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Solution and full classification of generalized binary functional equations of the type $(3;3;0)$

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Abstract. Generalized binary functional quasigroup equations in two individual variables with three appearances are under consideration. There exist five classes of the equations (two equations belong to the same class if there exists a relation between sets of their solutions). The quasigroup solution sets of equations from every class are given. In addition, it is proved that every parastrophe of a quasigroup has an orthogonal mate if the quasigroup has an orthogonal mate.

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1 Introduction

We continue the classification of generalized functional quasigroup equations [16, 18]. This problem was considered in many articles, in particular in [2, 3, 5, 8–13, 16, 18, 19].

One can assign a generalized functional quasigroup equation to every quasigroup identity by replacing all quasigroup operations and all their parastrophes with pairwise different functional variables. Note that two identities are pastrophically equivalent, i.e. define parastrophic varieties, if the corresponding generalized functional quasigroup equations are parastrophically primarily equivalent, i.e., they can be obtained one from the other by renaming functional variables or applying identities which define quasigroups. Therefore, a classification of generalized functional quasigroup equations up to parastrophically-primary equivalence, say ϵ_0, implies some classification of the corresponding quasigroup identities up to parastrophic equivalence.

If all solutions of a functional equation satisfy a property being invariant under parastrophic transformations, then all solutions of arbitrary functional equations from the same class satisfy this property. Such properties ('to be isotopic to the same group'; 'to have a two-sided neutral element' . . .) can be found in [17, 18].

In this article, a functional equation is a formula which is an equality of two terms consisting only of functional and individual variables, all functional variables are free and all individual variables are universally quantified. The *length* of a functional equation is the number of functional variables appearances. The *type* of a functional equation is the sequence $(m_1, \dots, m_n)$, where n is the number of different individual variables and m_i is the number of occurrences of the i-th individual variable.

A functional equation is called: *quasigroup* if all its functional variables represent quasigroup operations; *generalized* if all functional variables are pairwise different; *binary* if all functional variables are binary, that is, represent binary operations. A functional quasigroup equation is called *trivial* if it has only the solutions defined on a singleton. Note a functional quasigroup equation is trivial if it has one appearance of an individual variable. Therefore, non-trivial binary functional quasigroup equations of the length four can have at most three individual variables and be of the following types: $(6;0;0)$, $(4;2;0)$, $(3;3;0)$, $(2;2;2)$.

Functional equations in three different functional variables, i.e., functional equations of the type $(2;2;2)$, were investigated in [11–13]. Classification and solutions of functional equation of the type $(4;2;0)$ were given in [9] by the first author. Here, four length functional equations of the type $(3;3;0)$ are under consideration.

In [8, 10] it was stated that every generalized non-trivial binary functional quasigroup equation of the type $(3;3;0)$ is parastrophically primarily equivalent to at least one of the equations (2) – (7). The equations (2) – (7) were given by R. Koval in her PhD thesis [8]. Full proof of this fact is given by the first author in [10], where parastrophically-primary equivalence of (3) and (4), (5) and (7) was not studied.

In this paper, the equations (2) – (7) are solved. The equations (5) and (7) are parastrophically primarily equivalent and the equations (3) and (4) are not. As a consequence, a full classification of generalized non-trivial binary functional quasigroup equations of the type $(3;3;0)$ up to parastrophically-primary equivalence is obtained.

2 Preliminaries

An operation f, defined on a *carrier* Q, is said to be *left-invertible* (*right-invertible*) if each of its right (left) translations is a permutation of Q. An operation is called *invertible* or *quasigroup* operation if it is left- and right-invertible simultaneously. In other words, the equation $f(x;a) = b$ (respectively, $f(a;y) = b$) has a unique solution for all a, $b \in Q$ and it is denoted by ${}^{\ell}f(b;a)$ (respectively, by ${}^{r}f(a;b)$). It is easy to see that ${}^{\ell}f$ and ${}^{r}f$ are invertible. They are called *left* and *right divisions* of f. A quasigroup operation f, its divisions, divisions of the divisions, ... are called *parastrophes* of f. It is easy to verify that every quasigroup operation has at most six different parastrophes. A groupoid $(Q; f)$ is called a *quasigroup* if f is invertible. So, the equalities

$$\begin{aligned} &F({}^{\ell}F(x;y), y) = x, \quad {}^{\ell}F(F(x;y), y) = x, \\ &F(x; {}^{r}F(x;y)) = y, \quad {}^{r}F(x; F(x;y)) = y \end{aligned} \tag{1}$$

are *quasigroup hyperidentities* (see [14]), i.e., they hold for all carrier Q, for all values of F in the set Δ of all quasigroup operations defined on Q and for all $x, y \in Q$.

