$\max_{1\le i\le N}\int_0^\infty \frac{\lambda_i - \psi_i(x)}{\lambda_i}\,dx \ \le\ C, \quad i = 1,2,\dots,N \tag{45}$$

hold. Thus, we have proved the following assertion.

Lemma 2. *Under the conditions of theorem 1 if the kernels also possess the property* (8)*, then the function, representing the difference between the solution of the system* (12) *and its limit, is summable*

$$\lambda_i - \psi_i \in L_1(\mathbb{R}^+), \quad i = 1,2,\dots,N, \tag{46}$$

and integral estimates (45) *hold.*

3 Auxiliary boundary value problem for the system (1)

Let us return to the system (1) of singular integral equations. The solution of the original system will be obtained by an odd extension of the solution of the corresponding system on the positive axis. Thus we consider the following auxiliary boundary value problem

$$f_i^m(x) = (\mu_i(x) - 1)\, f_i^n(x) + \sum_{j=1}^{N} \int_0^\infty (K_{ij}(x-t) - K_{ij}(x+t))\; f_j(t)\, dt, \quad x \in \mathbb{R}^+ \quad (47)$$

for continues on $\mathbb{R}^+$ function $f(x) = (f_1(x), f_2(x), \dots, f_N(x))^T$. To the system of equations (47) we add the boundary conditions

$$\lim_{x\to+\infty} f_i(x) = \lambda_i, \quad i = 1, 2, \dots, N, \quad (48)$$

where λ_i is defined by (28).

We introduce the successive approximations for the system of equations (47)

$$\left(f_i^{(s+1)}(x)\right)^m = (\mu_i(x) - 1)\left(f_i^{(s)}(x)\right)^n + \\ + \sum_{j=1}^{N} \int_0^\infty (K_{ij}(x-t) - K_{ij}(x+t))\, f_j^{(s)}(t)\, dt, \quad (49)$$

$$f_i^{(0)}(x) = \psi_i(x), \quad s = 0, 1, 2, \dots, \quad x \in \mathbb{R}^+, \; i = 1, 2, \dots, N,$$

where $\psi(x) = (\psi_1(x), \psi_2(x), \dots, \psi_N(x))^T$ is the solution of system (12).

Monotonicity of iterations (49) **with respect** s**.** Taking into account (3), (4), (19) and (12) in (49) we have

$$\left(f_i^{(1)}(x)\right)^m \geq \sum_{j=1}^{N} \int_0^\infty (K_{ij}(x-t) - K_{ij}(x+t))\; \psi_j(t)\, dt = \psi_i^m(x), \\ x \in \mathbb{R}^+, \; i = 1, 2, \dots, N. \quad (50)$$

As noted above, for odd m the function x^m is continuous and monotonically increasing on $\mathbb{R}$. Then from (50) it follows that $f_i^{(1)}(x) \geq \psi_i(x) = f_i^{(0)}(x)$. Then, taking into account (3) and assuming that $f_i^{(s)}(x) \geq f_i^{(s-1)}(x)$, $i = 1, 2, \dots, N$, $x \in \mathbb{R}^+$ at some $s \in \mathbb{N}$ in (49), we get

$$\left(f_i^{(s+1)}(x)\right)^m \geq (\mu_i(x) - 1)\left(f_i^{(s-1)}(x)\right)^n + \\ + \sum_{j=1}^{N} \int_0^\infty (K_{ij}(x-t) - K_{ij}(x+t))\, f_j^{(s-1)}(t)\, dt = \left(f_i^{(s)}(x)\right)^m,$$

whence $f_i^{(s+1)}(x) \geq f_i^{(s)}(x)$, $\forall i = 1, 2, \dots, N$, $x \in \mathbb{R}^+$. So, the functional sequence $\{f_i^{(s)}(x)\}_{s=0}^\infty$ is monotone in s and $\psi_i(x) \leq f_i^{(s)}(x)$, $x \in \mathbb{R}^+$, $\forall i = 1, 2, \dots, N$.

