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Quotient Structure and Chain Conditions on Quasi Modules

Sandip Jana, Supriyo Mazumder

Abstract. Quasi module is an axiomatisation of the hyperspace structure based on a module. We initiated this structure in our paper [2]. It is a generalisation of the module structure in the sense that every module can be embedded into a quasi module and every quasi module contains a module. The structure of quasi module is a conglomeration of a commutative semigroup with an external ring multiplication and a compatible partial order. In the entire structure partial order has an intrinsic effect and plays a key role in any development of the theory of quasi modules. In the present paper we have discussed the quotient structure of a quasi module by introducing a congruence suitably. Also we introduce the concept of chain conditions on quasi modules and prove some theorems related to chain conditions.

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1 Introduction

Quasi module is an axiomatisation of the hyperspace structure based on a module. We proposed this structure in our paper [2], while we were studying the family $\mathscr{C}(M)$ of all nonempty compact subsets of a Hausdorff topological module M over some topological unitary ring R. This family, commonly known as *hyperspace*, is closed under usual addition of two sets and the ring multiplication of a set defined by : $A + B := \{a + b : a \in A, b \in B\}$ and $rA := \{ra : a \in A\}$, for any $A, B \in \mathscr{C}(M)$ and $r \in R$. Moreover, in the semigroup $\mathscr{C}(M)$ singletons are the only invertible elements, $\{\theta\}$ acting as the identity (θ being the identity in M). Considering these singletons as the minimal elements of $\mathscr{C}(M)$ with respect to the usual set-inclusion as partial order, we can identify the collection $\{\{m\} : m \in M\}$ of all minimal elements of $\mathscr{C}(M)$ with the module M through the isomorphism $\{m\} \longmapsto m$ $(m \in M)$. Again for any two $r, s \in R$ and $A, B \in \mathscr{C}(M)$ we have $(r + s)A \subseteq rA + sA$ and $rA \subseteq rB$, whenever $A \subseteq B$. We have axiomatised these properties of the hyperspace $\mathscr{C}(M)$ and introduced the concept of *quasi module* whose definition is as follows :

Definition 1.1 (see [2])**.** Let $(X, \leq)$ be a partially ordered set, '+' be a binary operation on X [called *addition*] and '$\cdot$': $R \times X \longrightarrow X$ be another composition [called *ring multiplication*, R being a unitary ring]. If the operations and partial order satisfy the following axioms then $(X,+,\cdot,\leq)$ is called a *quasi module* (in short *qmod*) over R.

A_1 : $(X,+)$ is a commutative semigroup with identity θ.

A_2 : $x \leq y\ (x,y \in X) \Rightarrow x+z \leq y+z,\ r\cdot x \leq r\cdot y,\ \forall\, z\in X, \forall\, r \in R$.

A_3 : (i) $r\cdot(x+y) = r\cdot x + r\cdot y$,

(ii) $r\cdot(s\cdot x) = (rs)\cdot x$,

(iii) $(r+s)\cdot x \leq r\cdot x + s\cdot x$,

(iv) $1\cdot x = x$, '1' being the multiplicative identity of R,

(v) $0\cdot x = \theta$ and $r\cdot\theta = \theta$,

$\forall\, x,y \in X,\ \forall\, r,s\in R$.

A_4 : $x + (-1)\cdot x = \theta$ if and only if $x \in X_0 := \{z \in X : y \nleq z, \forall\, y \in X \setminus \{z\}\}$.

A_5 : For each $x\in X, \exists\ y \in X_0$ such that $y \leq x$.

The elements of the set X_0 are the minimal elements of X with respect to the partial order of X. We call these elements of X_0 as '*one order*' elements of X. In [2] we have shown that this X_0 becomes a module over the same unitary ring R. This fact shows that every quasi module contains a module; also in the same paper we have shown through the example 1.3, given below for clarity, that every module can be embedded into a quasi module. In the present paper we have constructed another example which also explains the fact. We cite below two examples which will be needed later.

Example 1.2 (see [2])**.** Let $\mathbb{Z}$ be the ring of integers and $\mathbb{Z}^+ := \{n \in \mathbb{Z} : n \geq 0\}$. Then under the usual addition, $\mathbb{Z}^+$ is a commutative semigroup with the identity 0. Also it is a partially ordered set with respect to the usual order $(\leq)$ of integers. If we define the ring multiplication '$\cdot$' : $\mathbb{Z}\times\mathbb{Z}^+ \longrightarrow \mathbb{Z}^+$ by $(m,n) \longmapsto |m|n$, then it is a routine work to verify that $(\mathbb{Z}^+,+,\cdot,\leq)$ is a quasi module over $\mathbb{Z}$. Here the set of all one order elements is given by $[\mathbb{Z}^+]_0 = \{0\}$.

Example 1.3 (see [2])**.** Let M be a module over a unitary ring R. Let $\widetilde{M} := M \bigcup \{\omega\}$ $(\omega \notin M)$. Define '$+$', '$\cdot$' and the partial order '$\leq_p$' as follows :
(i) The operation '$+$' between any two elements of M is same as in the module M and $x+\omega := \omega$ and $\omega + x := \omega$, $\forall\, x \in \widetilde{M}$.
(ii) The operation '$\cdot$' when applied on $R\times M$ is same as in the module M and $r\cdot\omega := \omega$, if $r(\neq 0)\in R$ and $0\cdot\omega := \theta$, θ being the identity element in M.
(iii) $x \leq_p \omega$, $\forall\, x \in M$ and $x \leq_p x$, $\forall\, x\in\widetilde{M}$.

Then $\left(\widetilde{M},+,\cdot,\leq_p\ \right)$ is a quasi module over R. Here the set of all one order elements is M. In other words, M can be embedded into $\widetilde{M}$, $x \mapsto x$ being the embedding.

In this example if we consider $M = \mathbb{C}$, the vector space of all complex numbers as a module over itself then the extended complex plane $\mathbb{C}_\infty := \mathbb{C}\cup\{\infty\}$ becomes a quasi module over $\mathbb{C}$, provided we define $0\cdot\infty = 0$ and $z < \infty$, $\forall\, z\in\mathbb{C}$; here the set of all one order elements is $\mathbb{C}$. □

In the present paper we have introduced the concept of chain conditions on quasi modules and discuss some of their properties. Before doing this in the last section, we have developed the theory of congruence and quotient structure on quasi module.

2 Prerequisites

We start this section with an example explaining the fact that every module can be embedded into a quasi module which is different from the previous example 1.3.