Let W, V be terms and $[W]$ denote the set of all individual variables appearing in W. Let

$$\{x_1, \dots, x_n\} := [W] \cup [V];$$

then the formula

$$(\forall x_1) \dots (\forall x_n) \quad W = V$$

is called a *functional equation* if it has functional variables. As usual, the universal quantifiers are omitted. A sequence of operations, defined on a set Q, is called a *solution on Q of a functional equation* $W = V$ if $W = V$ becomes a true proposition after replacing all functional variables with the operations from the sequence [1]. If all components of a solution are invertible, then it is called a *quasigroup solution.* The set of all solutions on Q will be called *solution set* of the equation. Here we consider binary functional quasigroup equations that have neither individual nor functional constants.

Following Sade [15], an operation will be called *diagonal* if $f(x;x)$ is a permutation of the carrier set. A binary functional variable will be called *diagonal* if it takes its values in the set of all diagonal operations.

Two functional equations are said to be *parastrophically primarily equivalent* [9,12,16,18] if one can be obtained from the other in a finite number of the following steps:

1) application of quasigroup hyperidentities (1);
2) changing sides of the equation;
3) renaming individual variables;
4) renaming functional variables;
5) replacing a sub-term $F(x;x)$ with $\delta_F(x)$ if F is a diagonal functional variable and vice versa.

Lemma 1 (see [8])**.** *Let generalized functional equations $\omega = \upsilon$ and $\omega' = \upsilon'$ with m functional variables be parastrophically primarily equivalent. Then for each solution $(f_1, \dots, f_m)$ of $\omega = \upsilon$ there exist permutations $\sigma_1, \dots, \sigma_m$ of $\{1,2,3\}$ and a permutation τ of $\{1, \dots, m\}$ such that the tuple $({}^{\sigma_{1\tau}}f_{1\tau}, \dots, {}^{\sigma_{m\tau}}f_{m\tau})$ is a solution of $\omega' = \upsilon'$.*

From Lemma 1 the following corollary follows.

Corollary 1. *If generalized binary functional quasigroup equations $\omega = \upsilon$ and $\omega' = \upsilon'$ of the type $(3;3;0)$ are parastrophically primarily equivalent, then each of these equations has four functional variables, there exist permutations $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ of $\{1,2,3\}$ and a permutation τ of $\{1,2,3,4\}$ such that for each solution (f_1, f_2, f_3, f_4) of the equation $\omega = \upsilon$ the tuple $({}^{\sigma_{1\tau}}f_{1\tau}, {}^{\sigma_{2\tau}}f_{2\tau}, {}^{\sigma_{3\tau}}f_{3\tau}, {}^{\sigma_{4\tau}}f_{4\tau})$ is a solution of $\omega' = \upsilon'$.*

A functional quasigroup equation will be called *trivial* if it has solutions only on one-element carrier.

Lemma 2 (see [8,10])**.** *All generalized non-trivial binary functional quasigroup equations of the type* $(3;3;0)$ *are parastrophically-primarily equivalent to at least one of the following equations:*

$$F_1(x;y) = F_2(F_3(x;y);F_4(x;y)), \tag{2}$$
$$F_1(x;F_2(x;y)) = F_3(F_4(x;y);y), \tag{3}$$
$$F_1(x;F_2(y;y)) = F_3(x;F_4(x;y)), \tag{4}$$
$$F_1(x;y) = F_2(F_3(x;x);F_4(y;y)), \tag{5}$$
$$F_1(x;F_2(x;x)) = F_3(y;F_4(y;y)), \tag{6}$$
$$F_1(x;F_2(y;y)) = F_3(y;F_4(x;x)). \tag{7}$$

The equations (2) – (7) were written by R. Koval in her PhD Thesis [8]. Full proof of this fact is given by the first author in [10].

