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Later developments based on some ideas of Andrunachievici: Special radicals and The Lemma*

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Abstract. This is a survey of subsequent work on two topics derived from fundamental publications of V. A. Andrunachievici in the 1950s and 1960s: special radicals and the result which has come to be known as the Andrunachievici Lemma.

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1 Introduction

A (Kurosh-Amitsur) *radical class of rings* is a non-empty homomorphically closed class $\mathcal{R}$ such that for all rings A we have

$$\mathcal{R}(A) := \sum\{I \triangleleft A : I \in \mathcal{R}\} \in \mathcal{R} \text{ and } \mathcal{R}(A/\mathcal{R}(A)) = 0.$$

For many purposes a more convenient characterization of a radical class is as a non-empty class $\mathcal{R}$ satisfying the following conditions.
(i) $\mathcal{R}$ is homomorphically closed.
(ii) If $I \triangleleft A$ and both I and A/I are in $\mathcal{R}$ then $A \in \mathcal{R}$. ($\mathcal{R}$ is *closed under extensions*.)
(iii) If $\{I_\lambda : \lambda \in \Lambda\}$ is a chain of ideals of a ring A and each $I_\lambda \in \mathcal{R}$, then $\bigcup I_\lambda = \sum I_\lambda \in \mathcal{R}$.

The ideal $\mathcal{R}(A)$ will sometimes be called *the radical* of A, and ideals, subrings etc. of A which are in $\mathcal{R}$ will sometimes be called *radical ideals, radical subrings* etc. when this causes no ambiguity.

A class $\mathcal{K}$ of rings, radical or not, is *hereditary* if whenever $I \triangleleft A \in \mathcal{K}$ we have I in $\mathcal{K}$.

With each radical class $\mathcal{R}$ is associated the class of $\mathcal{R}$ - *semi-simple* rings, those rings A for which $\mathcal{R}(A) = 0$. In general a non empty class $\mathcal{S}$ is a *semi-simple class*, i.e. the class of $\mathcal{R}$ -semi-simple rings for some radical class $\mathcal{R}$, if and only if it is hereditary, closed under extensions and closed under subdirect products.

We note in passing that whereas the characterization of radical classes given above is valid, *mutatis mutandis*, for groups, non-associative rings and other structures for which radical theory can be developed, semi-simple classes in some such contexts need not be hereditary, and indeed there are settings in which they need

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not be describable by closure properties at all: in [2] it is shown that for arbitrary (not necessarily associative) algebras over a field, intersections of semi-simple classes need not be semi-simple classes.

For each class $\mathcal{C}$ of rings there is a smallest radical class containing $\mathcal{C}$; this is called the *lower radical class* defined by $\mathcal{C}$, and we shall denote it by $L(\mathcal{C})$. There is also a largest radical class whose semi-simple class contains $\mathcal{C}$, called the *upper radical class* defined by $\mathcal{C}$; this we shall denote by $U(\mathcal{C})$. In particular, if $\mathcal{S}$ is the class of $\mathcal{R}$ -semi-simple rings, then $\mathcal{R} = U(\mathcal{S})$.

Our remark above about semi-simple classes and closure properties suggests that upper radical classes might be problematical in some situations; in fact, in the class of all non-associative (= not necessarily associative) rings, some classes do not define upper radical classes; see, e.g., [2].

We shall often abbreviate "radical class" to "radical". Mostly our terminology (apart from some notation given in this Introduction) conforms to that of [3], to which we refer for further information about radical theory. In particular proofs of claims made above can be found there unless we have given another reference. The best general reference for radicals of structures other than associative rings is [4].

Although later in the paper we discuss various kinds of non-associative rings and algebras, in the absence of an explicit indication to the contrary, all rings are associative. We emphasize that we mean"rings" and not "rings with identity", the latter being equipped with an extra fundamental nullary operation.

