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On statistical convergence in generalized Lacunary sequence spaces

K. Raj, R. Anand

Abstract. In the present paper we introduce and study some generalized Lacunary sequence spaces of Musielak-Orlicz function using infinite matrix over n-normed spaces. We also make an effort to study some inclusion relations, topological and geometric properties of these spaces. Finally, we study statistical convergence on these spaces.

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1 Introduction and Preliminaries

Let w be the set of all real or complex sequences and l_∞, c and c_0 respectively, be the Banach spaces of bounded, convergent and null sequences $x = (x_k)$, normed by $\|x\| = \sup_k |x_k|$, where $k \in \mathbb{N}$. Let X and Y be two sequence spaces and $A = (a_{ik})$ be an infinite matrix of real or complex numbers a_{ik}, where $i, k \in \mathbb{N}$. Then we say that A defines a matrix mapping from X into Y if for every sequence $x = (x_i) \in X$, the sequence $Ax = \{A_i(x)\}$, the A-transform of x, is in Y, where

$$A_i(x) = \sum_{k=1}^{\infty} a_{ik} x_k \quad (i \in \mathbb{N}). \tag{1}$$

The matrix domain X_A of an infinite matrix A in a sequence space X is defined by

$$X_A = \{x = (x_k) : Ax \in X\}. \tag{2}$$

The approach of constructing a new sequence space by means of the matrix domain of a particular limitation method has been employed by several authors (see [24] and references therein).

The notion of difference sequence spaces was introduced by Kızmaz [14], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [7] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$. Another type of generalization of the difference sequence spaces is due to Tripathy and Esi [27], who studied the spaces $l_\infty(\Delta_m)$, $c(\Delta_m)$ and $c_0(\Delta_m)$.

Tripathy et al. [26] generalized the above notions and unified these as follows:
Let m, n be non-negative integers, then for $Z = l_\infty$, c and c_0, we have sequence spaces

$$Z(\Delta^n_{(m)}) = \{x = (x_k) \in w : (\Delta^n_{(m)} x_k) \in Z\},$$

where $\Delta^n_{(m)} x = (\Delta^n_{(m)} x_k) = (\Delta^{n-1}_{(m)} x_k - \Delta^{n-1}_{(m)} x_{k-m})$ and $\Delta^0_{(m)} x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta^n_{(m)} x_k = \sum_{i=0}^{n} (-1)^i \begin{pmatrix} n \\ i \end{pmatrix} x_{k+mi}.$$

Let m and n be non-negative integers and $v = (v_k)$ be a sequence of non-zero scalars. Then for Z, a given sequence space, we have

$$Z(\Delta^n_{(mv)}) = \{x = (x_k) \in w : (\Delta^n_{(mv)} x_k) \in Z\}, \text{ for } Z = l_\infty, c \text{ and } c_0,$$

where $\Delta^n_{(mv)} x_k = (\Delta^{n-1}_{(mv)} x_k - \Delta^{n-1}_{(mv)} x_{k-m})$ and $\Delta^0_{(mv)} x_k = v_k x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation:

$$\Delta^n_{(mv)} x_k = \sum_{i=0}^{n} (-1)^i \begin{pmatrix} n \\ i \end{pmatrix} v_{k-mi} x_{k-mi}.$$

In this expansion it is important to note that we take $v_{k-mi} = 0$ and $x_{k-mi} = 0$ for non-positive values of $k - mi$. Dutta [6] showed that these spaces can be made BK-spaces under the norm

$$||x|| = \sup_k |\Delta^n_{(mv)} x_k|.$$

For $n = 1$ and $v_k = 1$ for all $k \in \mathbb{N}$, we get the spaces $l_\infty(\Delta_m), c(\Delta_m)$ and $c_0(\Delta_m)$.
If $m = 1$ and $v_k = 1$ for all $k \in \mathbb{N}$, we get the spaces $l_\infty(\Delta^n), c(\Delta^n)$ and $c_0(\Delta^n)$.
Taking $m = n = 1$ and $v_k = 1$ for all $k \in \mathbb{N}$, we get the spaces $l_\infty(\Delta), c(\Delta)$ and $c_0(\Delta)$.

Parasar and Choudhary [20], Güngör et al. [12], Çolak et al. [4], and others used Orlicz functions for defining some new sequence spaces.

The concept of 2-normed spaces was initially developed by Gähler [11] in the mid of 1960's, while that of n-normed spaces can be seen in Misiak [16]. Since then many others have studied this concept and obtained various results (see [13]). Let $n \in \mathbb{N}$ and X be a linear space over the field $\mathbb{R}$ of reals of dimension d, where $d \geq n \geq 2$. A real valued function $||\cdot, \cdots, \cdot||$ on X^n satisfying the following four conditions:

1. $||x_1, x_2, \cdots, x_n|| = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent in X,

2. $||x_1, x_2, \cdots, x_n||$ is invariant under permutation,

3. $||\alpha x_1, x_2, \cdots, x_n|| = |\alpha| \; ||x_1, x_2, \cdots, x_n||$ for any $\alpha \in \mathbb{R}$, and

4. $||x+x', x_2, \cdots, x_n|| \leq ||x, x_2, \cdots, x_n|| + ||x', x_2, \cdots, x_n||$

is called an n-norm on X and the pair $(X, ||\cdot, \cdots, \cdot||)$ is called a n-normed space over the field $\mathbb{R}$.