If an operation is denoted by f and an element is denoted by a, then we agree to denote the corresponding left and right translations by L_a^f and R_a^f respectively, i.e.,

$$\begin{gathered} L_a^f(x) := f(a;x), \quad R_a^f(x) := f(x;a), \\ M_a^f(x) = y \Leftrightarrow f(x;y) = a. \end{gathered} \tag{8}$$

Operations f, g are called *orthogonal* ($f \perp g$) if the system

$$\begin{cases} f(x;y) = a, \\ g(x;y) = b \end{cases}$$

has a unique solution for all $a, b \in Q$.

Recall that the left multiplication $\underset{\ell}{\oplus}$ and the right multiplication $\underset{r}{\oplus}$ of binary operations are defined by

$$(g \underset{\ell}{\oplus} h)(x;y) := g(h(x;y);y),$$
$$(g \underset{r}{\oplus} h)(x;y) := g(x;h(x;y)).$$

Lemma 3 (see [4])**.** *Let* g*,* h *be invertible operations; then the following assertions are true:*

$$g \underset{\ell}{\oplus} h \textit{is invertible} \;\Leftrightarrow\; g \perp {}^{\ell}h,$$
$$g \underset{r}{\oplus} h \textit{is invertible} \;\Leftrightarrow\; g \perp {}^{r}h.$$

3 Invariants of parastrophic transformations

In this article we need the following statement.

Theorem 1. *If a quasigroup has a quasigroup orthogonal mate, then each of its parastrophes has a quasigroup orthogonal mate.*

Proof. Let $(Q; f)$ be a quasigroup and $|Q| = m < \infty$. Consider invertible function f as a ternary relation f':

$$(a_1, a_2, a_3) \in f' :\Leftrightarrow \; f(a_1, a_2) = a_3.$$

Therefore, a transversal of f is an m-element subset t of f' such that for all $i = 1, 2, 3$ i-th components of the triplets from t are pairwise different. Theorem 5.1.1 from [7] implies that a quasigroup $(Q; f)$ has an orthogonal mate if and only if f' is a union of disjoint transversals:

$$f' = t_1 \sqcup t_2 \sqcup \ldots \sqcup t_m,$$

where $A_1 \sqcup \ldots \sqcup A_n = A_1 \cup \ldots \cup A_n$ and $A_i \cap A_j =$ for all $i, j \in \overline{1, 2}$. Since

$${}^{\sigma}f' = \{(a_{1\sigma}, a_{2\sigma}, a_{3\sigma}) \mid (a_1, a_2, a_3) \in f'\},$$

then a transversal t in f' becomes a transversal ${}^{\sigma}t$ in ${}^{\sigma}f'$, where

$${}^{\sigma}t := \{(a_{1\sigma}, a_{2\sigma}, a_{3\sigma}) \mid (a_1, a_2, a_3) \in t\}.$$

Therefore, every parastrophe of some disjoint union of transversals is also union of disjoint transversals:

$${}^{\sigma}f' = {}^{\sigma}t_1 \sqcup {}^{\sigma}t_2 \sqcup \ldots \sqcup {}^{\sigma}t_m.$$

That is why if a quasigroup has an orthogonal mate, then each of its parastrophes has an orthogonal mate. □

Corollary 2. *A property "to have an orthogonal mate" is invariant under parastrophy.*

Proof. The proof follows from the statement 1. □

Example 1. Let us construct a quasigroup $(Q; \circ)$ on a five element set $Q := \{0, 1, 2, 3, 4\}$ which has no orthogonal mate. For this aim we use the J. Wanless [21] formula (see also [7, p.120]):

$$l_{ij} = \begin{cases} 1 & \text{if } (i, j) = (0, 0) \text{ or } (1, 4) \\ 0 & \text{if } (i, j) = (1, 0) \text{ or } (2, 4) \\ j + 2 & \text{if } i = 0 \text{ and } j = \{1, 3\} \\ j & \text{if } i = 2 \text{ and } j = \{1, 3\} \\ i + j & \text{otherwise} \end{cases}$$

The constructed quasigroup $(Q; \circ)$ is the following:

$(\circ)$	0	1	2	3	4
0	1	3	2	0	4
1	0	2	3	4	1
2	2	1	4	3	0
3	3	4	0	1	2
4	4	0	1	2	3

Using the permutation $\theta = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 & 0 \end{pmatrix}$ of columns, we obtain

$(\cdot)$	0	1	2	3	4
0	4	2	1	0	3
1	1	3	0	4	2
2	0	4	2	3	1
3	2	0	3	1	4
4	3	1	4	2	0

.