The boundedness of iterations (49). Note that from (4) and (6) it follows

$$M_i := \int_0^\infty (\mu_i(t)-1)\,dt \cdot \sup_{x\in\mathbb{R}} \sum_{j=1}^N K_{ij}(x) < +\infty, \; i=1,2,\dots,N. \tag{51}$$

Denoting

$$M=\max\{M_1,\, M_2,\, \dots,\, M_N\}, \quad \lambda^*=\max\{\lambda_1,\lambda_2,\dots,\lambda_N\}, \tag{52}$$

we prove that for any $s\in\{0,1,2,\dots\}$ the inequalities hold

$$f_i^{(s)}(x)\le (\lambda^*+M)^{\frac{1}{m-1}}\,\mu_i^{\frac{1}{n}}(x), \quad x\in\mathbb{R}^+, \; i=1,2,\dots,N. \tag{53}$$

In case $s=0$ inequality (53) holds by (49), (28), (52) and condition (3). Suppose that (53) holds for certain $s\in\mathbb{N}$. By (2), (3), (19), (51), (52) and the induction hypothesis in (49) we obtain

$$\left(f_i^{(s+1)}(x)\right)^m \le (\mu_i(x)-1)\,(\lambda^*+M)^{\frac{n}{m-1}}\;\mu_i(x)+$$

$$+\,(\lambda^*+M)^{\frac{1}{m-1}}\sum_{j=1}^N \int_0^\infty (K_{ij}(x-t)-K_{ij}(x+t))\;\mu_j^{\frac{1}{n}}(t)\,dt \le$$

$$\le (\lambda^*+M)^{\frac{1}{m-1}}\sum_{j=1}^N \int_0^\infty K_{ij}(x-t)\,\mu_j(t)\,dt + (\mu_i(x)-1)\,(\lambda^*+M)^{\frac{n}{m-1}}\;\mu_i(x)\le$$

$$\le (\lambda^*+M)^{\frac{1}{m-1}}\left(\sum_{j=1}^N \int_0^\infty K_{ij}(x-t)\;(\mu_j(t)-1)\;dt + \sum_{j=1}^N \int_0^\infty K_{ij}(x-t)\,dt\right)+$$

$$+\,(\lambda^*+M)^{\frac{n}{m-1}}\,(\mu_i(x)-1)\,\mu_i(x)\le(\lambda^*+M)^{\frac{m}{m-1}}+(\lambda^*+M)^{\frac{m}{m-1}}\,(\mu_i(x)-1)\;\mu_i(x)\le$$

$$\le(\lambda^*+M)^{\frac{m}{m-1}}\;\left(\mu_i(x)+\mu_i^2(x)-\mu_i(x)\right)\le(\lambda^*+M)^{\frac{m}{m-1}}\;\mu_i^{\frac{m}{n}}(x).$$

It follows the validity of (53) for $s+1$, and hence, for any $s\in\{0,1,2,\dots\}$.

Further, by induction on s it is easy to verify that all functions of the sequence $\{f_i^{(s)}(x)\}_{s=0}^\infty$ $(i=1,2,\dots,N)$ are measurable by x on $\mathbb{R}^+$.

Based on the above properties of iterations (49) (monotonicity and boundedness with respect to s) we conclude that the sequence $\{f_i^{(s)}(x)\}_{s=0}^\infty$ converges pointwise as $s\to+\infty$: $\lim\limits_{s\to+\infty} f_i^{(s)}(x)=f_i(x)$, $x\in\mathbb{R}^+$. According to (3)–(7) and the Levi theorem the limit function $f(x)=(f_1(x),f_2(x),\dots,f_N(x))^T$ satisfies the equation (47). Then, by (53) there are two-sided estimates:

$$\psi_i(x)\le f_i(x)\le(\lambda^*+M)^{\frac{1}{m-1}}\,\mu_i^{\frac{1}{n}}(x), \quad x\in\mathbb{R}^+, \quad i=1,2,\dots,N. \tag{54}$$

Thus, the following lemma is proved.