Example 2.1. Let M be a module over the unitary ring $\mathbb{Z}$ of integers. Let $X := \mathbb{Z}^+ \times M$. We define addition on X component-wise and ring multiplication on X by $(n,(m,a)) \longmapsto (|n|m, na)$, $\forall\, n \in \mathbb{Z}$ and $\forall\,(m,a) \in X$. We define a partial order '$\preceq$' on X as follows : $(n,a) \preceq (m,b)$ if and only if $n \leq m$ and $a = b$. We now show that X with aforesaid operations and partial order is a quasi module over $\mathbb{Z}$.
$\mathbf{A_1}$: Clearly, X is a commutative semigroup with identity $(0,\theta)$, θ being the additive identity of M.
$\mathbf{A_2}$: Let $(n,a) \preceq (m,b)$ $\big((n,a),(m,b) \in X\big) \Rightarrow n \leq m$ and $a = b$. Then for any $(p,c) \in X$ we have $n+p \leq m+p$ and $a+c = b+c \Rightarrow (n+p, a+c) \preceq (m+p, b+c)$ $\Rightarrow (n,a)+(p,c) \preceq (m,b)+(p,c)$.

Again for any $q \in \mathbb{Z}$ we have $|q|n \leq |q|m$ and $qa = qb \Rightarrow (|q|n, qa) \preceq (|q|m, qb)$ $\Rightarrow q \cdot (n,a) \preceq q \cdot (m,b)$.
$\mathbf{A_3}$: Let $(p,a),(q,b) \in X$ and $n,m \in \mathbb{Z}$. Then
(i) $n\big((p,a)+(q,b)\big) = n(p+q, a+b) = \big(|n|(p+q), n(a+b)\big) = (|n|p+|n|q, na+nb) = (|n|p, na) + (|n|q, nb) = n(p,a) + n(q,b)$.
(ii) $n\big(m(p,a)\big) = n(|m|p, ma) = (|nm|p, nma) = nm(p,a)$.
(iii) $(n+m)(p,a) = \big(|n+m|p, (n+m)a\big) \preceq (|n|p, na)+(|m|p, ma) = n(p,a)+m(p,a)$.
(iv) $1(p,a) = (p,a)$.
(v) $0(p,a) = (0,\theta)$ and $n(0,\theta) = (0,\theta)$, for any $n \in \mathbb{Z}$ and any $(p,a) \in X$.
$\mathbf{A_4}$: Let X_0 be the set of all minimal elements of X. Then $X_0 = \big\{(0,r) : r \in M\big\}$. Now, $(n,r) - (n,r) = (0,\theta) \Leftrightarrow (2n,\theta) = (0,\theta) \Leftrightarrow n = 0 \Leftrightarrow (n,r) \in X_0$.
$\mathbf{A_5}$: For any $(n,a) \in X$, $\exists\,(0,a) \in X_0$ such that $(0,a) \preceq (n,a)$.

Thus X is a quasi module over $\mathbb{Z}$ with respect to the aforesaid operations and partial order. Here the set of all one order elements X_0 can be identified with M through the map $(0,x) \mapsto x$ $(x \in M)$.

Looking at the operations and partial order on $\mathbb{Z}^+ \times M$ and also the arguments used to prove this as a quasi module, we can construct another 'bigger' quasi module viz., the Cartesian product $[0,\infty) \times M := \big\{(r,x) : r \in [0,\infty), x \in M\big\}$ over the ring of integers $\mathbb{Z}$, which also clarifies the embedding of the module M into a quasi module. One more fact is clear from this discussion that every module can be embedded into a variety of quasi modules. □

Example 2.2. If M is a module over the ring $\mathbb{C}$ of complex numbers then $[0,\infty) \times M$ becomes a quasi module over $\mathbb{C}$, the operations and partial order being same as in the above example 2.1. If M is a module over the ring of real numbers $\mathbb{R}$, then also $[0,\infty) \times M$ becomes a quasi module over $\mathbb{R}$.

Definition 2.3 (see [2])**.** A subset Y of a qmod X is said to be a *sub quasi module* (*subqmod* in short) if Y itself is a quasi module with all the compositions of X being restricted to Y.

Note 2.4 (see [2])**.** A subset Y of a qmod X over a unitary ring R is a sub quasi module iff Y satisfies the following :
(i) $rx + sy \in Y,\ \forall\, r, s \in R$ and $\forall\, x, y \in Y$.
(ii) $Y_0 \subseteq X_0 \bigcap Y$, where $Y_0 := \{z \in Y : y \nleq z, \forall\, y \in Y \smallsetminus \{z\}\}$.
(iii) $\forall\, y \in Y$, $\exists\, y_0 \in Y_0$ such that $y_0 \leq y$.

If Y is a subqmod of X then actually $Y_0 = X_0 \cap Y$, since for any $Y \subseteq X$ we have $X_0 \cap Y \subseteq Y_0$.

Definition 2.5 (see [2])**.** A mapping $f : X \longrightarrow Y$ (X, Y being two quasi modules over a unitary ring R) is called an *order-morphism* if
(i) $f(x + y) = f(x) + f(y)$, $\forall\, x, y \in X$
(ii) $f(rx) = rf(x)$, $\forall\, r \in R$, $\forall\, x \in X$
(iii) $x \leq y$ $(x, y \in X) \Rightarrow f(x) \leq f(y)$
(iv) $p \leq q$ $\big(p, q \in f(X)\big) \Rightarrow f^{-1}(p) \subseteq\downarrow f^{-1}(q)$ and $f^{-1}(q) \subseteq\uparrow f^{-1}(p)$, where $\uparrow A := \{x \in X : x \geq a$ for some $a \in A\}$ and $\downarrow A := \{x \in X : x \leq a$ for some $a \in A\}$ for any $A \subseteq X$.

A surjective (injective, bijective) order-morphism is called an *order-epimorphism* (*order-monomorphism*, *order-isomorphism* respectively).

If $f : X \longrightarrow Y$ be an order-morphism and θ, θ' be the identity elements of X, Y respectively then $f(\theta) = f(0.\theta) = 0.f(\theta) = \theta'$. It is also clear that $f(X_0) \subseteq Y_0$ and hence $X_0 \subseteq f^{-1}(Y_0)$. If $X_0 = f^{-1}(Y_0)$ we call such f as "*normal*" order-morphism. Clearly each order-monomorphism is normal.

Proposition 2.6 (see [2])**.** *If $f : X \longrightarrow Y$ (X, Y being two quasi modules over a unitary ring R) be an order-morphism then $f(M) := \{f(m) : m \in M\}$ is a subqmod of Y, for any subqmod M of X.*

Proposition 2.7. *If $f : X \longrightarrow Y$ (X, Y be two qmods over R) be a normal order-morphism then $f^{-1}(N) := \{x \in X : f(x) \in N\}$ is a subqmod of X, for any subqmod N of $f(X)$* [By above Proposition 2.6, $f(X)$ is a subqmod of Y].