The quasigroup $(Q;\cdot)$ is diagonal and has no orthogonal mate, because it is an isotope to $(Q;\circ)$ (see also [20, p.]).

Notation. J. Wanless formula implies that entry 0 from the cell $(1,0)$ belong to none transversal of the quasigroup $(Q;\circ)$. So, the entry 0 from the cell $(1,2)$ belongs to none transversal of the quasigroup $(Q;\cdot)$. That is why $(Q;\circ)$ has no orthogonal quasigroup mate.

An orthogonal quasigroup mate can be constructed according to the algorithm described in [6].

4 Solution of equations of the type $(3;3;0)$

In this section, we find solutions of generalized functional equations (2) – (7) on the set of invertible binary operations.

Proposition 1. *A quadruple* (f_1, f_2, f_3, f_4) *of invertible binary functions is a quasigroup solution of* (2) *if and only if the relationship*

$$f_1(x;y) = f_2(f_3(x;y); f_4(x;y)) \tag{9}$$

holds.

Proof. Evident. ☐

Proposition 2. *A quadruple* (f_1, f_2, f_3, f_4) *of binary functions is a quasigroup solution of* (3) *if and only if there exists an invertible function* f *such that*

$$f_2(x;y) = {}^{r}f_1(x; f(x;y)), \quad f_4(x;y) = {}^{\ell}f_3(f(x;y); y), \tag{10}$$

f_1, f_2 *are invertible and*

$${}^{r}f_1 \perp {}^{r}f, \qquad {}^{\ell}f_3 \perp {}^{\ell}f. \tag{11}$$

Proof. Let (f_1, f_2, f_3, f_4) be a quasigroup solution of (3), that is, the identity

$$f_1(x; f_2(x;y)) = f_3(f_4(x;y); y) \tag{12}$$

is true. Determine an operation f:

$$f(x;y) := f_1(x; f_2(x;y)), \tag{13}$$

then according to (12), we have

$$f(x;y) = f_3(f_4(x;y);y). \tag{14}$$

(13) and (14) can be written in the form of translations:

$$L_x^f = L_x^{f_1} L_x^{f_2}, \quad R_y^f = R_y^{f_3} R_y^{f_4}.$$

Since the operations f_1, f_2, f_3, f_4 are invertible, then for all x, y, the traslations L_x^f and R_y^f are bijections of the carrier set Q, i.e., the function f is invertible. Using the definition of the right division of f_1 and the left division of f_3, we obtain (10). From this equalities, according to Lemma 3, we have (11).

Vice versa, let (10) and (11) be true, then according to Lemma 3, the functions f_2 and f_4 are invertible. Besides

$$f_1(x; f_2(x;y)) = f_1(x; {}^r f_1(x; f(x;y))) = f(x;y)$$

$$f_3(f_4(x;y);y) = f_3({}^\ell f_3((f(x;y);y);y) = f(x;y).$$

Thus, (12) is identity, therefore the quadruple (f_1, f_2, f_3, f_4) is a quasigroup solution of (3). □

Corollary 3. *Each parastrophe of each solution of* (3) *has an orthogonal mate.*

Proof. Equalities (10) can be written as follows:

$$f_2(x; {}^r f(x;y)) = {}^r f_1(x;y), \quad f_4(x; {}^\ell f(x;y)) = {}^\ell f_3(x;y).$$

According to Lemma 3 and Proposition 2, each of the functions f_2 f_4, ${}^r f_1$, ${}^\ell f_3$ has an orthogonal mate. By virtue of Corollary 2, all parastrophes of f_1, f_2, f_3, f_4 have orthogonal mates. □

Proposition 3. *A quadruple (f_1, f_2, f_3, f_4) of binary functions is a quasigroup solution of* (4) *if and only if f_2 is diagonal invertible function, f_1, f_3 are invertible and the following relationships:*

$$^r f_3 \perp {}^r f_1; \tag{15}$$

$$f_4(x;y) = {}^r f_3(x; f_1(x; \delta_{f_2} y)) \tag{16}$$

hold.

Proof. Let (f_1, f_2, f_3, f_4) be a quasigroup solution of (4), that is, the identity

$$f_1(x; f_2(y;y)) = f_3(x; f_4(x;y)) \tag{17}$$

is true. Put $x = a \in Q$ in (17):

$$\delta_{f_2} = \left(L_a^{f_1}\right)^{-1} L_a^{f_3} L_a^{f_4}.$$

δ_{f_2} is a bijection of the set Q, because it is the composition of translations of invertible functions. In means that f_2 is diagonal, so from (17) we have

$$f_1(x;\delta_{f_2}y) = f_3(x;f_4(x;y)). \tag{18}$$

Using the right division of f_3, we obtain (16). According to Lemma 3, the equality (18) implies (15).