Example 1. We may take $X = \mathbb{R}^n$ being equipped with the n-norm $||x_1, x_2, \cdots, x_n||_E$ = the volume of the n-dimensional parallelepiped spanned by the vectors $x_1, x_2, \cdots, x_n$ which may be given explicitly by the formula

$$||x_1, x_2, \cdots, x_n||_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots, n$. Let $(X, ||\cdot, \cdots, \cdot||)$ be an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \cdots, a_n\}$ be linearly independent set in X. Then the function $||\cdot, \cdots, \cdot||_\infty$ on X^{n-1} defined by

$$||x_1, x_2, \cdots, x_{n-1}||_\infty = \max\{||x_1, x_2, \cdots, x_{n-1}, a_i|| : i = 1, 2, \cdots, n\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1, a_2, \cdots, a_n\}$.

A sequence (x_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to converge to some $L \in X$ if

$$\lim_{k\to\infty} ||x_k - L, z_1, \cdots, z_{n-1}|| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

An Orlicz function M is a function which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$. If the convexity of an Orlicz function is replaced by subadditivity, we call it a modulus function introduced by Nakano [19].

Lindenstrauss and Tzafriri [15] used the idea of Orlicz function to define the following sequence space,

$$\ell_M = \Big\{x = (x_k) \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty, \text{ for some } \rho > 0\Big\},$$

which is called an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$||x|| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1\Big\}.$$

A sequence $\mathcal{M} = (M_i)$ of Orlicz functions is called a Musielak-Orlicz function. A Musielak-Orlicz function $\mathcal{M} = (M_i)$ is said to satisfy Δ_2-*condition* if there exist constants a, $K > 0$ and a sequence $c = (c_i)_{i=1}^{\infty} \in l_+^1$ (the positive cone of l^1) such that the inequality

$$M_i(2u) \leq KM_i(u) + c_i$$

holds for all $i \in \mathbb{N}$ and $u \in \mathbb{R}^+$, whenever $M_i(u) \leq a$. For more details about sequence spaces see [2, 17, 21–23] and references therein.

An increasing sequence of non-negative integers $h_r = (i_r - i_{r-1}) \to \infty$ as $r \to \infty$ can be made through *lacunary sequence* $\theta = (i_r)$, $r = 0, 1, 2, \cdots$, where $i_0 = 0$. The intervals determined by θ are denoted by $I_r = (i_{r-1}, i_r]$ and the ratio i_r/i_{r-1} will be denoted by q_r. Freedman [9] defined the space of lacunary strongly convergent sequences N_θ as:

$$N_\theta = \Big\{x = (x_k) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |x_k - L| = 0 \ \text{ for some } \ L\Big\}.$$

Let X be a sequence space. Then X is called

(i) Solid (or normal) if $(\alpha_i x_i) \in X$ whenever $(x_i) \in X$ and for all sequences (α_i) of scalars with $|\alpha_i| \leq 1$, for all $i \in \mathbb{N}$;

(ii) Monotone provided X contains the canonical preimages of all its step spaces. If X is normal, then X is monotone.

Let $A = (a_{ik})$ be an infinite matrix of complex numbers. Let $\mathcal{M} = (M_i)$ be a Musielak-Orlicz function, $p = (p_i)$ be a bounded sequence of positive real numbers, $u = (u_i)$ be a sequence of strictly positive real numbers. We define the following sequence spaces in the present paper:
$\nu[A, u, \Delta^n_{(mv)}, \theta, p, M_i, ||., ..., .||] =$

$$\Big\{x : \lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} \Big[u_i M_i\Big(\Big|\Big|\frac{\Delta^n_{(mv)} A_i(x) - s}{\rho}, z_1, ..., z_{n-1}\Big|\Big|\Big)\Big]^{p_i} = 0 \ \text{ for some } s, \rho > 0\Big\},$$

$\nu_0[A, u, \Delta^n_{(mv)}, \theta, p, M_i, ||., ..., .||] =$

$$\Big\{x : \lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} \Big[u_i M_i\Big(\Big|\Big|\frac{\Delta^n_{(mv)} A_i(x)}{\rho}, z_1, ..., z_{n-1}\Big|\Big|\Big)\Big]^{p_i} = 0, \ \text{ for some } \rho > 0\Big\}$$

and
$\nu_\infty[A, u, \Delta^n_{(mv)}, \theta, p, M_i, ||., ..., .||] =$

$$\Big\{x : \sup_r \frac{1}{h_r} \sum_{i\in I_r} \Big[u_i M_i\Big(\Big|\Big|\frac{\Delta^n_{(mv)} A_i(x)}{\rho}, z_1, ..., z_{n-1}\Big|\Big|\Big)\Big]^{p_i} < \infty, \ \text{ for some } \rho > 0\Big\}.$$

The following inequality will be used throughout the paper. If $0 < p_i \leq \sup p_i = H$, $K = \max(1, 2^{H-1})$, then

$$|a_i + b_i|^{p_i} \quad \leq \quad K\{|a_i|^{p_i} + |b_i|^{p_i}\} \tag{3}$$

for all i and $a_i, b_i \in \mathbb{C}$. Also, $|a|^{p_i} \leq \max(1, |a|^H)$, for all $a \in \mathbb{C}$.