Proof. Let $x_1, x_2 \in f^{-1}(N)$ and $r, s \in R$. Then $f(x_1), f(x_2) \in N \Rightarrow rf(x_1) + sf(x_2) \in N \Rightarrow f(rx_1 + sx_2) \in N \Rightarrow rx_1 + sx_2 \in f^{-1}(N)$.

To show that $[f^{-1}(N)]_0 \subseteq X_0 \bigcap f^{-1}(N)$, let $t \in [f^{-1}(N)]_0 \subseteq f^{-1}(N)$. We claim that $f(t) \in N_0$. In fact, for any $t' \in N$ with $t' \leq f(t)$ we have $f^{-1}f(t) \subseteq\uparrow f^{-1}(t')$ $\big[$note that $t', f(t) \in N \subseteq f(X)\big] \Rightarrow t \geq p$, for some $p \in f^{-1}(t') \subseteq f^{-1}(N)$. So $t \in [f^{-1}(N)]_0 \Rightarrow t = p \Rightarrow f(t) = f(p) = t' \Rightarrow f(t) \in N_0 \Rightarrow t \in f^{-1}(N_0)$. Thus $[f^{-1}(N)]_0 \subseteq f^{-1}(N_0)$. Again $f^{-1}(N_0) = f^{-1}(N \cap [f(X)]_0) = f^{-1}(N \cap f(X) \cap Y_0) = f^{-1}(N \cap Y_0) = f^{-1}(N) \cap f^{-1}(Y_0) = f^{-1}(N) \cap X_0$ $\big[\because X_0 = f^{-1}(Y_0)$, for f is normal$\big]$. Also we know that $f^{-1}(N) \cap X_0 \subseteq [f^{-1}(N)]_0$. Thus it follows that $[f^{-1}(N)]_0 = f^{-1}(N_0) = f^{-1}(N) \cap X_0$.

Now let $x \in f^{-1}(N)$. Then $f(x) \in N$. So $\exists\, p \in N_0$ such that $p \leq f(x)$. Now $p \in N_0 \subseteq f(X) \Rightarrow f^{-1}f(x) \subseteq\uparrow f^{-1}(p) \Rightarrow \exists\, q \in f^{-1}(p)$ such that $x \geq q$. Since $[f^{-1}(N)]_0 = f^{-1}(N_0)$ we have, $q \in f^{-1}(p) \subseteq [f^{-1}(N)]_0$. Thus it follows that $f^{-1}(N)$ is a subqmod of X. □

The normality condition of the order-morphism f in the above proposition is not necessary, as the following example shows.

Example 2.8. Let $\{X_\alpha : \alpha \in \Lambda\}$ be an arbitrary family of qmods over a common unitary ring R and $X := \prod\limits_{\alpha\in\Lambda} X_\alpha$ be the Cartesian product which becomes a qmod over R with respect to the following operations and partial order (see Section 4 of [2]):
For $x = (x_\alpha), y = (y_\alpha) \in X$ and $r \in R$ we define (i) $x + y := (x_\alpha + y_\alpha)$, (ii) $rx := (rx_\alpha)$, (iii) $x \leq y$ if $x_\alpha \leq y_\alpha$, $\forall\, \alpha \in \Lambda$. Also the β-th projection map $p_\beta : X \longrightarrow X_\beta$ is an order-epimorphism, for each $\beta \in \Lambda$. However, each projection map can never be normal. In fact, $p_\beta^{-1}([X_\beta]_0) = \prod\limits_{\alpha\in\Lambda} Y_\alpha$, where $Y_\alpha = X_\alpha$, for $\alpha \neq \beta$ and $Y_\beta = [X_\beta]_0$. But $X_0 = \prod\limits_{\alpha\in\Lambda} [X_\alpha]_0$. Thus $p_\beta^{-1}([X_\beta]_0) \neq X_0$.

We now show that for any subqmod Z of $X_\beta\ (= p_\beta(X))$, $p_\beta^{-1}(Z)$ must be a subqmod of X. In fact, it is a routine verification that $p_\beta^{-1}(Z)$ is closed under addition and ring multiplication. Again,

$$\begin{aligned}
\left[p_\beta^{-1}(Z)\right]_0 &= \Big[\prod_{\alpha\in\Lambda} Y_\alpha\Big]_0, \text{ where } Y_\alpha = \begin{cases} X_\alpha, \text{ if } \alpha \neq \beta \\ Z, \text{ if } \alpha = \beta \end{cases} \\
&= \prod_{\alpha\in\Lambda} [Y_\alpha]_0 \\
&= X_0 \bigcap p_\beta^{-1}(Z), \text{ since } Z \cap [X_\beta]_0 = Z_0.
\end{aligned}$$

Also $x = (x_\alpha) \in p_\beta^{-1}(Z) \Rightarrow x_\beta \in Z$ and $x_\alpha \in X_\alpha$, for any $\alpha \neq \beta \Rightarrow \exists\, t_\alpha \in [X_\alpha]_0$, for $\alpha \neq \beta$ and $t_\beta \in Z_0$ such that $x_\alpha \geq t_\alpha$, $\forall\, \alpha \in \Lambda \Rightarrow x \geq t$, where $t = (t_\alpha) \in \left[p_\beta^{-1}(Z)\right]_0$. Thus it follows that $p_\beta^{-1}(Z)$ is a subqmod of X, for each $\beta \in \Lambda$, although p_β is not normal.

Definition 2.9 (see [3])**.** Let $\{A_i\}_{i\in\mathbb{Z}}$ be a sequence of qmods over a common unitary ring and $\{f_i \in (A_{i+1})^{A_i} : i \in \mathbb{Z}\}$ be a sequence of order-morphisms. Then by a *sequence of qmods* and *order-morphisms* we shall mean the diagram

$$\cdots \longrightarrow A_i \xrightarrow{f_i} A_{i+1} \xrightarrow{f_{i+1}} A_{i+2} \longrightarrow \cdots$$

This sequence is said to be *exact* if

$$(f_i \times f_i)(A_i \times A_i) \bigcup \Delta(A_{i+1}) = \ker f_{i+1},\ \forall\, i, \text{ where } \Delta(A_{i+1}) := \big\{(b, b) : b \in A_{i+1}\big\}.$$

Result 2.10 (see [3])**.** (i) *$f : A \longrightarrow B$ is an order-monomorphism if and only if the sequence $0 \xrightarrow{i} A \xrightarrow{f} B$ is exact.*

(ii) $f : A \longrightarrow B$ *is an order-epimorphism if and only if* $A \xrightarrow{f} B \xrightarrow{\underline{0}} 0$ *is an exact sequence.* [Here 0 denotes the trivial qmod containing the additive identity only, i is the inclusion map and $\underline{0}$ is the zero map.]

3 Congruence and Quotient on quasi module

In this section we shall introduce the concept of congruence and quotient on quasi module.