Vice versa, according to Lemma 3, the function f_4 is invertible. From (16) we define f_1 using the right division of f_3, as a result we obtain (18). Replacing δ_{f_2} with $f_2(y;y)$, we have the identity (17). Thus, the quadruple (f_1, f_2, f_3, f_4) is a quasigroup solution of (4). □

Proposition 4. *A quadruple (f_1, f_2, f_3, f_4) of binary functions is a quasigroup solution of (5) if and only if f_2, f_3, f_4 are invertible, f_3, f_4 are diagonal and the following relationship*

$$f_1(x;y) = f_2\left(\delta_{f_3}x;\delta_{f_4}y\right) \tag{19}$$

holds.

Proof. Let (f_1, f_2, f_3, f_4) be a quasigroup solution of (5), that is, the identity

$$f_1(x;y) = f_2(f_3(x;x);f_4(y;y)) \tag{20}$$

is true. Substitute $x = c \in Q$ and $y = c$ in (20) in turns, we obtain

$$\delta_{f_3} = \left(L^{f_2}_{f_3(c;c)}\right)^{-1} L^{f_1}_c, \quad \delta_{f_4} = \left(R^{f_2}_{f_4(c;c)}\right)^{-1} R^{f_1}_c. \tag{21}$$

δ_{f_3} and δ_{f_4} are bijections of the set Q, because they are the compositions of translations of invertible functions.

The operations f_3, f_4 are diagonal which follows from (21). Substitute δ_{f_3} and δ_{f_4} in (20) for f_3, f_4, as a result we obtain (19).

Vice versa, according to Lemma 3, the function f_1 is invertible. Verify the truth of the identity (20):

$$f_2(f_3(x;x);f_4(y;y)) = f_2(\delta_{f_3}x;\delta_{f_4}y) = f_1(x;y).$$

Thus, the quadruple (f_1, f_2, f_3, f_4) is a quasigroup solution of (5). □

Proposition 5. *A quadruple (f_1, f_2, f_3, f_4) of invertible binary functions is a quasigroup solution of (6) if and only if the functions f_2, f_4 are diagonal and the following relationships*

$$\delta_{f_2} = M^{f_1}_a, \quad \delta_{f_4} = M^{f_3}_a \tag{22}$$

hold for some a from Q.

Proof. Let (f_1, f_2, f_3, f_4) be a quasigroup solution of (6), that is, the identity

$$f_1\left(x; \delta_{f_2}(x)\right) = f_3\left(y; \delta_{f_4}(y)\right) \tag{23}$$

is true. Let us replace x with an element a of the carrier, we obtain

$$f_1(x; \delta_{f_2}(x)) = a, \qquad f_3(y; \delta_{f_4}(y)) = a.$$

These equalities are equivalent to (22). □

Proposition 6. *A quadruple (f_1, f_2, f_3, f_4) of invertible binary functions is a quasigroup solution of* (7) *if and only if the functions f_2, f_4 are diagonal and the following relationship*

$$f_1(x; y) = f_3\left(\delta_{f_2}^{-1}(y); \delta_{f_4}(x)\right) \tag{24}$$

holds.

Proof. Let (f_1, f_2, f_3, f_4) be a quasigroup solution of (7), that is, the identity

$$f_1(x; f_2(y; y)) = f_3(y; f_4(x; x)) \tag{25}$$

is true. Put $x = a \in Q$ and $y = a$ in (25):

$$\delta_{f_2} = \left(L_a^{f_1}\right)^{-1} R^{f_3}_{f_4(a;a)}, \quad \delta_{f_4} = \left(L_a^{f_3}\right)^{-1} R^{f_1}_{f_2(a;a)}.$$

Transformations δ_{f_2} and δ_{f_4} are bijections of Q, because they are compositions of translations of invertible operations, i.e., the functions f_2, f_4 are diagonal. Substitute δ_{f_2} and δ_{f_4} for f_2, f_4 in (25). As a result we obtain (24).