The main objective of this paper is to introduce the concept of generalized Lacunary sequence spaces of Museilak-Orlicz function using infinite matrix over n-normed spaces. We also make an effort to study some topological properties and prove some inclusion relations between these sequence spaces. Finally, by using these concepts we study statistical convergence of these spaces.

2 Main Results

Theorem 1. *Let $\mathcal{M} = (M_i)$ be a Musielak-Orlicz function, $p = (p_i)$ be a bounded sequence of positive real numbers, $u = (u_i)$ be a sequence of strictly positive real numbers and $A = (a_{ik})$ be an infinite matrix of complex numbers. Then, the spaces $\nu[A, u, \Delta^n_{(mv)}, \theta, p, M_i, ||., ..., .||]$, $\nu_0[A, u, \Delta^n_{(mv)}, \theta, p, M_i, ||., ..., .||]$ and $\nu_\infty[A, u, \Delta^n_{(mv)}, \theta, p, M_i, ||., ..., .||]$ are linear spaces over the complex field $\mathbb{C}$.*

Proof. Let x and $y \in \nu_0[A, u, \Delta^n_{(mv)}, \theta, p, M_i, ||., ..., .||]$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive real numbers ρ_1 and ρ_2 such that

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} \Big[u_i M_i\Big(\Big|\Big|\frac{\Delta^n_{(mv)} A_i(x)}{\rho_1}, z_1, ..., z_{n-1}\Big|\Big|\Big)\Big]^{p_i} = 0$$

and

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} \Big[u_i M_i\Big(\Big|\Big|\frac{\Delta^n_{(mv)} A_i(y)}{\rho_2}, z_1, ..., z_{n-1}\Big|\Big|\Big)\Big]^{p_i} = 0.$$

Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $||., ..., .||$ is a n-norm on X and $\mathcal{M} = (M_i)$ is non-decreasing and convex function for each i, so by inequality (3), we have

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} \Big[u_i M_i\Big(\Big|\Big|\frac{\Delta^n_{(mv)} A_i(\alpha x + \beta y)}{\rho_3}, z_1, ..., z_{n-1}\Big|\Big|\Big)\Big]^{p_i}$$

$$\begin{aligned}
&\leq \lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} \Big[u_i M_i\Big(\Big|\Big|\frac{\Delta^n_{(mv)} A_i(\alpha x)}{\rho_3}, z_1, ..., z_{n-1}\Big|\Big| + \Big|\Big|\frac{\Delta^n_{(mv)} A_i(\beta y)}{\rho_3}, z_1, ..., \\
&\quad z_{n-1}\Big|\Big|\Big)\Big]^{p_i} \\
&\leq K \lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} \frac{1}{2^{p_i}} \Big[u_i M_i\Big(\Big|\Big|\frac{\Delta^n_{(mv)} A_i(x)}{\rho_1}, z_1, ..., z_{n-1}\Big|\Big|\Big)\Big]^{p_i} \\
&+ K \lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} \frac{1}{2^{p_i}} \Big[u_i M_i\Big(\Big|\Big|\frac{\Delta^n_{(mv)} A_i(y)}{\rho_2}, z_1, ..., z_{n-1}\Big|\Big|\Big)\Big]^{p_i} \\
&= 0.
\end{aligned}$$

Thus, we have $\alpha x+\beta y \in \nu_0[A,u,\Delta^n_{(mv)},\theta,p,M_i,||.,...,.||]$. Hence, $\nu_0[A,u,\Delta^n_{(mv)},\theta,p,M_i,||.,...,.||]$ is a linear space. Simultaneously, it can be proved that $\nu[A,u,\Delta^n_{(mv)},\theta,p,M_i,||.,...,.||]$ and $\nu_\infty[A,u,\Delta^n_{(mv)},\theta,p,M_i,||.,...,.||]$ are linear spaces. □

Theorem 2. *Let $\mathcal{M}=(M_i)$ be a Musielak-Orlicz function, $p=(p_i)$ be a bounded sequence of positive real numbers, $u=(u_i)$ be a sequence of strictly positive real numbers and $A=(a_{ik})$ be an infinite matrix of complex numbers. If $\sup_i (M_i(x))^{p_i} < \infty\ \forall$ fixed $x>0$, then $\nu[A,u,\Delta^n_{(mv)},\theta,p,M_i,||.,...,.||] \subset \nu_\infty[A,u,\Delta^n_{(mv)},\theta,p,M_i,||.,...,.||]$.*

Proof. Let $x \in \nu[A,u,\Delta^n_{(mv)},\theta,p,M_i,||.,...,.||]$. Then there exists some positive ρ_1 such that