Lemma 3. *Assuming the conditions* (2)–(7) *hold the system of equations* (47) *has a non-trivial measurable solution* $f(x)=(f_1(x),f_2(x),\dots,f_N(x))^T$ *on the positive semi-axis satisfying the two-sided estimate* (54).

4 Integral asymptotic of constructed solution

With the further presentation, the following fact will be useful to us. The function $f-\psi$ representing difference between solutions of systems of nonlinear integral equations (12) and (47) is summable.

First note that $f_i \in L_1^{loc}(\mathbb{R}^+), \ \ \forall i=1,2,\dots,N$. Indeed, for any $0<r<+\infty$, if $x\in(0,r)$, from (54) we have

$$0 \le \psi_i(x) \le f_i(x) \le (\lambda^*+M)^{\frac{1}{m-1}}\left(\mu_i^{\frac{1}{n}}(x)-1\right)+(\lambda^*+M)^{\frac{1}{m-1}} \le$$
$$\le (\lambda^*+M)^{\frac{1}{m-1}}\left(\mu_i(x)-1\right)+(\lambda^*+M)^{\frac{1}{m-1}}, \quad i=1,2,\dots,N.$$

From this and (3), (4) it follows $f_i\in L_1(0,r)$, $\forall r\in\mathbb{R}^+$, hence $f_i\in L_1^{loc}(\mathbb{R}^+)$.

Since $\psi_i\in C([0,\,+\infty))$, then $\psi_i\in L_1(0,r)$ for any $r\in\mathbb{R}^+$ and $i\in\{1,2,\dots,N\}$. So, $f_i-\psi_i\in L_1^{loc}(\mathbb{R}^+)$, from where it follows that

$$f-\psi\in\underbrace{L_1^{loc}(\mathbb{R}^+)\times L_1^{loc}(\mathbb{R}^+)\times\ldots\times L_1^{loc}(\mathbb{R}^+)}_{N}.$$

From the properties (26), (27) of function ψ, as the solution of boundary problem (12) and (28), the existence of numbers $r_i>0,\ i\in\{1,2,\dots,N\}$ follows, such that

$$\psi_i(r_i)=\sqrt[m-1]{\frac{c_i}{m}}, \qquad c_i\equiv\sqrt[m-1]{m}\sum_{j=1}^{N}a_{ij},\ \forall i=1,2,\dots,N. \tag{55}$$

Then there exists the value $r_0=\max\{r_1,\,r_2,\,\dots,\,r_N\}$, at witch the inequalities hold at the same time

$$\psi_i(r_0)\ge\sqrt[m-1]{\frac{c_i}{m}},\ \forall i=1,2,\dots,N. \tag{56}$$

Below we show that $f_i-\psi_i\in L(r_0,+\infty)$. To do this, first by induction on s we prove

$$f_i^{(s)}-\psi_i\in L(r_0,+\infty),\quad s=0,1,2,\dots,\quad i=1,2,\dots,N. \tag{57}$$

In the case $s=0$ the inclusion (57) is obviously satisfied. Suppose that (57) holds for certain $s\in\mathbb{N}$. Then, in view of the monotonicity of iterations (49) with respect to s, (12), (53), for any $i\in\{1,2,\dots,N\}$ we get

$$m\psi_i^{m-1}(x)\left(f_i^{(s+1)}(x)-\psi_i(x)\right)\le\left(f_i^{(s+1)}(x)\right)^m-\psi_i^m(x)=$$

$$=(\mu_i(x)-1)\left(f_i^{(s)}(x)\right)^n+\sum_{j=1}^{N}\int_0^\infty(K_{ij}(x-t)-K_{ij}(x+t))\left(f_j^{(s)}(t)-\psi_j(t)\right)dt\le$$

$$\le(\lambda^*+M)^{\frac{n}{m-1}}\,\mu_i(x)\,\left(\mu_i(x)-1\right)+\sum_{j=1}^{N}\int_0^\infty(K_{ij}(x-t)-K_{ij}(x+t))\left(f_j^{(s)}(t)-\psi_j(t)\right)dt.$$