Vice versa, from this assumption follows that the functions f_2, f_4 are orthogonal. So,

$$f_1(x; \delta_{f_2} y) = f_3\left(y; \delta_{f_4}(x)\right),$$

that is the identity (25) is true.

Thus, the quadruple (f_1, f_2, f_3, f_4) is a quasigroup solution of (7). □

5 Full classification of functional equations of the type $(3; 3; 0)$

In this section, we have proved that the equations (5) and (7) are parastrophically primarily equivalent and the equations (3) and (4) are not. As a consequence, we obtain a full classification of generalized non-trivial binary functional quasigroup equations of the type $(3; 3; 0)$ up to parastrophically-primary equivalence.

Theorem 2. *Every generalized non-trivial binary functional quasigroup equation of the type* $(3; 3; 0)$ *is parastrophically primarily equivalent to exactly one of* (2) – (6).

Proof. According to Lemma 2, all equations satisfying the theorem conditions are parastrophically primarily equivalent to at least one equation from (2) – (7).

The equations (5) and (7) are parastrophically primarily equivalent. Indeed, (5) can be written as follows:

$$F_1(x;y) = {}^sF_2(\delta_{F_4}(y), F_3(x;x)).$$

Since the functional variable F_4 is diagonal in (5), then δ_{F_4} is invertible and the obtained equation is parastrophically primarily equivalent to

$$F_1(x;\delta_{F_4}^{-1}(y)) = {}^sF_2(y, F_3(x;x)).$$

Denote $F_2'(y;y) := \delta_{F_4}^{-1}(y)$, $F_3' := {}^sF_2$, $F_4' = F_3$, as a result we obtain

$$F_1(x;F_2'(y;y)) = F_3'(y, F_4'(x;x))$$

which coincides with (7).

Now we will prove parastrophically-primary non-equivalence of the equations (2) – (6). For this purpose, we introduce the following notations: 1) $|Q| > 1$; 2) $(Q;h)$ is an arbitrary Steiner quasigroup, $(Q;f)$ a TS-quasigroup, $(Q;g,e)$ a Steiner loop, $(Q;p)$ is an arbitrary idempotent none TS-quasigroup; 3) $\mathbb{Z}_5$ is the ring modulo five and $f_1(x;y) := 4x+y$, $f_3(x;y) := 2x+3y$, f_2 is a quasigroup operation $(\cdot)$ defined in Example 1, f_4 is defined by (16).

	(3)	(4)	(5)	(6)
(2)	(h,h,h,h)	(h,h,h,h)	(f,p,f,f)	(h,h,h,h)
(3)	$\times$	(f_1,f_2,f_3,f_4)	(h,h,h,h)	(g,h,g,h)
(4)	$\times$	$\times$	(h,h,h,h)	(g,h,g,h)
(5)	$\times$	$\times$	$\times$	(h,h,h,h)

We will prove that in the intersection of the (i)-th row and the (j)-th column of this table, there is a quadruple of functions which is a solution of one of the equations (i), (j) and is not a solution of the other.

1. The quadruple (h,h,h,h) is a solution of the equations (2) and (5). Indeed, putting this quadruple into each of (2) and (5):

$$xy = xy \cdot xy, \qquad xy = x^2y^2.$$

This equalities are true because $(Q;\cdot)$ is idempotent.

Replacing every functional variable with $(\cdot)$ in each of (3), (4), (6), we obtain:

$$x \cdot xy = xy \cdot y, \quad xy^2 = x \cdot xy, \quad xx^2 = yy^2.$$

Each of these equalities implies $|Q| = 1$. A contradiction with assumption. Therefore according to Corollary 1, a quadruple (h,h,h,h) is a solution of none of the equations (3), (4), (6). Thus, we have proved parastrophically-primary pairwise non-equivalence of equations (2), (5), (3), (4), (6) except the pair (2) and (5).

2. It is easy to see, (f, p, f, f) is a solution of (2). Suppose (2) and (5) are parastrophically primarily equivalent. Corollary 1 implies that for some $\sigma \in S_3$ at least one of the quadruples

$$({}^{\sigma}p, f, f, f); \qquad (f, {}^{\sigma}p, f, f); \qquad (f, f, {}^{\sigma}p, f); \qquad (f, f, f, {}^{\sigma}p) \tag{26}$$

is a solution of (5). In other words, the identities

$${}^{\sigma}p(x; y) = f(f(x; x); f(y; y)) \text{ (i)}, \qquad f(x; y) = {}^{\sigma}p(f(x; x); f(y; y)) \text{ (ii)},$$

$$f(x; y) = f({}^{\sigma}p(x; x); f(y; y)) \text{ (iii)}, \qquad f(x; y) = f(f(x; x); {}^{\sigma}p(y; y)) \text{ (iv)}$$

are true.