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r}\Big[u_iM_i\Big(\Big|\Big|\frac{\Delta^n_{(mv)}A_i(x)-s}{\rho_1},z_1,...,z_{n-1}\Big|\Big|\Big)\Big]^{p_i}=0.$$

Define $\rho=2\rho_1$. Since (M_i) is non-decreasing, convex and by using inequality (3), we have

$$\begin{aligned}
&\lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r}\Big[u_iM_i\Big(\Big|\Big|\frac{\Delta^n_{(mv)}A_i(x)}{\rho},z_1,...,z_{n-1}\Big|\Big|\Big)\Big]^{p_i}\\
&= \lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r}\Big[u_iM_i\Big(\Big|\Big|\frac{\Delta^n_{(mv)}A_i(x)-s+s}{2\rho_1},z_1,...,z_{n-1}\Big|\Big|\Big)\Big]^{p_i}\\
&\le K\lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r}\frac{1}{2^{p_i}}\Big[u_iM_i\Big(\Big|\Big|\frac{\Delta^n_{(mv)}A_i(x)-s}{\rho_1},z_1,...,z_{n-1}\Big|\Big|\Big)\Big]^{p_i}\\
&+ K\lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r}\frac{1}{2^{p_i}}\Big[u_iM_i\Big(\Big|\Big|\frac{s}{\rho_1},z_1,...,z_{n-1}\Big|\Big|\Big)\Big]^{p_i}\\
&< K\lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r}\Big[u_iM_i\Big(\Big|\Big|\frac{\Delta^n_{(mv)}A_i(x)-s}{\rho_1},z_1,...,z_{n-1}\Big|\Big|\Big)\Big]^{p_i}\\
&+ K\lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r}\Big[u_iM_i\Big(\Big|\Big|\frac{s}{\rho_1},z_1,...,z_{n-1}\Big|\Big|\Big)\Big]^{p_i}\\
&< \infty.
\end{aligned}$$

Hence, $x\in\nu_\infty[A,u,\Delta^n_{(mv)},\theta,p,M_i,||.,...,.||]$. This completes the proof. □

Theorem 3. *The sequence spaces $\nu_0[A,u,\Delta^n_{(mv)},\theta,p,M_i,||.,...,.||]$ and $\nu_\infty[A,u,\Delta^n_{(mv)},\theta,p,M_i,||.,...,.||]$ are solid and so monotone.*

Proof. Suppose $x\in\nu_\infty[A,u,\Delta^n_{(mv)},\theta,p,M_i,||.,...,.||]$. Then

$$\sup_r\frac{1}{h_r}\sum_{i\in I_r}\Big[u_iM_i\Big(\Big|\Big|\frac{\Delta^n_{(mv)}A_i(x)}{\rho},z_1,...,z_{n-1}\Big|\Big|\Big)\Big]^{p_i}<\infty,\ \text{ for some } \rho>0.$$

Let $\alpha = (\alpha_i)$ be a sequence of scalars such that $|\alpha_i| \leq 1$ for all $i \in \mathbb{N}$. Then, we have

$$\begin{aligned}
&\sup_r \frac{1}{h_r} \sum_{i \in I_r} \Big[u_i M_i\Big(\Big|\Big|\frac{\Delta^n_{(mv)} A_i(\alpha x)}{\rho}, z_1, ..., z_{n-1}\Big|\Big|\Big)\Big]^{p_i} \\
&\qquad \leq \sup_r \frac{1}{h_r} \sum_{i \in I_r} \Big[u_i M_i\Big(\Big|\Big|\frac{\Delta^n_{(mv)} A_i(x)}{\rho}, z_1, ..., z_{n-1}\Big|\Big|\Big)\Big]^{p_i} \\
&\qquad < \infty,
\end{aligned}$$

which leads us to the desired result. □

Theorem 4. *Let $\mathcal{M}' = (M'_i)$ and $\mathcal{M}'' = (M''_i)$ be Musielak-Orlicz functions. Then* $\nu_0[A, u, \Delta^n_{(mv)}, \theta, p, M'_i, ||., ..., .||] \cap \nu_0[A, u, \Delta^n_{(mv)}, \theta, p, M''_i, ||., ..., .||] \subseteq \nu_0[A, u, \Delta^n_{(mv)}, \theta, p, (M'_i + M''_i), ||., ..., .||]$.