If $x \in [r_0, +\infty)$, then from the received inequality in view of the monotonicity of the function ψ (see (26)), for any $i \in \{1, 2, \dots, N\}$, we have

$$0 \le m\psi_i^{m-1}(r_0)\left(f_i^{(s+1)}(x) - \psi_i(x)\right) \le (\lambda^* + M)^{\frac{n}{m-1}}\left(\mu_i^2(x) - \mu_i(x)\right) + \\ + \sum_{j=1}^{N}\int_0^\infty (K_{ij}(x-t) - K_{ij}(x+t))\left(f_j^{(s)}(t) - \psi_j(t)\right) dt.$$

Then by (56) and the monotonicity of the sequence $\{f_i^{(s)}(x)\}_{s=0}^{\infty}$ on s we obtain

$$0 \le c_i\left(f_i^{(s+1)}(x) - \psi_i(x)\right) \le (\lambda^* + M)^{\frac{n}{m-1}}\left((\mu_i(x)-1)^2 + (\mu_i(x)-1)\right) + \\ + \sum_{j=1}^{N}\int_0^\infty (K_{ij}(x-t) - K_{ij}(x+t))\left(f_j^{(s+1)}(t) - \psi_j(t)\right) dt. \tag{58}$$

Using the conditions (4), (6), we get

$$f_i^{(s)} - \psi_i(x) \in L(r_0, +\infty), \quad s = 0, 1, 2, \dots, \quad i = 1, 2, \dots, N.$$

Then integrating both sides of the inequality (58) with respect to x from r_0 to $+\infty$, for any $i \in \{1, 2, \dots, N\}$ we obtain

$$c_i\int_{r_0}^{\infty}\left(f_i^{(s+1)}(x) - \psi_i(x)\right) dx \le \\ \le (\lambda^* + M)^{\frac{n}{m-1}}\left(\int_{r_0}^{\infty}(\mu_i(x)-1)^2\, dx + \int_{r_0}^{\infty}(\mu_i(x)-1)\, dx\right) + \mathfrak{I}_i = \\ = (\lambda^* + M)^{\frac{n}{m-1}}\left(\|\mu_i - 1\|^2_{L_2(r_0,+\infty)} + \|\mu_i - 1\|_{L_1(r_0,+\infty)}\right) + \mathfrak{I}_i, \tag{59}$$

where is denoted

$$\mathfrak{I}_i \equiv \sum_{j=1}^{N}\int_{r_0}^{\infty}\int_0^{\infty}(K_{ij}(x-t) - K_{ij}(x+t))\left(f_j^{(s+1)}(t) - \psi_j(t)\right) dt\, dx. \tag{60}$$

By (53), (3), (7), (8) for $\mathfrak{I}_i$ we obtain the following chain of inequalities

$$\mathfrak{I}_i \le \sum_{j=1}^{N}(\lambda^* + M)^{\frac{1}{m-1}}\int_{r_0}^{\infty}\int_0^{r_0}K_{ij}(x-t)\mu_j^{\frac{1}{n}}(t)\, dt\, dx + \\ + \sum_{j=1}^{N}\int_{r_0}^{\infty}\int_{r_0}^{\infty}K_{ij}(x-t)\left(f_j^{(s+1)}(t) - \psi_j(t)\right) dt\, dx \le$$

$$\leq (\lambda^*+M)^{\frac{1}{m-1}}\sum_{j=1}^{N}\left(\int\limits_{r_0}^{\infty}\int\limits_{0}^{r_0}K_{ij}(x-t)\,(\mu_j(t)-1)\,dt\,dx+\int\limits_{r_0}^{\infty}\int\limits_{0}^{r_0}K_{ij}(x-t)dt\,dx\right)+$$

$$+\sum_{j=1}^{N}\int\limits_{r_0}^{\infty}\left(f_j^{(s+1)}(t)-\psi_j(t)\right)\int\limits_{-\infty}^{t}K_{ij}(u)dudt\leq$$