Definition 3.1. Let E be an equivalence relation on a qmod X over a unitary ring R. Then E is said to be a *congruence* on X if it satisfies the following :

$$\begin{aligned}
&(i)\ (a,b) \in E \Longrightarrow (x+a, x+b) \in E,\ \forall\, x \in X\\
&(ii)\ (a,b) \in E \Longrightarrow (ra, rb) \in E,\ \forall\, r \in R\\
&(iii)\ x \le y \le z\ \&\ (x,z) \in E \Longrightarrow (x,y) \in E\ \text{[and hence } (y,z) \in E]\\
&(iv)\ a \le x \le b\ \&\ (x,y) \in E \Longrightarrow \exists\, c,d \in X \text{ such that } c \le y \le d \text{ and}\\
&\qquad\qquad\qquad\qquad (a,c) \in E, (b,d) \in E.
\end{aligned}$$

Any congruence E on a qmod X (over a unitary ring R) produces the quotient set $X/E := \{[x] : x \in X\}$, where $[x]$ denotes the equivalence class containing x (with respect to E), i.e

$$[x] := \{y \in X : (x,y) \in E\}.$$

We show that X/E becomes a qmod over R with respect to the following operations and partial order.

$$\begin{aligned}
&(i)\ [x] + [y] := [x+y],\ \forall\, [x],[y] \in X/E\\
&(ii)\ r[x] := [rx],\ \forall\, [x] \in X/E, \forall\, r \in R\\
&(iii)\ [x] \preccurlyeq [y] \Longleftrightarrow \text{ for any } x' \in [x],\ \exists\, y' \in [y] \text{ such that } x' \le y' \text{ and}\\
&\qquad\qquad\qquad \text{for any } y'' \in [y],\ \exists\, x'' \in [x] \text{ such that } x'' \le y''.
\end{aligned}$$

Thus $[x] \preccurlyeq [y]$ in $X/E \Longleftrightarrow [x] \subseteq\downarrow [y]$ and $[y] \subseteq\uparrow [x]$, where for the set-inclusion relation, $[x]$ is considered as a subset of X.

Theorem 3.2. *X/E is a quasi module over R with respect to the aforesaid operations and partial order, for any quasi module X over R and any congruence E on X.*

Proof. First we have to check whether the aforesaid operations are well-defined and the order '$\preccurlyeq$' is truly a partial order. For this let $(x,x'),(y,y') \in E \Rightarrow (x+y, x'+y),(x'+y, x'+y') \in E \Rightarrow (x+y, x'+y') \in E \Rightarrow [x+y] = [x'+y']$. Also $(x,x') \in E \Rightarrow (\alpha x, \alpha x') \in E \Rightarrow [\alpha x] = [\alpha x']$. Thus the operations are well-defined.

Clearly the order '$\preccurlyeq$' on X/E is reflexive and transitive. To check that it is anti-symmetric, let $[x] \preccurlyeq [y]$ and $[y] \preccurlyeq [x] \Rightarrow$ for any $x' \in [x], \exists\, y' \in [y]$ such that $x' \le y'$

and for this $y' \in [y], \exists\, x'' \in [x]$ such that $y' \leq x'' \Rightarrow x' \leq y' \leq x'' \Rightarrow (x', y') \in E$ [by axiom (iii) of the definition of congruence 3.1] $\Rightarrow [x] = [x'] = [y'] = [y]$.
$\mathbf{A_1}$: Obviously $(X/E, +)$ is a commutative semigroup with identity $[\theta]$, θ being the identity in X.
$\mathbf{A_2}$: Let $[x] \preccurlyeq [y]$. Then for x, $\exists\, y' \in [y]$ such that $x \leq y'$ (by definition of '$\leq$' in X/E) $\Rightarrow x + z \leq y' + z$, for any $z \in X$. So by axiom (iv) of the definition of congruence 3.1, for any $z' \in [x+z]$, $\exists\, y'' \in [y'+z]$ such that $z' \leq y''$. Now note that $(y', y) \in E \Rightarrow (y'+z, y+z) \in E$ and hence $[y'+z] = [y+z]$. Conversely, $[x] \preccurlyeq [y] \Rightarrow$ for $y, \exists\, x' \in [x]$ such that $x' \leq y$ (by definition of '$\leq$' in X/E) $\Rightarrow x' + z \leq y + z$, for any $z \in X$. So by axiom (iv) of the definition of congruence 3.1, for any $z'' \in [y+z]$, $\exists\, x'' \in [x'+z]$ such that $x'' \leq z''$. Now note that $(x', x) \in E \Rightarrow (x'+z, x+z) \in E$ and hence $[x'+z] = [x+z]$. Thus we have $[x+z] \preccurlyeq [y+z] \Rightarrow [x] + [z] \preccurlyeq [y] + [z]$, $\forall\, [z] \in X/E$.

Now let $[x] \preccurlyeq [y]$ and $r \in R$. We show that $[rx] \preccurlyeq [ry]$. For $x, \exists\, y' \in [y]$ such that $x \leq y' \Rightarrow rx \leq ry'$. Then for any $x' \in [rx], \exists\, y'' \in [ry']$ such that $x' \leq y''$ [by axiom (iv) of the definition of congruence 3.1]. Now $(y, y') \in E \Rightarrow (ry, ry') \in E \Rightarrow [ry] = [ry']$.
Conversely, $[x] \preccurlyeq [y] \Rightarrow$ for $y, \exists\, x' \in [x]$ such that $x' \leq y \Rightarrow rx' \leq ry$. So for any $y''' \in [ry], \exists\, x'' \in [rx']$ such that $x'' \leq y'''$ [by axiom (iv) of the definition of congruence 3.1]. Again $(x, x') \in E \Rightarrow (rx, rx') \in E \Rightarrow [rx] = [rx']$. Thus we have $r[x] = [rx] \preccurlyeq [ry] = r[y]$, for any $r \in R$.
$\mathbf{A_3}$: Let $[x], [y] \in X/E$ and $r, s \in R$. Then

(i) $r([x] + [y]) = [r(x+y)] = [rx + ry] = r[x] + r[y]$.

(ii) $r(s[x]) = [(rs)x] = (rs)[x]$.

(iii) $(r+s)[x] = [(r+s)x]$. We have to show that $[(r+s)x] \preccurlyeq [rx] + [sx]$. For this we first show that $[u] \preccurlyeq [v]$, whenever $u \leq v$ $(u, v \in X)$. For each $u' \in [u]$, we have by axiom (iv) of the definition of congruence 3.1, some $v' \in [v]$ $(\because u \leq v)$ such that $u' \leq v'$. Also by same axiom, for each $v'' \in [v]$, $\exists\, u'' \in [u]$ such that $u'' \leq v''$. This justifies that $[u] \preccurlyeq [v]$, whenever $u \leq v$. Thus $(r+s)x \leq rx + sx \Rightarrow [(r+s)x] \preccurlyeq [rx + sx] = r[x] + s[x]$.