If f is idempotent, then (i), (ii) mean that ${}^{\sigma}p = f$. Therefore, $(Q; p)$ is totally symmetric. A contradiction to assumption. If $(Q; f, e)$ is a loop, then $f(x, x) = e$. Therefore, (iii) and (iv) mean that $f(x; y) = x$ and $(Q; f)$ is not a quasigroup. A contradiction. Thus, (2) and (5) are not parastrophically primarily equivalent.

3. The quadruple (g, h, g, h) is a solution of (6).

Suppose each of the pairs of equations (3), (6) and (4), (6) are parastrophically primarily equivalent. By virtue of Corollary 1, a solution of the equations (3) and (4) is at least one of the following quadruples:

$$(h, h, g, g); \ \ (h, g, h, g); \ \ (h, g, g, h); \ \ (g, h, h, g); \ \ (g, h, g, h); \ \ (g, g, h, h). \tag{27}$$

Putting (27) in (3), we obtain the identities

$$(i_1): \ h(x, h(x; y)) = g(g(x; y); y); \quad (i_2): \ h(x, g(x; y)) = h(g(x; y); y);$$

$$(i_3): \ h(x, g(x; y)) = g(h(x; y); y); \quad (i_4): \ g(x, h(x; y)) = h(g(x; y); y);$$

$$(i_5): \ g(x, h(x; y)) = g(h(x; y); y); \quad (i_6): \ g(x, g(x; y)) = h(h(x; y); y),$$

and inserting (27) into (4), we have

$$(i_7): \ h(x, h(y; y)) = g(x; g(x; y)); \quad (i_8): \ h(x, g(y; y)) = h(x; g(x; y));$$

$$(i_9): \ h(x, g(y; y)) = g(x; h(x; y)); \quad (i_{10}): \ g(x, h(y; y)) = h(x; g(x; y));$$

$$(i_{11}): \ g(x, h(y; y)) = g(x; h(x; y)); \quad (i_{12}): \ g(x, g(y; y)) = h(x; h(x; y)).$$

Using the totally-symmetric property of the functions h and g from each of the identities (i_1), (i_6), (i_7), (i_{12}), we obtain $|Q| = 1$. A contradiction to the assumption.

The identities (i_2), (i_5), (i_8), (i_{11}) imply $|Q| = 1$, for this we replace: $g(x; y)$ with x in (i_2); $h(x; y)$ with x in (i_5); $g(y; y)$ with e in (i_8); and $h(y; y)$ with y in (i_{11}).

Putting $x = y$ in (i_3), (i_4), (i_9), (i_{10}) and using the properties of the Steiner quasigroup and the Steiner loop, we obtain

$$h(x; e) = f(x; x) \quad \text{or} \quad f(x; x) = h(e; x),$$

i.e., $h(x; e) = e$ or $h(e; x) = e$. It means that $|Q| = 1$.

Thus, (3) and (6), (4) and (6) are parastrophically primarily non-equivalent.

4. Equalities ${}^r f_1(x;y) = x+y$, ${}^r f_3(x;y) = x+2y$ follow from the definition of the functions f_1, f_3. Since $\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \neq 0$, then (17) is true. Note that f_2 is an invertible diagonal operation and

$$f_2(x;x) = \delta_{f_2}(x) := \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 & 0 \end{pmatrix}(x).$$

Therefore the quadruple (f_1, f_2, f_3, f_4) is a quasigroup solution of (4). Besides according to the construction, f_2 has no orthogonal mate.

Suppose that (3) and (4) are parastrophically primarily equivalent. According to Corollary 1, the quadruple $({}^{\sigma_{1\tau}}f_{1\tau}, {}^{\sigma_{2\tau}}f_{2\tau}, {}^{\sigma_{3\tau}}f_{3\tau}, {}^{\sigma_{4\tau}}f_{4\tau})$ is a solution of (3). But by virtue of Corollary 3, f_2 has an orthogonal mate. A contradiction.

Thus, (3) and (4) are parastrophically-primarily non-equivalent. □

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