$$\leq (\lambda^*+M)^{\frac{1}{m-1}}\sum_{j=1}^{N}\left(\int\limits_{0}^{\infty}(\mu_j(t)-1)\,dt\int\limits_{-\infty}^{+\infty}K_{ij}(u)du+\int\limits_{r_0}^{\infty}\int\limits_{x-r_0}^{x}K_{ij}(u)du\,dx\right)+$$

$$+\sum_{j=1}^{N}a_{ij}\int\limits_{r_0}^{\infty}\left(f_j^{(s+1)}(t)-\psi_j(t)\right)dt\leq$$

$$\leq (\lambda^*+M)^{\frac{1}{m-1}}\sum_{j=1}^{N}\left(a_{ij}\int\limits_{0}^{\infty}(\mu_j(t)-1)\,dt+\int\limits_{0}^{\infty}uK_{ij}(u)\,du\right)+$$

$$+\sum_{j=1}^{N}a_{ij}\int\limits_{r_0}^{\infty}\left(f_j^{(s+1)}(t)-\psi_j(t)\right)dt\leq$$

$$\leq (\lambda^*+M)^{\frac{1}{m-1}}\sum_{j=1}^{N}\left(a_{ij}\|\mu_j-1\|_{L_1(\mathbb{R}^+)}+\upsilon_{ij}\right)+\sum_{j=1}^{N}a_{ij}\int\limits_{r_0}^{\infty}\left(f_j^{(s+1)}(t)-\psi_j(t)\right)dt.$$

We substitute these inequalities for $\mathfrak{I}_i$ into (59). Then summing over i from 1 to N, taking into account the notation c_i, defined by formula (55), we get

$$0\leq \sqrt[m-1]{m}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\int\limits_{r_0}^{\infty}\left(f_i^{(s+1)}(x)-\psi_i(x)\right)\,dx\leq(\lambda^*+M)^{\frac{n}{m-1}}\sum_{i=1}^{N}\|\mu_i-1\|^2_{L_2(r_0,+\infty)}+$$

$$+(\lambda^*+M)^{\frac{n}{m-1}}\sum_{i=1}^{N}\|\mu_i-1\|_{L_1(r_0,+\infty)}+(\lambda^*+M)^{\frac{1}{m-1}}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(a_{ij}\|\mu_j-1\|_{L_1(\mathbb{R}^+)}+\upsilon_{ij}\right)+$$

$$+\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\int\limits_{r_0}^{\infty}\left(f_j^{(s+1)}(t)-\psi_j(t)\right)dt,\quad s=0,1,2\ldots,$$

or

$$\sqrt[m-1]{m}\cdot N\left(\min_{1\leq i,j\leq n}a_{ij}\right)\sum_{i=1}^{N}\int\limits_{r_0}^{\infty}\left(f_i^{(s+1)}(x)-\psi_i(x)\right)\,dx\leq$$

$$\leq(\lambda^*+M)^{\frac{n}{m-1}}\sum_{i=1}^{N}\|\mu_i-1\|^2_{L_2(r_0,+\infty)}+(\lambda^*+M)^{\frac{n}{m-1}}\sum_{i=1}^{N}\|\mu_i-1\|_{L_1(r_0,+\infty)}+$$

$$+(\lambda^*+M)^{\frac{1}{m-1}}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(a_{ij}\|\mu_j-1\|_{L_1(\mathbb{R}^+)}+\upsilon_{ij}\right)+$$

$$+N\left(\max_{1\le i,j\le n}a_{ij}\right)\sum_{j=1}^{N}\int\limits_{r_0}^{\infty}\left(f_j^{(s+1)}(t)-\psi_j(t)\right)dt,\quad s=0,1,2\dots\ .$$