Axioms **(iv)** and **(v)** are immediate.
$\mathbf{A_4}$: $[x] - [x] = [\theta] \Leftrightarrow [x - x] = [\theta] \Leftrightarrow (x - x, \theta) \in E$. Let $Y := \{[x] : (x - x, \theta) \in E\}$. We claim that $Y = [X/E]_0$. First of all, $[p] \in Y, \forall\, p \in X_0$. Let $[x] \in Y$ and $[y] \preccurlyeq [x]$ $\Rightarrow$ for x, $\exists\, y' \in [y]$ such that $y' \leq x \Rightarrow y' - x \leq x - x \Rightarrow [y' - x] \preccurlyeq [x - x] = [\theta] \Rightarrow$ for θ, $\exists\, z \in [y' - x]$ such that $z \leq \theta \Rightarrow z = \theta \Rightarrow [y' - x] = [z] = [\theta] \Rightarrow [y' - x] + [x] = [\theta] + [x]$ $\Rightarrow [y'] + [-x + x] = [x] \Rightarrow [y'] + [\theta] = [x] \Rightarrow [y] = [x]$ $(\because (y, y') \in E) \Rightarrow [x] \in [X/E]_0$. Conversely, if $[x] \in [X/E]_0$ then for any $p \in X_0$ with $p \leq x$ we must have $[p] = [x]$ $\Rightarrow (p, x) \in E \Rightarrow (p - p, x - x) \in E \Rightarrow (\theta, x - x) \in E \Rightarrow [x] \in Y$. Thus we have $[X/E]_0 = \{[x] \in X/E : (x - x, \theta) \in E\}$ and hence $[x] - [x] = [\theta] \Leftrightarrow [x] \in [X/E]_0$.
$\mathbf{A_5}$: As X is a qmod so for each $x \in X, \exists\, p \in X_0$ such that $p \leq x \Rightarrow [p] \preccurlyeq [x]$, where $[p] \in [X/E]_0$.

Therefore $(X/E, +, ., \preccurlyeq)$ is a qmod over R. □

Definition 3.3. Let E be a congruence on a qmod X. Then a subqmod Y of X is called E-*compatible* if $(x, y) \in E$ and $x \in Y$ implies $y \in Y$.

Thus Y is E-compatible iff $Y = \bigcup_{y \in Y} [y]$, where $[y] := \{x \in X : (x, y) \in E\}$.

Theorem 3.4. *Let E be a congruence on a qmod X (over a unitary ring R) and Y be an E-compatible subqmod of X. Then $Y/E := \{[y] : y \in Y\}$ is a subqmod of X/E.*

Proof. Let $[x], [y] \in Y/E$ and $r, s \in R$. Then $x, y \in Y \Rightarrow rx + sy \in Y \Rightarrow r[x] + s[y] = [rx + sy] \in Y/E$, since Y is E-compatible.

To show that $[Y/E]_0 \subseteq [X/E]_0 \bigcap (Y/E)$ it is enough to note that $[Y/E]_0 = \{[y] \in Y/E : (y - y, \theta) \in E\}$ whose justification follows the same line of proof of $[X/E]_0 = \{[x] \in X/E : (x - x, \theta) \in E\}$ as done above. It is also easy to note that $[p] \in [Y/E]_0$, $\forall\, p \in Y_0$.

For any $[y] \in Y/E$, $\exists\, p \in Y_0$ such that $p \leq y \Rightarrow [p] \preccurlyeq [y]$, where $[p] \in [Y/E]_0$. Thus in view of 2.4 we can say that Y/E is a subqmod of X/E. ☐

We now consider the canonical map $\pi : X \longrightarrow X/E$ defined by $\pi(x) := [x]$, $\forall\, x \in X$. Then (i) $\pi(x + y) = [x + y] = [x] + [y] = \pi(x) + \pi(y)$, $\forall\, x, y \in X$
(ii) $\pi(rx) = [rx] = r[x] = r\pi(x)$, for any $r \in R$, $\forall\, x \in X$
(iii) $x \leq y$ $(x, y \in X) \Longrightarrow \pi(x) = [x] \preccurlyeq [y] = \pi(y)$ [justified during the proof of X/E to be qmod]
(iv) $\pi(x) \preccurlyeq \pi(y)$ in $X/E \Longrightarrow [x] \preccurlyeq [y]$ in $X/E \Longrightarrow [x] \subseteq\downarrow [y]$ and $[y] \subseteq\uparrow [x]$ which implies $\pi^{-1}\pi(x) \subseteq\downarrow \pi^{-1}\pi(y)$ and $\pi^{-1}\pi(y) \subseteq\uparrow \pi^{-1}\pi(x)$, since $\pi^{-1}\pi(x) = [x]$, $\forall\, x \in X$.

Thus it follows that $\pi : X \longrightarrow X/E$ becomes an order-epimorphism, since clearly it is surjective. Also π will be normal iff $\pi^{-1}([X/E]_0) = X_0$ iff $\bigcup\{[x] : (x - x, \theta) \in E\} = X_0$. Thus we have the following equivalent statements.

Result 3.5. *The canonical map $\pi : X \longrightarrow X/E$ is normal iff any one of the following holds.*
(i) $(x - x, \theta) \in E \Longleftrightarrow x - x = \theta$
(ii) $(x - x, \theta) \in E \Longleftrightarrow x \in X_0$
(iii) $(x, y) \in E$ *with* $y \in X_0 \Longrightarrow x \in X_0$
(iv) $\bigcup_{p \in X_0} [p] = X_0$.

It is thus reasonable to call a congruence E on a qmod X to be '*normal*' if any one of the above four assertions is satisfied by E.

4 Chain Conditions

In this section we introduce the concept of chain conditions on quasi modules and define Noetherian and Artinian quasi modules. Also we discuss various ways of construction of Noetherian and Artinian quasi modules.

Definition 4.1. Let X be a qmod over a unitary ring R. Then X is said to be *Noetherian* if for every ascending chain $Y_1 \subseteq Y_2 \subseteq Y_3 \subseteq \cdots$ of subqmods of X, $\exists\, n \in \mathbb{N}$ such that $Y_k = Y_n$, $\forall\, k \geq n$.

X is said to satisfy *maximum chain condition* if every nonempty collection of subqmods of X has a maximal element with respect to the usual set-inclusion as partial order in the collection of all subqmods of X.