Proof. Suppose that $x \in \nu_0[A, u, \Delta^n_{(mv)}, \theta, p, M'_i, ||., ..., .||] \cap \nu_0[A, u, \Delta^n_{(mv)}, \theta, p, M''_i, ||., ..., .||]$. Then

$$\begin{aligned}
&\lim_{r\to\infty} \frac{1}{h_r} \sum_{i \in I_r} \Big[u_i (M'_i + M''_i)\Big(\Big|\Big|\frac{\Delta^n_{(mv)} A_i(x)}{\rho}, z_1, ..., z_{n-1}\Big|\Big|\Big)\Big]^{p_i} \\
&\qquad = \lim_{r\to\infty} \frac{1}{h_r} \sum_{i \in I_r} \Big[u_i M'_i\Big(\Big|\Big|\frac{\Delta^n_{(mv)} A_i(x)}{\rho}, z_1, ..., z_{n-1}\Big|\Big|\Big)\Big]^{p_i} \\
&\qquad + \lim_{r\to\infty} \frac{1}{h_r} \sum_{i \in I_r} \Big[u_i M''_i\Big(\Big|\Big|\frac{\Delta^n_{(mv)} A_i(x)}{\rho}, z_1, ..., z_{n-1}\Big|\Big|\Big)\Big]^{p_i} \\
&\qquad \leq K \lim_{r\to\infty} \frac{1}{h_r} \sum_{i \in I_r} \Big[u_i M'_i\Big(\Big|\Big|\frac{\Delta^n_{(mv)} A_i(x)}{\rho}, z_1, ..., z_{n-1}\Big|\Big|\Big)\Big]^{p_i} \\
&\qquad + K \lim_{r\to\infty} \frac{1}{h_r} \sum_{i \in I_r} \Big[u_i M''_i\Big(\Big|\Big|\frac{\Delta^n_{(mv)} A_i(x)}{\rho}, z_1, ..., z_{n-1}\Big|\Big|\Big)\Big]^{p_i} \\
&\qquad \to 0 \text{ as } r \to \infty.
\end{aligned}$$

Thus, $x \in \nu_0[A, u, \Delta^n_{(mv)}, \theta, p, (M'_i + M''_i), ||., ..., .||]$. Hence, the proof is complete. □

Theorem 5. *Let $\mathcal{M} = (M_i)$ and $\mathcal{M}' = (M'_i)$ be two Musielak-Orlicz functions satisfying Δ_2- condition. Then*
(i) $\nu_0[A, u, \Delta^n_{(mv)}, \theta, p, M'_i, ||., ..., .||] \subseteq \nu_0[A, u, \Delta^n_{(mv)}, \theta, p, M_i \circ M'_i, ||., ..., .||]$.
(ii) $\nu[A, u, \Delta^n_{(mv)}, \theta, p, M'_i, ||., ..., .||] \subseteq \nu[A, u, \Delta^n_{(mv)}, \theta, p, M_i \circ M'_i, ||., ..., .||]$.
(iii) $\nu_\infty[A, u, \Delta^n_{(mv)}, \theta, p, M'_i, ||., ..., .||] \subseteq \nu_\infty[A, u, \Delta^n_{(mv)}, \theta, p, M_i \circ M'_i, ||., ..., .||]$.

Proof. We consider the first case only. Rests can be proved in a similar way. Let $x \in \nu_0[A, u, \Delta^n_{(mv)}, \theta, p, M'_i, ||., ..., .||]$. Then

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{i \in I_r} \Big[u_i M'_i\Big(\Big|\Big|\frac{\Delta^n_{(mv)} A_i(x)}{\rho}, z_1, ..., z_{n-1}\Big|\Big|\Big)\Big]^{p_i} = 0.$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M_i(t) < \epsilon$ for $0 \le t \le \delta$.
Let $y_i = \Big[u_i M_i'\Big(\Big|\Big|\frac{\Delta^n_{(mv)} A_i(x)}{\rho}, z_1, ..., z_{n-1}\Big|\Big|\Big)\Big]$. Thus, we can write

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} [M_i(y_i)]^{p_i} = \lim_{r\to\infty} \frac{1}{h_r} \sum_{y_i\le\delta} [M_i(y_i)]^{p_i} + \lim_{r\to\infty} \frac{1}{h_r} \sum_{y_i>\delta} [M_i(y_i)]^{p_i}.$$

Since $\mathcal{M} = (M_i)$ satisfies Δ_2- condition, we have

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{y_i\le\delta} [M_i(y_i)]^{p_i} \le \max\{1, M_i(1)^H\} \lim_{r\to\infty} \frac{1}{h_r} \sum_{y_i\le\delta} [(y_i)]^{p_i}. \tag{4}$$

For $y_i > \delta$, we use the fact that $y_i < \frac{y_i}{\delta} < 1 + \frac{y_i}{\delta}$. Since $\mathcal{M} = (M_i)$ is non-decreasing and convex, it follows that

$$M_i(y_i) < M_i(1 + \frac{y_i}{\delta}) < \frac{1}{2} M_i(2) + \frac{1}{2}(\frac{2y_i}{\delta}).$$

Since $\mathcal{M} = (M_i)$ satisfies Δ_2- condition and $\frac{y_i}{\delta} > 1$, there exists $K > 0$ such that

$$M_i(y_i) < \frac{1}{2} K \frac{y_i}{\delta} M_i(2) + \frac{1}{2} K \frac{y_i}{\delta} M_i(2) = K \frac{y_i}{\delta} M_i(2).$$

Therefore, we have

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{y_i>\delta} [u_i M_i(y_i)]^{p_i} \le \max\{1, (K \frac{M_i(2)}{\delta})^H\} \lim_{r\to\infty} \frac{1}{h_r} \sum_{y_i>\delta} [(y_i)]^{p_i}. \tag{5}$$