Whence, by the additional condition (31) for the matrix $A=(a_{ij})_{i,j=1}^{N\times N}$, we get the uniform estimate

$$\sum_{i=1}^{N}\int\limits_{r_0}^{\infty}\left(f_i^{(s+1)}(x)-\psi_i(x)\right)dx\le$$

$$\le\widetilde{C}\,(\lambda^*+M)^{\frac{n}{m-1}}\sum_{i=1}^{N}\left(\|\mu_i-1\|^2_{L_2(r_0,+\infty)}+\|\mu_i-1\|_{L_1(r_0,+\infty)}\right)+$$

$$+\,\widetilde{C}\,(\lambda^*+M)^{\frac{1}{m-1}}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(a_{ij}\|\mu_j-1\|_{L_1(\mathbb{R}^+)}+\upsilon_{ij}\right),\quad\forall s=0,1,2\dots,$$

here denoted

$$\widetilde{C}\equiv\frac{1}{N\left(\sqrt[m-1]{m}\cdot\min\limits_{1\le i,j\le n}a_{ij}-\max\limits_{1\le i,j\le n}a_{ij}\right)}>0.$$

Passing to limit as $s\to\infty$ with applying the Levi theorem, we have

$$\sum_{i=1}^{N}\int\limits_{r_0}^{\infty}\left(f_i(x)-\psi_i(x)\right)dx\le$$

$$\le\widetilde{C}\,(\lambda^*+M)^{\frac{n}{m-1}}\sum_{i=1}^{N}\left(\|\mu_i-1\|^2_{L_2(r_0,+\infty)}+\|\mu_i-1\|_{L_1(r_0,+\infty)}\right)+$$

$$+\,\widetilde{C}\,(\lambda^*+M)^{\frac{1}{m-1}}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(a_{ij}\|\mu_j-1\|_{L_1(\mathbb{R}^+)}+\upsilon_{ij}\right),\quad\forall s=0,1,2\dots\ .$$

Since $f_i(x)-\psi_i(x)\ge0,\quad x\in\mathbb{R}^+$, then $f_i-\psi_i\in L(r_0,+\infty)$. On the other hand, $f_i-\psi_i\in L_1^{loc}(\mathbb{R}^+)$, consequently, $f_i-\psi_i\in L_1(\mathbb{R}^+),\quad i=1,2,\dots,N$.

So, we have proved the following result.

Lemma 4. *Under the conditions of Lemma 3 and* (31)*, for the solution $f(x)$ of the system* (47)*, constructed through the iterations* (49)*, the inclusion hold*

$$f-\psi\in L_1^{\times N}(\mathbb{R}^+)\equiv\underbrace{L_1(\mathbb{R}^+)\times L_1(\mathbb{R}^+)\times\dots\times L_1(\mathbb{R}^+)}_{N},$$

where $\psi(x)$ is solution of boundary problem (12) *and* (28) *with the properties* (35)*.*

Corollary. *From Lemma 4 and inclusion* (35) *it easy follows that* $\lambda-f\in L_1^{\times N}(\mathbb{R}^+)$.

5 Properties of the solution of the system (47). Limit and odd continuation on the negative semi-axis

First we show that for the constructed solution $f(x)$, the limit relation (48) holds, from which the basic result of the present paper will follow.

Thus, we prove that

$$\lim_{x\to+\infty}(f_i(x)-\psi_i(x))=0,\quad i=1,2,\dots,N. \tag{61}$$

Using the two-sided estimate (54), from (12) and (47) at $x\in\mathbb{R}^+$ we have

$$0\le f_i^m(x)-\psi_i^m(x)\le(\lambda^*+M)^{\frac{n}{m-1}}\ \ \mu_i(x)\ (\mu_i(x)-1)+$$

$$+\sum_{j=1}^N\int_0^\infty K_{ij}(x-t)\,\frac{f_j(t)-\psi_j(t)}{\mu_j(t)}dt+\sum_{j=1}^N\int_0^\infty K_{ij}(x-t)\,\frac{\mu_j(t)-1}{\mu_j(t)}\,(f_j(t)-\psi_j(t))\,dt\le$$

$$\le(\lambda^*+M)^{\frac{n}{m-1}}\ \ \mu_i(x)\ (\mu_i(x)-1)+\sum_{j=1}^N\int_0^\infty K_{ij}(x-t)\,\frac{f_j(t)-\psi_j(t)}{\mu_j(t)}dt+$$