Example 4.2. (i) The qmod $(\mathbb{Z}^+, +, ., \leq)$ over the unitary ring $\mathbb{Z}$, explained in Example 1.2, is Noetherian, since any subqmod of this qmod is of the form $n\mathbb{Z}^+$, for $n \in \mathbb{N}$ and hence for any increasing sequence of subqmods, there are only finitely many subqmods between the first term and $\mathbb{Z}^+$.

(ii) The qmod $\mathbb{Z}^+ \times M$ for any module M over $\mathbb{Z}$, explained in Example 2.1, is Noetherian iff M is Noetherian, since any subqmod of $\mathbb{Z}^+ \times M$ is of the form $n\mathbb{Z}^+ \times N$, for some submodule N of M and any $n \in \mathbb{N}$.

Theorem 4.3. *Let X be a qmod over a unitary ring R. Then the following conditions are equivalent :*
(i) *X is Noetherian.*
(ii) *X satisfies maximum chain condition.*

Proof. (i) $\Rightarrow$ (ii) : Let X be Noetherian and $\mathscr{F}$ be a nonempty collection of subqmods of X. To prove that $\mathscr{F}$ contains a maximal element let us first consider an arbitrary chain $Y_1 \subseteq Y_2 \subseteq Y_3 \subseteq \cdots$ in $\mathscr{F}$. X being Noetherian, there is an $n \in \mathbb{N}$ such that $Y_k = Y_n$, $\forall\, k \geq n$. This shows that Y_n is an upper bound of the above chain. Then by Zorn's lemma, $\mathscr{F}$ has a maximal element. Consequently, X satisfies maximum chain condition.
(ii) $\Rightarrow$ (i) : Let $Y_1 \subseteq Y_2 \subseteq Y_3 \subseteq \cdots$ be an ascending chain of subqmods of X. Then by condition (ii), the collection $\mathscr{T} := \{Y_i : i \in \mathbb{N}\}$ has a maximal element Y_0 (say). Since $Y_0 \in \mathscr{T}$ and Y_0 is maximal, it follows that $Y_i = Y_0$ for all $i \geq p$ for some $p \in \mathbb{N}$. This shows that X is Noetherian. □

Definition 4.4. Let X be a qmod over a unitary ring R. Then X is said to be *Artinian* if for every descending chain $Y_1 \supseteq Y_2 \supseteq Y_3 \supseteq \cdots$ of subqmods of X, $\exists\, n \in \mathbb{N}$ such that $Y_k = Y_n$, $\forall\, k \geq n$.

X is said to satisfy *minimum chain condition* if every nonempty collection of subqmods of X has a minimal element with respect to the usual set-inclusion as partial order in the collection of all subqmods of X.

Example 4.5. (i) The qmod $(\mathbb{C}_\infty, +, \cdot, \leq_p)$, explained in example 1.3, over the unitary ring $\mathbb{C}$ is Artinian, since it has no proper subqmod. It thus follows that $(\mathbb{C}_\infty, +, \cdot, \leq_p)$ is Noetherian also.

(ii) The qmod $(\mathbb{Z}^+, +, ., \leq)$ in example 1.2 is not Artinian, since $\mathbb{Z}^+ \supset 2\mathbb{Z}^+ \supset 4\mathbb{Z}^+ \supset 8\mathbb{Z}^+ \supset \cdots$ is a descending chain of subqmods of $\mathbb{Z}^+$ which can never be stationary after a finite stage.

(iii) The qmod $[0, \infty) \times M$, explained in example 2.2, is Artinian iff M is Artinian, since any subqmod of $[0, \infty) \times M$ is of the form $[0, \infty) \times N$, for some submodule N of M.

Theorem 4.6. *Let X be a qmod over a unitary ring R. Then the following conditions are equivalent :*
(i) *X is Artinian.*
(ii) *X satisfies minimum chain condition.*

Proof. (i) $\Rightarrow$ (ii) : Let $\mathscr{F}$ be a nonempty collection of subqmods of X and $Y_1 \in \mathscr{F}$. If Y_1 is not minimal in $\mathscr{F}$ there is $Y_2 \in \mathscr{F}$ such that $Y_1 \supsetneq Y_2$. If Y_2 is not minimal in $\mathscr{F}$ then there is $Y_3 \in \mathscr{F}$ such that $Y_2 \supsetneq Y_3$. Continuing in this way we get a descending chain $Y_1 \supsetneq Y_2 \supsetneq Y_3 \supsetneq \cdots$ of subqmods of X. So X being Artinian, $\exists\, n \in \mathbb{N}$ such that $Y_k = Y_n$, $\forall\, k \geq n$. This shows that Y_n is a minimal element of $\mathscr{F}$.
(ii) $\Rightarrow$ (i) : Let $Y_1 \supseteq Y_2 \supseteq Y_3 \supseteq \cdots$ be a descending chain of subqmods of X. Then by condition (ii), the collection $\mathscr{T} := \{Y_i : i \in \mathbb{N}\}$ has a minimal element Y_0 (say). Since $Y_0 \in \mathscr{T}$ and Y_0 is minimal, it follows that $Y_i = Y_0$ for all $i \geq p$ for some $p \in \mathbb{N}$. This shows that X is Artinian. □

We now discuss how exact sequences influence on Noetherian and Artinian qmods.

Theorem 4.7. *Let X, X', X'' be three quasi modules over a unitary ring R and $0 \to X' \xrightarrow{\alpha} X \xrightarrow{\beta} X'' \to 0$ be an exact sequence. Then X is Noetherian if both X', X'' are so. Also if X is Noetherian then X' must be so.*

Proof. Let X' and X'' be Noetherian. We are to show X is Noetherian. For this let $X_1 \subseteq X_2 \subseteq X_3 \subseteq \cdots$ be an ascending chain of subqmods of X. Then $\beta(X_1) \subseteq \beta(X_2) \subseteq \beta(X_3) \subseteq \cdots$ forms an ascending chain of subqmods of X'' [by Proposition 2.6]. So X'' being Noetherian, $\exists\, p \in \mathbb{N}$ such that $\beta(X_i) = \beta(X_p), \forall\, i \geq p$. If $X_i = X_p$, $\forall\, i \geq p$ we are done. If not, we need some more arguments. In fact, the given sequence being exact we have $\ker \beta = \Delta(X) \cup (\alpha \times \alpha)(X' \times X')$. This ensures that for distinct $x, y \in X$, $\beta(x) = \beta(y)$ if and only if both $x, y \in \alpha(X')$. Thus for any $i \geq p$ for which $X_i \neq X_p$, we must have $X_i \subseteq \alpha(X')$. So $X_1 \subseteq X_2 \subseteq X_3 \subseteq \cdots$ being an increasing chain we can say that $X_i \subseteq \alpha(X')$, $\forall\, i \in \mathbb{N}$. Therefore $\alpha^{-1}(X_1) \subseteq \alpha^{-1}(X_2) \subseteq \alpha^{-1}(X_3) \subseteq \cdots$ forms an ascending chain of subqmods in X' [by Proposition 2.7, since α being injective is normal]. So X' being Noetherian we can find some $q\,(\geq p)$ such that $\alpha^{-1}(X_i) = \alpha^{-1}(X_q)$, $\forall\, i \geq q \Rightarrow X_i = \alpha\alpha^{-1}(X_i) = \alpha\alpha^{-1}(X_q) = X_q$, $\forall\, i \geq q$ [$\because$ each $X_i \subseteq \alpha(X')$]. Thus X becomes Noetherian.