2 Special radicals

Special radicals, the principal concern of both parts of Andrunachievici's fundamental paper [1], are the upper radicals defined by certain classes of prime rings. A *special class* is a non-empty class $\mathcal{P}$ of prime rings which is hereditary for non-zero ideals (we don't count the one-element ring as a prime ring) and closed under essential extensions. (If $I \triangleleft A$ then I is an *essential ideal* if $I \cap J \neq 0$ for all non-zero ideals J of A; A is then called an *essential extension* of I). In the original definition a condition involving annihilators was used, and this was subsequently shown to be equivalent to closure under essential extensions [5], [6]. (In the latter case the proof is attributed to E. H. Connell.)

A radical class $\mathcal{R}$ is *special*, if $\mathcal{R} = U(\mathcal{P})$ for some special class $\mathcal{P}$ of prime rings.

Some of the best known radicals are special: for the prime, Jacobson and Brown-McCoy radicals we can take $\mathcal{P}$ to be the classes of all prime rings, all left (or right) primitive rings and all simple rings with identity, respectively. The locally nilpotent and nil radicals are also special. Special radicals can also be characterized intrinsically.

Theorem 1. *(See [1]) A radical class $\mathcal{R}$ is special if and only if every $\mathcal{R}$-semi-simple ring is a subdirect product of prime $\mathcal{R}$-semi-simple rings. Consequently all nilpotent rings are contained in every special radical class.*

In [1] there was an extra condition: that $\mathcal{R}$ be hereditary, but it was subsequently shown by Beidar [7] that this is implied by by the other conditions.

So special radical classes are hereditary and contain all nilpotent rings. Radical classes with these properties are said to be *supernilpotent* (though this terminology is not universally used). Is the subdirect product representation condition also redundant? In other words, are there supernilpotent radicals which are not special? Yes, there are, so that condition is *not* redundant.

Theorem 2. *Supernilpotent radicals need not be special.*

The first examples were produced by Ryabukhin.

Example 1. [8] The class $\mathcal{T}$ of boolean rings without ideals isomorphic to the two-element field has an upper radical which is supernilpotent but not special, essentially because the only prime homomorphic images of rings in $\mathcal{T}$ are in $U(\mathcal{T})$.

Example 2. [9] For every infinite cardinal $\mathfrak{m}$ the class $\mathcal{M}_{\mathfrak{m}}$ of rings A with $|A| \leq \mathfrak{m}$ defines a lower radical class $L(\mathcal{M}_{\mathfrak{m}})$ such that if $\mathcal{R}$ is a hereditary radical class and $\mathcal{B} \subsetneq \mathcal{R} \subseteq L(\mathcal{M}_{\mathfrak{m}})$then $\mathcal{R}$ is supernilpotent but not special because there are $\mathcal{R}$-semi-simple rings whose only prime homomorphic images are small enough to be in $\mathcal{R}$. (Here $\mathcal{B}$ is the prime radical class, the lower radical class defined by the class of all nilpotent; also the upper radical class defined by the class of all prime rings.) In particular the radicals $L(\mathcal{M}_{\mathfrak{m}})$ are supernilpotent and non-special. They are also *strongly hereditary* in the sense that all subrings of radical rings are themselves radical. This is because the lower radical defined by a strongly hereditary class is itself strongly hereditary [10].

Theorem 3. *A strongly hereditary supernilpotent radical need not be special.*

We mention one more example, or family of examples, due to Beidar and Salavová [11].

Example 3. Let $\mathcal{R}_n$ be the upper radical class defined by the class of semiprime rings which do not have ideals which are prime rings, and which satisfy the standard identity of degree $2n$ but not that of degree $2(n-1)$. Let $\mathcal{S}_n$ be the corresponding semi-simple class. Then all the $\mathcal{R}_n$ are supernilpotent and non-special. Also $\mathcal{S}_m \cap \mathcal{S}_n = 0$ when $m \neq n$.

The radicals $\mathcal{R}_n$ in Example 3 are pairwise incomparable. A similar collection of radicals for algebras over a field was found by Ryabukhin [12].

Examples 1-3 are based on semiprime rings which don't have ideals which are prime rings of some kind. This might suggest that rings with no ideals at all which are prime rings might be worth looking at. That very class of rings, under the name *prime essential rings* (as they are also characterized by the property that all their prime ideals are essential ideals) was introduced by Rowen [13], but for other purposes. Connections between such rings and radical theory were given some attention by Gardner and Stewart [14], but the definitive result is due to France-Jackson [15].