$$+(\lambda^*+M)^{\frac{1}{m-1}}\sum_{j=1}^N\int_0^\infty K_{ij}(x-t)\,\frac{\mu_j(t)-1}{\mu_j^{\frac{n-1}{n}}(t)}dt\le(\lambda^*+M)^{\frac{n}{m-1}}\ \ \mu_i(x)\ (\mu_i(x)-1)+$$

$$+\sum_{j=1}^N\left(\int_0^\infty K_{ij}(x-t)\,\frac{f_j(t)-\psi_j(t)}{\mu_j(t)}dt+(\lambda^*+M)^{\frac{1}{n-1}}\int_0^\infty K_{ij}(x-t)\ (\mu_j(t)-1)\ dt\right).$$

So, for all $i\in\{1,2,\dots,N\}$ we have

$$0\le f_i^m(x)-\psi_i^m(x)\le(\lambda^*+M)^{\frac{n}{m-1}}\ \ \mu_i(x)\ (\mu_i(x)-1)+$$
$$+\sum_{j=1}^N\left(\int_0^\infty K_{ij}(x-t)\frac{f_j(t)-\psi_j(t)}{\mu_j(t)}dt+(\lambda^*+M)^{\frac{1}{n-1}}\int_0^\infty K_{ij}(x-t)\,(\mu_j(t)-1)\,dt\right). \tag{62}$$

Note that

$$\frac{f_j-\psi_j}{\mu_j}\in L_1(\mathbb{R}^+)\cap L_\infty(\mathbb{R}^+),\quad j=1,2,\dots,N, \tag{63}$$

because

$$0\le\frac{f_j(t)-\psi_j(t)}{\mu_j(t)}\le(\lambda^*+M)^{\frac{1}{m-1}}\ \frac{1}{\mu_j^{\frac{n-1}{n}}(t)}\le(\lambda^*+M)^{\frac{1}{m-1}}<+\infty$$

and

$$0\le\frac{f_j(t)-\psi_j(t)}{\mu_j(t)}\le f_j(t)-\psi_j(t)$$

(inclusion $f_j - \psi_j \in L_1(\mathbb{R}^+)$ holds by Lemma 4).

But, on the other hand, $K_{ij} \in L_1(\mathbb{R}) \cap C_M(\mathbb{R})$, $i, j = 1, 2, \ldots, N$ (see (6)). Therefore, according to (63) and Lemma 5 from the paper [14], we have

$$\lim_{x\to+\infty} \int_0^\infty K_{ij}(x-t)\, \frac{f_j(t) - \psi_j(t)}{\mu_j(t)} dt = 0, \quad \forall i, j = 1, 2, \ldots, N. \tag{64}$$

On the basis of the above mentioned limit relation for the convolution operation we have

$$\lim_{x\to+\infty} \int_0^\infty K_{ij}(x-t)\, (\mu_j(t) - 1)\, dt = \lim_{x\to+\infty} (\mu_j(x) - 1) \int_{-\infty}^{+\infty} K_{ij}(u) du = 0. \tag{65}$$

Then by (64), (65) and (3) from (62) we obtain

$$\lim_{x\to+\infty} (f_i^m(x) - \psi_i^m(x)) = 0,$$

from which the validity of the statement (61) follows.

Further, since $\lim\limits_{x\to+\infty} \psi_i(x) = \lambda_i$ (see (28)) then by the obvious inequality

$$0 \le |f_i(x) - \lambda_i| \le |f_i(x) - \psi_i(x)| + |\psi_i(x) - \lambda_i|$$

we get $\lim\limits_{x\to+\infty} f(x) = \lambda$.

So, based on the obtained results we can assert.