Hence, by equations (4) and (5), we have

$$\begin{aligned}
&\lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} \Big[u_i (M_i \circ M_i')\Big(\Big|\Big|\frac{\Delta^n_{(mv)} A_i(x)}{\rho}, z_1, ..., z_{n-1}\Big|\Big|\Big)\Big]^{p_i} \\
&= \lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} [u_i M_i(y_i)]^{p_i} \\
&\le D \lim_{r\to\infty} \frac{1}{h_r} \sum_{y_i\le\delta} [y_i]^{p_i} \\
&+ G \lim_{r\to\infty} \frac{1}{h_r} \sum_{y_i>\delta} [y_i]^{p_i},
\end{aligned}$$

where $D = \max\{1, M_i(1)^H\}$ and $G = \max\{1, (K\frac{M_i(2)}{\delta})^H\}$.
Hence, $\nu_0[A, u, \Delta^n_{(mv)}, \theta, p, M_i', ||., ..., .||] \subseteq \nu_0[A, u, \Delta^n_{(mv)}, \theta, p, M_i \circ M_i', ||., ..., .||]$. □

Theorem 6. *Let $0 \le p_i \le q_i$ for all i and $(\frac{q_i}{p_i})$ be bounded. Then $\nu[A, u, \Delta^n_{(mv)}, \theta, q, M_i, ||., ..., .||] \subset \nu[A, u, \Delta^n_{(mv)}, \theta, p, M_i, ||., ..., .||]$.*

Proof. Let $x \in \nu[A,u,\Delta^n_{(mv)},\theta,q,M_i,||.,...,.||]$. Write $t_i = \Big[u_i M_i\Big(\Big|\Big|\frac{\Delta^n_{(mv)}A_i(x)-s}{\rho}, z_1, ..., z_{n-1}\Big|\Big|\Big)\Big]^{q_i}$ and $\mu_i = \frac{p_i}{q_i}$ for all $i \in \mathbb{N}$. Then $0 < \mu_i \leq 1$ for every $i \in \mathbb{N}$. Take $0 < \mu < \mu_i$ for every $i \in \mathbb{N}$. Define the sequences (a_i) and (b_i) as follows:
For $t_i \geq 1$, let $a_i = t_i$ and $b_i = 0$ and for $t_i < 1$, let $a_i = 0$ and $b_i = t_i$. Then clearly for all $i \in \mathbb{N}$, we have $t_i = a_i + b_i, t_i^{\mu_i} = a_i^{\mu_i} + b_i^{\mu_i}$. Now, it follows that $a_i^{\mu_i} \leq a_i \leq t_i$ and $b_i^{\mu_i} \leq b_i^{\mu}$. Therefore,

$$\begin{aligned}\lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r} t_i^{\mu_i} &= \lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r}(a_i^{\mu_i}+b_i^{\mu_i}) \\ &\leq \lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r} t_i + \lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r} b_i^{\mu}.\end{aligned}$$

Now for each i,

$$\begin{aligned}\lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r} b_i^{\mu} &= \lim_{r\to\infty}\sum_{i\in I_r}\Big(\frac{1}{h_r}b_i\Big)^{\mu}\Big(\frac{1}{h_r}\Big)^{1-\mu} \\ &\leq \lim_{r\to\infty}\Big(\sum_{i\in I_r}\Big[\Big(\frac{1}{h_r}b_i\Big)^{\mu}\Big]^{\frac{1}{\mu}}\Big)^{\mu}.\lim_{r\to\infty}\Big(\sum_{i\in I_r}\Big[\Big(\frac{1}{h_r}\Big)^{1-\mu}\Big]^{\frac{1}{1-\mu}}\Big)^{1-\mu} \\ &= \lim_{r\to\infty}\Big(\frac{1}{h_r}\sum_{i\in I_r} b_i\Big)^{\mu}\end{aligned}$$

and so

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r} t_i^{\mu_i} \leq \lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r} t_i + \lim_{r\to\infty}\Big(\frac{1}{h_r}\sum_{i\in I_r} b_i\Big)^{\mu}.$$

Hence, $x \in \nu[A,u,\Delta^n_{(mv)},\theta,p,M_i,||.,...,.||]$. □

Theorem 7. *(i) If* $0 \leq \inf p_i \leq p_i \leq 1$ *for all* i, *then* $\nu[A,u,\Delta^n_{(mv)},\theta,M_i,||.,...,.||] \subset \nu[A,u,\Delta^n_{(mv)},\theta,p,M_i,||.,...,.||]$.
(ii) If $1 < p_i \leq \sup p_i = H < \infty$, *then* $\nu[A,u,\Delta^n_{(mv)},\theta,p,M_i,||.,...,.||] \subset \nu[A,u,\Delta^n_{(mv)},\theta,M_i,||.,...,.||]$.