Theorem 4. *[15] Every supernilpotent radical which properly contains the prime radical and for which all prime essential rings are semi-simple is non-special.*

The same author constructed infinitely many supernilpotent radicals for which there are *no* prime semi-simple rings [16]. Clearly these are not special. Note that having no prime semi-simple rings is not the same as having all prime rings in the radical class; the latter condition requires that all rings be radical, as the radical class must contain all free rings.

Jaegermann and Sands [17] gave an example of a non-special N-*radical*. An N radical is a supernilpotent radical which is left hereditary and left strong, i.e. left ideals of radical rings are radical and the radical of any ring contains all radical left ideals. Thus we have

Theorem 5. *A left hereditary, left strong supernilpotent radical need not be special.*

A radical class $\mathcal{R}$ is *strict* if for every ring A, $\mathcal{R}(A)$ contains all subrings of A which are in $\mathcal{R}$. As a contrast to Theorems 3 and 5, it appears not to be known whether a strict supernilpotent radical must be special.

There are some conditions on a supernilpotent radical which do force it to be special. Stewart [18] showed that a supernilpotent radical class $\mathcal{R}$ with the property that a ring A is in $\mathcal{R}$ if and only if all its one-generator subrings are in $\mathcal{R}$ must be special. The same author [19] showed that a supernilpotent radical whose semi-simple class $\mathcal{S}$ has the property that $A \in \mathcal{S}$ if and only if all one-generator subrings are in $\mathcal{S}$ must be special, but such radicals are probably very few. A supernilpotent radical whose semi-simple class is closed under (semi)prime homomorphic images must be special [20]. This includes the upper radicals defined by semi-simple radical classes, but these are obviously special, since a semi- simple radical class is the variety generated by some finite set of finite fields.

Since Snider [21] observed that various classes of radicals are large complete lattices (or "lattices": they are not sets) and initiated their investigation, the lattice of special radicals has attracted much attention. If $\mathcal{R}_i, i \in I$ are special radicals, we get our lattice operations by defining $\bigwedge \mathcal{R}_i$ to be $\bigcap \mathcal{R}_i$ and $\bigvee \mathcal{R}_i$ to be the smallest special radical class containing $\bigcup \mathcal{R}_i$. Such a smallest special radical exists for every class; a construction is given in [22]. In other cases (all radicals, all supernilpotent radicals and so on) the operations are defined analogously. Snider showed ([21], pp. 210-211) that although the special radicals form a lattice, it is not a sublattice of the lattice of all radicals.

It might seem plausible that the upper radical defined by a single simple ring with identity is a *coatom* in the lattice of special radicals, but this is not necessarily so. For example take the field $\mathbb{Q}$ of rational numbers. The polynomial ring $\mathbb{Q}[X]$ is a subdirect product of copies of $\mathbb{Q}$ and so is semi-simple with respect to $U(\mathbb{Q})$. But $\mathbb{Q} \in U(\mathbb{Q}[X])$, so $U(\mathbb{Q}) \subsetneq U(\mathbb{Q}[X])$. The coatoms were described by Krachilov [23].

Theorem 6. *[23] A radical $\mathcal{R}$ is a coatom in the lattice of special radicals if and only if $\mathcal{R} = U(M_n(F))$, where $M_n(F)$ is the ring of $n \times n$ matrices over a finite field F, for some n.*

The *atoms* in the lattice of special radicals have proved to be much more elusive, and have not yet been completely characterized. Before we give an account of what is currently known about atoms we should explain some of the terminology which will be used.

Every class $\mathcal{C}$ of rings is contained in a *smallest special radical class* which we shall call $L_s(\mathcal{C})$ (or $L_s(A)$ if $\mathcal{C}$ has a single member A). A construction is given in [22]. Each prime ring A is contained in a *smallest special class* denoted by π_A, described as follows.