Theorem 2. *Under conditions* (2)–(8), (31) *the odd extension of a nontrivial measurable solution* $f(x)$ *of boundary problem* (47) *and* (48) *on* $(-\infty, 0)$*:*

$$F(x) = \begin{cases} f(x) & \text{if } x \ge 0, \\ -f(-x) & \text{if } x < 0 \end{cases} \tag{66}$$

satisfies the system (1) *almost everywhere on* $\mathbb{R}$ *and the boundary conditions*

$$\lim_{x\to\pm\infty} F_i(x) = \pm\,\lambda_i, \quad i = 1, 2, \ldots, N.$$

Moreover, this solution possess the properties

I. $f_i(x) \le F_i(x) \le (\lambda^* + M)^{\frac{1}{m-1}}\, \mu_i^{\frac{1}{n}}(x)$ *for* $x > 0$, $i = 1, 2, \ldots, N$,
$-(\lambda^* + M)^{\frac{1}{m-1}}\, \mu_i^{\frac{1}{n}}(x) \le F_i(x) \le f_i(-x)$ *for* $x < 0$, $i = 1, 2, \ldots, N$,
where $\lambda^* = \max\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$,

II. $\lambda_i - F_i \in L_1(\mathbb{R}^+)$, $\lambda_i + F_i \in L_1(\mathbb{R}^-)$, $i = 1, 2, \ldots, N$.

Proof. The proof of the theorem is implemented by direct checking with using Lemma 3 and Lemma 4. □

Remark. It is easy to see that

$$F_i(\pm 0) \equiv \lim_{x\to\pm 0} F_i(x) = \pm\infty, \quad i = 1, 2, \ldots, N.$$

Indeed, taking into account that $\mu_i(x) \geq 1$, $x \in \mathbb{R}^+$, $K_{ij}(x) > 0$, $x \in \mathbb{R}$, and $f_i(x) \geq F_i(x) > 0$, $x \in \mathbb{R}^+$ $(i, j = 1, 2, \ldots, N)$, from system (47) we get

$$f_i^m(x) \geq (\mu_i(x) - 1)\, f_i^n(x), \quad x \in \mathbb{R}^+,$$

which implies

$$f_i(x) \geq (\mu_i(x) - 1)^{\frac{1}{m-n}}, \qquad x \in \mathbb{R}^+, \qquad i = 1, 2, \ldots, N. \tag{67}$$

On the other hand, according to (54) we have

$$f_i(x) \leq (\lambda^* + M)^{\frac{1}{m-1}}\, \mu_i^{\frac{1}{n}}(x), \quad x \in \mathbb{R}^+, \;\; i = 1, 2, \ldots, N. \tag{68}$$

Hence, from (67) and (68) taking into account $\mu_i(0) = +\infty$, we get $f_i(+0) = +\infty$. Then, bearing in mind (66), we obtain $F_i(\pm 0) = \pm\infty$.

At the end of the paper we give some examples of functions $\{\mu_i\}_{i=1}^N$ and $\{K_{ij}\}_{i,j=1}^{N\times N}$, satisfying conditions (3), (4) and (5)–(8)

- $\mu_i(x) = 1 + \frac{b_i}{|x|^{\alpha_i}} e^{-x^2}, \quad \alpha_i \in \left(0, \frac{1}{2}\right), \quad b_i > 0, \quad x \in \mathbb{R}^+;$
- $\mu_i(x) = 1 + \frac{\gamma_i}{|x|^{\frac{1}{4}}} \cdot \frac{1}{1+x^2}, \quad \gamma_i > 0, \quad x \in \mathbb{R}^+;$
- $K_{ij}(x) = \frac{a_{ij}}{\sqrt{4\pi\sigma}} e^{-\frac{x^2}{4\sigma}},, \quad x \in \mathbb{R}, \quad \sigma > 0, \quad A = (a_{ij})_{i,j=1}^{N\times N}, \quad r(A) = 1;$
- $K_{ij}(x) = \int\limits_a^b e^{-|x|s} G_{ij}(s)\, ds, \quad x \in \mathbb{R};$
  where $\quad a > 0, \;\; b \geq +\infty, \;\; G_{ij}(s) > 0, \;\; s \in [a, b], \;\; G_{ij} \in L_1(a, b),$

$$a_{ij} = 2\int\limits_a^b \frac{G_{ij}(s)}{s}\, ds < \infty, \quad A = (a_{ij})_{i,j=1}^{N\times N}, \quad r(A) = 1.$$

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