To show that X' is Noetherian whenever X is Noetherian, let $X'_1 \subseteq X'_2 \subseteq X'_3 \subseteq \cdots$ be an ascending chain of subqmods of X'. Then by Proposition 2.6, $\alpha(X'_1) \subseteq \alpha(X'_2) \subseteq \alpha(X'_3) \subseteq \cdots$ forms an ascending chain of subqmods in X. So X being Noetherian, $\exists\, q \in \mathbb{N}$ such that $\alpha(X'_k) = \alpha(X'_q)$, $\forall\, k \geq q \Rightarrow X'_k = \alpha^{-1}\alpha(X'_k) = \alpha^{-1}\alpha(X'_q) = X'_q$, $\forall\, k \geq q$ $\big[$since α is injective, by result 2.10(i)$\big]$. This justifies that X' is Noetherian. □

In the above exact sequence if X is Noetherian then X'' need not be so. However we have the following theorem.

Theorem 4.8. *If $f : X \to Y$ is a normal order-morphism, X, Y being two qmods over a unitary ring R, then $f(X)$ is Noetherian whenever X is so.*

Proof. Let $X_1'' \subseteq X_2'' \subseteq X_3'' \subseteq \cdots$ be an ascending chain of subqmods of $f(X)$. Then by Proposition 2.7, $f^{-1}(X_1'') \subseteq f^{-1}(X_2'') \subseteq f^{-1}(X_3'') \subseteq \cdots$ forms an ascending chain of subqmods in X, since f is normal. So X being Noetherian, $\exists\, p \in \mathbb{N}$ such that $f^{-1}(X_k'') = f^{-1}(X_p'')$, $\forall\, k \geq p \Rightarrow X_k'' = ff^{-1}(X_k'') = ff^{-1}(X_p'') = X_p''$, $\forall\, k \geq p$ $\big[$since $X_k'' \subseteq f(X)$, $\forall\, k\big]$. This justifies that $f(X)$ is Noetherian. □

In the exact sequence $0 \to X' \xrightarrow{\alpha} X \xrightarrow{\beta} X'' \to 0$, if X is Noetherian then X'' need not be Noetherian, since β cannot be normal. For, the sequence being exact we have $\ker\beta = \Delta(X) \cup (\alpha \times \alpha)(X' \times X')$ which implies that β sends exactly the entire $\alpha(X')$ to 0. This fact is contrary to the normality condition $X_0 = \beta^{-1}([X'']_0)$, since $\alpha(X')$ contains lots of non-one order elements and $[\alpha(X')]_0 = X_0 \cap \alpha(X')$, as $\alpha(X')$ is a subqmod of X.

Theorem 4.9. *Let X, X', X'' be three quasi modules over a unitary ring R and $0 \to X' \xrightarrow{\alpha} X \xrightarrow{\beta} X'' \to 0$ be an exact sequence. Then X is Artinian if both X', X'' are so. Also if X is Artinian then X' must be so.*

Proof. To prove that X is Artinian whenever both X', X'' are so let $X_1 \supseteq X_2 \supseteq X_3 \supseteq \cdots$ be a descending chain of subqmods of X. Then $\beta(X_1) \supseteq \beta(X_2) \supseteq \beta(X_3) \supseteq \cdots$ forms a descending chain of subqmods of X'' [by Proposition 2.6]. So X'' being Artinian, $\exists\, p \in \mathbb{N}$ such that $\beta(X_i) = \beta(X_p), \forall\, i \geq p$. If $X_i = X_p$, $\forall\, i \geq p$ we are nothing to prove. If $X_j \neq X_p$ for some $j \geq p$ then $\beta(X_j) = \beta(X_p)$ implies both $X_j, X_p \subseteq \alpha(X')$ $\big[$since $\ker\beta = \Delta(X) \cup (\alpha \times \alpha)(X' \times X')\big] \Rightarrow X_i \subseteq \alpha(X')$, $\forall\, i \geq p$, since $\{X_i\}_i$ is a descending chain. Therefore by Proposition 2.7 we can say that $\alpha^{-1}(X_p) \supseteq \alpha^{-1}(X_{p+1}) \supseteq \alpha^{-1}(X_{p+2}) \supseteq \cdots$ forms a descending chain of subqmods in X'. So X' being Artinian we have a $q \geq p$ such that $\alpha^{-1}(X_i) = \alpha^{-1}(X_q)$, $\forall\, i \geq q$ $\Rightarrow X_i = \alpha\alpha^{-1}(X_i) = \alpha\alpha^{-1}(X_q) = X_q$, $\forall\, i \geq q$ $\big[\because$ for each $i \geq q$, $X_i \subseteq \alpha(X')\big]$. Thus X becomes Artinian.

If X is Artinian then the fact that X' will be Artinian can be proved in the same line of proof as in the above Theorem 4.7; the only change that should be made is to consider an arbitrary descending chain instead of ascending one. □

We also have an analogue of Theorem 4.8 for Artinian qmod whose proof is similar with only replacing every ascending chain by a descending one.

Theorem 4.10. *If $f : X \to Y$ is a normal order-morphism, X, Y being two qmods over a unitary ring R, then $f(X)$ is Artinian whenever X is so.*

The property that "a qmod is Noetherian or Artinian is a hereditary property" is shown below.

Theorem 4.11. *Let X be a qmod over a unitary ring R and Y be subqmod of X. Then Y is Noetherian (Artinian) whenever X is Noetherian (Artinian).*

Proof. Y being a subqmod of X, the inclusion map $i : Y \to X$ is an order-monomorphism. So $0 \to Y \xrightarrow{i} X$ becomes an exact sequence (by result 2.10). Then by Theorem 4.7 we can say that Y is Noetherian whenever X is so (we have to use Theorem 4.9 to show the Artinian case). □

We now discuss Noetherian and Artinian qmods in the context of quotient structure on qmod.

Theorem 4.12. *If X is a Noetherian qmod over a unitary ring R and E is a normal congruence on X then X/E is also a Noetherian qmod over the same unitary ring.*

Proof. This theorem follows from Theorem 4.8, since E being normal the canonical map $\pi : X \longrightarrow X/E$ is a normal order-epimorphism (by result 3.5). □

The argument used to prove the above theorem shows that its analogue for Artinian qmod also holds.