Proposition 1. *[24] For a prime ring A, π_A consists of all prime rings which have a non-zero ideal isomorphic to an accessible subring of A.*

We also denote the class of all prime rings by π. As well as idempotent simple rings, three types of rings play roles in the discussion of special atoms.

(i) A $*$-ring [25] is a non-zero semiprime ring A such that $A/I \in \mathcal{B}$ for every non-zero ideal I. Such a ring must be prime.

(ii) A PEI-ring [26] is a prime ring A such that $A/I \in \pi_A$ for all prime ideals I.

(iii) A $**$ -ring [28] is a prime ring A for which each semiprime homomorphic image is in π_A.

Clearly $* \Rightarrow **$ and $** \Rightarrow PEI$. A ring constructed by Leavitt and van Leeuwen [29] which is prime and isomorphic to all its non-zero homomorphic images is a $**$ -ring and hence a PEI -ring but not a $*$ -ring. No other implications are known. Note, though, that idempotent simple rings are $*$ -rings.

The first result on special atoms is due to Ryabukhin [9].

Theorem 7. *[9] For every idempotent simple ring T, $L_s(T)$ is an atom in the lattice of special radicals.*

Further examples were provided by Korolczuk [25].

Theorem 8. *[25] If A is a $*$-ring then $L_s(A)$ is an atom in the lattice of special radicals. There are $*$-rings B such there are no simple rings T for which $L_s(B) = L_s(T)$.*

For a prime ring A, $\pi\backslash\pi_A$ is a special class, informally a rather large one, so its upper radical is a "rather small" special radical, and a plausible candidate for an atom.

Proposition 2. *For every prime ring A we have $U(\pi\backslash\pi_A) \subseteq L_s(A)$.*

Proof. If $0 \neq R \in U(\pi\backslash\pi_A)$ and I is a prime ideal of R, then $R/I \notin \pi\backslash\pi_A$, i.e. $R/I \in \pi_A$. Hence R/I has a non-zero ideal K/I isomorphic to an accessible subring of A and therefore in $L_s(A)$. Thus (by Theorem 4 and Proposition 5 of [22]) $R \in L_s(L_s(A)) = L_s(A)$. $\square$

Proposition 3. *For a prime ring A, $L_s(A) = U(\pi\backslash\pi_A)$ if and only if A is a PEI-ring.*

Proof. We show that $L_s(A) \subseteq U(\pi\backslash\pi_A)$ if and only if A is PEI. Since $U(\pi\backslash\pi_A)$ is special, this is equivalent to showing that $A \in U(\pi\backslash\pi_A)$ if and only if A is PEI. If $A \in U(\pi\backslash\pi_A)$ and J is a prime ideal of A, then A/J must be in π_A and hence A is

PEI. Conversely, if A is PEI, suppose $A \notin U(\pi\backslash\pi_A)$. Then as $\pi\backslash\pi_A$ is hereditary for non-zero ideals, A has a homomorphic image $B \in \pi\backslash\pi_A$. But since B is prime it must be in π_A. From this contradiction we conclude that $A \in U(\pi\backslash\pi_A)$. □

Proposition 4. *If A is a prime ring and $L_s(A)$ is an atom, then $L_s(A) = U(\pi\backslash\pi_A)$ or $U(\pi\backslash\pi_A) = \mathcal{B}$.*

Proof. By Proposition 2 $U(\pi\backslash\pi_A) \subseteq L_s(A)$, so it must be either $L_s(A)$ or $\mathcal{B}$. □

Corollary 1. *If A is prime and $L_s(A)$ is an atom in the lattice of special radicals, then either $L_s(A) = U(\pi\backslash\pi_A)$ or A is a subdirect product of rings in $\pi\backslash\pi_A$.*

Proof. If $L_s(A) \neq U(\pi\backslash\pi_A)$ then $U(\pi\backslash\pi_A) = \mathcal{B}$, so A, like every semiprime ring, is a subdirect product of rings in $\pi\backslash\pi_A$. Conversely, if A is such a subdirect product, then A is $U(\pi\backslash\pi_A)$-semi-simple, so $U(\pi\backslash\pi_A) \neq L_s(A)$. □

These results lead, though not immediately, to the following theorem, whose proof can be put together straightforwardly from the cited results of France-Jackson and Liang Zhian.