Proof. (i) Let $x \in \nu[A,u,\Delta^n_{(mv)},\theta,M_i,||.,...,.||]$. Since $0 < \inf p_i \leq 1$, we get

$$\begin{aligned}&\lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r}\Big[u_i M_i\Big(\Big|\Big|\frac{\Delta^n_{(mv)}A_i(x)-s}{\rho}, z_1, ..., z_{n-1}\Big|\Big|\Big)\Big] \\ &\leq \lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r}\Big[u_i M_i\Big(\Big|\Big|\frac{\Delta^n_{(mv)}A_i(x)-s}{\rho}, z_1, ..., z_{n-1}\Big|\Big|\Big)\Big]^{p_i}\end{aligned}$$

and hence $x \in \nu[A,u,\Delta^n_{(mv)},\theta,p,M_i,||.,...,.||]$.
(ii) Let $1 \leq p_i \leq \sup p_i = H < \infty$ and $x \in \nu[A,u,\Delta^n_{(mv)},\theta,p,M_i,||.,...,.||]$. Then for

each $0 < \epsilon < 1$, there exists a positive integer s_0 such that

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} \Big[u_i M_i\Big(\Big|\Big|\frac{\Delta^n_{(mv)} A_i(x) - s}{\rho}, z_1, ..., z_{n-1}\Big|\Big|\Big)\Big]^{p_i} \quad \leq \quad \epsilon < 1 \text{ for all } r \geq s_0.$$

This implies that

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} \Big[u_i M_i\Big(\Big|\Big|\frac{\Delta^n_{(mv)} A_i(x) - s}{\rho}, z_1, ..., z_{n-1}\Big|\Big|\Big)\Big]^{p_i}$$

$$\leq \quad \lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} \Big[u_i M_i\Big(\Big|\Big|\frac{\Delta^n_{(mv)} A_i(x) - s}{\rho}, z_1, ..., z_{n-1}\Big|\Big|\Big)\Big].$$

Therefore, $x \in \nu[A, u, \Delta^n_{(mv)}, \theta, M_i, ||., ..., .||]$. $\square$

3 Statistical Convergence

The notion of statistical convergence was introduced by Fast [8] and Schoenberg [25] independently. Over the years and under different names, statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Later on, it was further investigated from the sequence space point of view and linked with summability theory by Fridy [10], Connor [3], Mursaleen et al.[18] and many others. In recent years, generalizations of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on locally compact spaces. Statistical convergence and its generalizations are also connected with subsets of the Stone-Cech compactification of natural numbers. Moreover, statistical convergence is closely related to the concept of convergence in probability.

A complex number sequence $x = (x_i)$ is said to be statistically convergent to the number l if for every $\epsilon > 0$, $\lim_{n\to\infty} \frac{1}{n}|K(\epsilon)| = 0$, where $K(\epsilon)$ denotes the number of elements in $K(\epsilon) = \{i \in N : |x_i - l| \geq \epsilon\}$.

The set of statistically convergent sequences is denoted by S.

A sequence $x = (x_i)$ is said to be $S[A, u, \Delta^n_{(mv)}, \theta, \|., ..., .\|]$-statistically convergent to s if

$$\lim_{r\to\infty} \frac{1}{h_r} \Big|\Big\{i \in I_r : \Big|\Big|\frac{\Delta^n_{(mv)} A_i(x) - s}{\rho}, z_1, ..., z_{n-1}\Big|\Big| \geq \epsilon\Big\}\Big| = 0.$$

In this section we introduce $S[A, u, \Delta^n_{(mv)}, \theta, \|., ..., .\|]$-statistical convergence and give some relations between $S[A, u, \Delta^n_{(mv)}, \theta, \|., ..., .\|]$-statistically convergent sequences and $\nu[A, u, \Delta^n_{(mv)}, \theta, p, M_i, ||., ..., .||]$-convergent sequences.

Theorem 8. *Let $\mathcal{M} = (M_i)$ be a Musielak-Orlicz function. Then* $\nu[A, u, \Delta^n_{(mv)}, \theta, p, M_i, ||., ..., .||] \subset S[A, u, \Delta^n_{(mv)}, \theta, ||., ..., .||]$.

Proof. Let $x \in \nu[A, u, \Delta^n_{(mv)}, \theta, p, M_i, ||., ..., .||]$. Take $\epsilon > 0, \sum\limits_1$ denotes the sum over $i \leq n$ with $\Big|\Big|\frac{\Delta^n_{(mv)} A_i(x)-s}{\rho}, z_1, ..., z_{n-1}\Big|\Big| \geq \epsilon$ and $\sum\limits_2$ denotes the sum over $i \leq n$ with $\Big|\Big|\frac{\Delta^n_{(mv)} A_i(x)-s}{\rho}, z_1, ..., z_{n-1}\Big|\Big| < \epsilon$. Then for each $z_1, ..., z_{n-1}$, we obtain