Theorem 4.13. *If X is an Artinian qmod over a unitary ring R and E is a normal congruence on X then X/E is also an Artinian qmod over the same unitary ring.*

Proof. It follows from Theorem 4.10 using the canonical map $\pi : X \longrightarrow X/E$ which is a normal order-epimorphism, E being a normal congruence. □

We conclude this section with the findings regarding the effect of Cartesian product on Noetherian and Artinian qmods.

Theorem 4.14. *Let $\{X_j : j = 1, \ldots, n\}$ be a finite family of qmods over a common unitary ring R and $X := X_1 \times X_2 \times \cdots \times X_n$ be their Cartesian product. Then X is Noetherian iff each X_j is Noetherian.*

Proof. Let each X_j be Noetherian and $Y_1 \subseteq Y_2 \subseteq Y_3 \subseteq \cdots$ be an ascending chain of subqmods of X. Then for each $j \in \{1, 2, \ldots, n\}$, $p_j(Y_1) \subseteq p_j(Y_2) \subseteq p_j(Y_3) \subseteq \cdots$ becomes an ascending chain of subqmods of $p_j(X) = X_j$, by Proposition 2.6. So X_j being Noetherian, $\exists\, r_j \in \mathbb{N}$ such that $p_j(Y_i) = p_j(Y_{r_j})$, $\forall\, i \geq r_j$, $\forall\, j = 1, \ldots, n$.
Now let $r := \max\{r_j : j = 1, \ldots, n\} \in \mathbb{N}$. Then $p_j(Y_i) = p_j(Y_r)$, $\forall\, i \geq r$, $\forall\, j$. We claim that $Y_i = Y_r$, $\forall\, i \geq r$. Since $\{Y_i : i \in \mathbb{N}\}$ is an ascending chain, it is enough to show that $Y_i \subseteq Y_r$, $\forall\, i \geq r$. For this let $x = (x_j) \in Y_i$, whenever $i \geq r \Rightarrow p_j(x) = x_j \in p_j(Y_i) = p_j(Y_r)$, $\forall\, i \geq r$, $\forall\, j \Rightarrow x = (x_j) \in Y_r \Rightarrow Y_i \subseteq Y_r$, $\forall\, i \geq r$. Thus it follows that X is Noetherian.

Conversely, let X be Noetherian and $Z_1 \subseteq Z_2 \subseteq Z_3 \subseteq \cdots$ be an ascending chain of subqmods in X_j, where $j \in \{1, 2, \ldots, n\}$. Then $p_j^{-1}(Z_1) \subseteq p_j^{-1}(Z_2) \subseteq p_j^{-1}(Z_3) \subseteq \cdots$ forms an ascending chain of subqmods in X, by example 2.8. So X being Noetherian, $\exists\, m \in \mathbb{N}$ such that $p_j^{-1}(Z_i) = p_j^{-1}(Z_m)$, $\forall\, i \geq m \Rightarrow p_j p_j^{-1}(Z_i) = p_j p_j^{-1}(Z_m)$, $\forall\, i \geq m \Rightarrow Z_i = Z_m$, $\forall\, i \geq m$ [$\because$ p_j is surjective]. Arbitrariness of j proves the assertion. □

Theorem 4.15. *Let $\{X_j : j = 1, \ldots, n\}$ be a finite family of qmods over a common unitary ring R and $X := X_1 \times X_2 \times \cdots \times X_n$ be their Cartesian product. Then X is Artinian iff each X_j is Artinian.*

Proof. The proof of this theorem follows in a similar manner as that of the above Theorem 4.14. Here we just have to consider a descending chain of subqmods instead of ascending one. □

The above theorems fail in general if we consider infinite product of qmods, as the following example shows.

Example 4.16. (i) We have shown in example 4.2 that $(\mathbb{Z}^+, +)$ is a Noetherian qmod over $\mathbb{Z}$. If we consider its infinite product $X := (\mathbb{Z}^+)^{\aleph_0}$ it will not be Noetherian, since $\mathbb{Z}^+ \times \{0\} \times \{0\} \cdots \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+ \times \{0\} \cdots \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+ \times \mathbb{Z}^+ \times \{0\} \cdots \subseteq \cdots$ is an increasing sequence of subqmods in X which can never be stationary after a finite stage.

(ii) The countably infinite product $X := (\mathbb{C}_\infty)^{\aleph_0}$ of the Artinian qmod $\mathbb{C}_\infty$ (explained in example 4.5) is not Artinian, since $\mathbb{C} \times \mathbb{C}_\infty \times \mathbb{C}_\infty \cdots \supseteq \mathbb{C} \times \mathbb{C} \times \mathbb{C}_\infty \times \mathbb{C}_\infty \cdots \supseteq \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times \mathbb{C}_\infty \cdots \supseteq \cdots$ is a descending chain of subqmods of X which can not be stationary after a finite stage.

These examples are instructive to get the following theorem.

Theorem 4.17. *Let $\mathscr{F} := \{X_n : n \in \mathbb{N}\}$ be a countable family of Noetherian (Artinian) qmods over a unitary ring R. Then the product $X := \prod_{n\in\mathbb{N}} X_n$ cannot be Noetherian (Artinian).*

Proof. If $\mathscr{F}$ is a family of Noetherian qmods then $X_1 \times [X_2]_0 \times [X_3]_0 \cdots \subseteq X_1 \times X_2 \times [X_3]_0 \cdots \subseteq X_1 \times X_2 \times X_3 \times [X_4]_0 \cdots \subseteq \cdots$ is an increasing sequence of subqmods in X which can never be stationary after a finite stage.

If $\mathscr{F}$ is a family of Artinian qmods then considering the descending chain of subqmods $[X_1]_0 \times X_2 \times X_3 \cdots \supseteq [X_1]_0 \times [X_2]_0 \times X_3 \cdots \supseteq [X_1]_0 \times [X_2]_0 \times [X_3]_0 \times X_4 \cdots \supseteq \cdots$ we can see that it is non-terminating to a fixed subqmod of X and hence justifies our assertion. □

In view of Theorems 4.14, 4.15 and 4.17 we can make the following conclusion.

Theorem 4.18. *Let $\{X_\alpha : \alpha \in \Lambda\}$ be an arbitrary family of Noetherian (Artinian) qmods over a unitary ring R. Then the product $\prod\limits_{\alpha\in\Lambda} X_\alpha$ is Noetherian (Artinian) iff the index set Λ is finite.*

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Sandip Jana , Supriyo Mazumder
Department of Pure Mathematics
University of Calcutta
35, Ballygunge Circular Road
Kolkata-700019, India
E-mail: *sjpm12@gmail.com; supriyo88@gmail.com*

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