Theorem 9. *(Liang Zhian [26], Theorem 4, France-Jackson [27], Theorem 2) If A is a PEI-ring then $L_s(A)$ is an atom in the lattice of special radicals. Every special atom has the form $L_s(B)$ for a prime ring B such that either B is PEI or $U(\pi\backslash\pi_B) = \mathcal{B}$.*

France-Jackson and Groenewald [28] proved that $L_s(A)$ is a special atom for every $**$-ring, but $**$-rings have PEI.

Although subsequent work has raised possibilities, no examples of special atoms have yet come to light beyond those described in Theorems 7 and 8. The only known example of a PEI-ring which is not a $*$-ring (it is also a $**$-ring) is the ring of [29] mentioned above, call it V. Then V is isomorphic to all its non-zero homomorphic images and is prime. The proper ideals of V form a chain

$$0 = H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots \subseteq H_n \subseteq H_{n+1} \subseteq \dots,$$

where H_1 is (idempotent) simple and $H_{n+l}/H_n \cong H_1$ for each n. Since $H_1 \in L_s(V)$ and $L_s(V)$ is an atom, we have $L_s(V) = L_s(H_1)$. Another paper of France-Jackson [30] contains an example of a pure essential ring (so not a $*$-ring) which generates a special atom. This ring is a semigroup ring ring $\mathbb{Z}_2[D]$ over the two-element field $\mathbb{Z}_2$. But $\mathbb{Z}_2$ is a homomorphic image of $\mathbb{Z}_2[D]$, so $L_s(\mathbb{Z}_2[D]) = L_s(\mathbb{Z}_2)$.

In a rather imprecise sense, the smaller a special class, the larger the special radical (namely, its upper radical) it defines. For a prime ring A, π_A is a minimal special class, since $\pi_B = \pi_A$ for every $B \in \pi_A$, but only when A is a finite idempotent simple ring is $U(\pi_A)$ a coatom, as we saw in Theorem 6; in other cases there are plenty of subdirect products of rings in π_A that can aggregate to form special classes defining special radicals bigger than $U(\pi_A)$. There are two ways in which π_A can be "as small as possible". One is where π_A consists of all essential extensions of A, so that A is simple and π_A is the class of subdirectly irreducible rings with hearts isomorphic to A. The other, which we shall now consider, is where π_A consists of the non-zero ideals of A (so that A has no proper essential extensions, i.e. A has an identity).

We are able to characterize these rings only under an extra hypothesis. In [31] it is claimed that this characterization applies without the extra hypothesis, but there is an error in the proof and we don't know whether this hypothesis is needed or not. Here then is our theorem.

Theorem 10. *Let A be a prime ring such that every isomorphism between ideals of A extends to an automorphism of A. Then π_A is the set of non-zero ideals of A if and only if*

(i) A is a simple ring of characteristic 0 *with identity, or*

(ii) $pA = 0$ for some prime p and A is either a simple ring with identity or a certain ring which is not an algebra over any field but $\mathbb{Z}_p$ and has only one proper ideal, or

(iii) A is additively torsion-free and reduced and a principal ideal domain such that every proper homomorphic image is isomorphic to some $\mathbb{Z}_n$.

Proof. Let A be a prime ring such that π_A is the set of non-zero ideals of A. If I is a non-zero ideal of A , then the standard unital extension $I * \mathbb{Z}$ has a homomorphic image I_* such that I is (isomorphic to) an essential ideal of I_*. Hence I_* is isomorphic to an accessible subring of A. But I_* has an identity and A is prime, so $I_* \cong A$. Thus A has an identity. Also I_* is generated by $I \cup \{1\}$, so A is generated by $J \cup \{1\}$, where $J \triangleleft A$ and $J \cong I$. By our assumption, there is an automorphism f of A such that $f(I) = J$. But then $A = f(A) = f(\{j + n1 : j \in J, n \in \mathbb{Z}\}) = \{f(j) + n1 : j \in J, n \in \mathbb{Z}\}$, so A is generated by $I \cup \{1\}$. If now $K \triangleleft I$ then clearly $K \triangleleft A$. It now follows that *all accessible subrings of A are ideals* and A is generated by $I \cup \{1\}$ for every non-zero ideal I.