$$\begin{aligned}
&\lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} \Big[u_i M_i\Big(\Big|\Big|\frac{\Delta^n_{(mv)} A_i(x) - s}{\rho}, z_1, ..., z_{n-1}\Big|\Big|\Big)\Big]^{p_i} \\
&= \lim_{r\to\infty} \frac{1}{h_r} \sum_1 \Big[u_i M_i\Big(\Big|\Big|\frac{\Delta^n_{(mv)} A_i(x) - s}{\rho}, z_1, ..., z_{n-1}\Big|\Big|\Big)\Big]^{p_i} \\
&+ \lim_{r\to\infty} \frac{1}{h_r} \sum_2 \Big[u_i M_i\Big(\Big|\Big|\frac{\Delta^n_{(mv)} A_i(x) - s}{\rho}, z_1, ..., z_{n-1}\Big|\Big|\Big)\Big]^{p_i} \\
&\geq \lim_{r\to\infty} \frac{1}{h_r} \sum_1 \Big[u_i M_i\Big(\Big|\Big|\frac{\Delta^n_{(mv)} A_i(x) - s}{\rho}, z_1, ..., z_{n-1}\Big|\Big|\Big)\Big]^{p_i} \\
&\geq \lim_{r\to\infty} \frac{1}{h_r} \sum_1 \Big[u_i M_i(\epsilon)\Big]^{p_i} \\
&\geq \lim_{r\to\infty} \frac{1}{h_r} \sum_1 \min \Big[\{u_i M_i(\epsilon)\}^h, \{u_i M_i(\epsilon)\}^H\Big] \\
&= \lim_{r\to\infty} \frac{1}{h_r} \Big|\Big\{i \leq n : \Big|\Big|\frac{\Delta^n_{(mv)} A_i(x) - s}{\rho}, z_1, ..., z_{n-1}\Big|\Big| \geq \epsilon\Big\}\Big| \ \min \Big[\{u_i M_i(\epsilon)\}^h, \\
&\quad \{u_i M_i(\epsilon)\}^H\Big].
\end{aligned}$$

Hence, $x \in S[A, u, \Delta^n_{(mv)}, \theta, ||., ..., .||]$. This completes the proof of the theorem. □

Theorem 9. *Let $\mathcal{M} = (M_i)$ be a bounded Musielak-Orlicz function. Then* $S[A, u, \Delta^n_{(mv)}, \theta, ||., ..., .||] \subset \nu[A, u, \Delta^n_{(mv)}, \theta, p, M_i, ||., ..., .||]$.

Proof. Suppose that $\mathcal{M} = (M_i)$ be bounded. For given $\epsilon > 0, \sum\limits_1$ denotes the sum over $i \leq n$ with $\Big|\Big|\frac{\Delta^n_{(mv)} A_i(x)-s}{\rho}, z_1, ..., z_{n-1}\Big|\Big| \geq \epsilon$ and $\sum\limits_2$ denotes the sum over $i \leq n$ with $\Big|\Big|\frac{\Delta^n_{(mv)} A_i(x)-s}{\rho}, z_1, ..., z_{n-1}\Big|\Big| < \epsilon$. Since $\mathcal{M} = (M_i)$ is bounded, there exists an integer N such that $M_i < N$ for all i. Then for each $z_1, ..., z_{n-1}$, we obtain

$$\begin{aligned}
&\lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} \Big[u_i M_i\Big(\Big|\Big|\frac{\Delta^n_{(mv)} A_i(x) - s}{\rho}, z_1, ..., z_{n-1}\Big|\Big|\Big)\Big]^{p_i} \\
&= \lim_{r\to\infty} \frac{1}{h_r} \sum_1 \Big[u_i M_i\Big(\Big|\Big|\frac{\Delta^n_{(mv)} A_i(x) - s}{\rho}, z_1, ..., z_{n-1}\Big|\Big|\Big)\Big]^{p_i}
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \lim_{r\to\infty} \frac{1}{h_r} \sum_2 \Big[u_i M_i\Big(\Big|\Big|\frac{\Delta^n_{(mv)} A_i(x) - s}{\rho}, z_1, ..., z_{n-1}\Big|\Big|\Big)\Big]^{p_i} \\
&\leq \quad \lim_{r\to\infty} \frac{1}{h_r} \sum_1 \max\Big[\{u_i N^h\}, \{u_i N^H\}\Big] \\
&+ \quad \lim_{r\to\infty} \frac{1}{h_r} \sum_2 \Big[u_i M_i(\epsilon)^{p_i}\Big] \\
&\leq \quad \max\Big[\{u_i N^h\}, \{u_i N^H\}\Big] \lim_{r\to\infty} \frac{1}{h_r} \Big|\Big\{i \leq n : \Big|\Big|\frac{\Delta^n_{(mv)} A_i(x) - s}{\rho}, z_1, ..., z_{n-1}\Big|\Big| \geq \epsilon\Big\}\Big| \\
&+ \quad \max\Big[\{u_i M_i(\epsilon)^h\}, \{u_i M_i(\epsilon)^H\}\Big].
\end{aligned}
$$

Hence, $x \in \nu[A, u, \Delta^n_{(mv)}, \theta, p, M_i, ||., ..., .||]$. □

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K. Raj, R. Anand
Department of Mathematics
Shri Mata Vaishno Devi University
Katra-182320, J&K, India
E-mail: *kuldipraj68@gmail.com*, *renuanand71@gmail.com*

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