The rest of the proof more or less follows the second part of the proof offered for the theorem in [31] (which seems to be correct: we show that our rings are injectives in a certain category and invoke Theorems 2.5 and 2.13 of [33] and Theorem of [34] for their structure. □

The rings in (iii) of Theorem 10 form an interesting class. If A is such a ring then for every prime p, either $pA = A$ or A is a p-pure subring of the ring of p-adic integers ([34], Theorem). Since $pA \neq A$ for at least one p, it follows that $|A| \leq 2^{\aleph}_0$. Also every non-zero element of A is an integer multiple of a unit. The rings in (iii) are also the integral domains, other than fields, in which all accessible subrings are ideals: see Theorems 3.1 and 3.2 of [32].

For rings which are algebras over fields other than the $\mathbb{Z}_p$ we don't need the assumption concerning isomorphic ideals. Note that this includes torsion-free divisible rings.

Proposition 5. *Let A be a prime ring which is an algebra over a field K other than the fields $\mathbb{Z}_p$. If π_A is the set of non-zero ideals of A then A is a simple ring with identity.*

Proof. (i) If K has prime characteristic p, then $pA = 0$. If I is a non-zero ideal of A without an identity, we can adjoin the identity of $\mathbb{Z}_p$ to get the ring $I * \mathbb{Z}_p$ and this has a homomorphic image I_{*p} in which (up to isomorphism) I is an essential ideal. Hence $I_{*p} \in \pi_A$ and thus it is isomorphic to an accessible subring of A. Since A is prime and I_{*p} has an identity, we have $A \cong I_{*p}$. This means that A has an identity and A is generated by $J \cup \{1\}$ for some ideal J which is isomorphic to I. But then $A/J \cong \mathbb{Z}_p$, while as A has an identity, all its ideals are K-algebra ideals, so $\mathbb{Z}_p$ is a K-algebra. This is impossible, so there is no non-zero ideal without an identity. It follows that A is a simple ring with identity.

(ii) If K has characteristic 0 we argue similarly, using $I * \mathbb{Z}$ instead of $I * \mathbb{Z}_p$ ($\mathbb{Z}$ can't be a K-algebra). □

Thus although for torsion-free reduced rings our result is only valid modulo the assumption about isomorphic ideals, we now know a fair bit about when π_A is the set of non-zero ideals, since the requirement of primeness places some restrictions on the additive group of a ring.

Proposition 6. *If A is a prime ring then either $pA = 0$ for a unique prime p or A is torsion-free.*

For a proof see, e.g. [35], Theorem 4.1.1, p. 36. Note that a torsion-free prime ring need not be either additively divisible or reduced; e.g. consider the subring $\mathbb{Z}+X\mathbb{Q}[X]$ of the polynomial ring $\mathbb{Q}[X]$ over the rationals. However such a "blended" ring cannot generate a special class consisting of ideals.

Proposition 7. *Let A be a torsion-free prime ring such that π_A is the set of non-zero ideals of A. Then additively A is either divisible or reduced.*

Proof. Let A be a prime ring such that π_A is the set of non-zero ideals and $\mathcal{D}(A) \neq 0$, where $\mathcal{D}$ is the radical class of all divisible rings. Let $\mathcal{D}(A) * \mathbb{Q}$ be the ring obtained by the adjunction to $\mathcal{D}(A)$ of the identity of $\mathbb{Q}$. An argument by now familiar shows that $\mathcal{D}(A) * \mathbb{Q}$ has a homomorphic image $\mathcal{D}(A)_{*\mathbb{Q}}$ which has an identity and is in π_A, whence $A \cong \mathcal{D}(A)_{*\mathbb{Q}} \in \mathcal{D}$. We have shown that if A is not reduced it is divisible. □

It is not known whether there are any prime rings (necessarily torsion-free and reduced additively) which don't satisfy the requirement on isomorphic ideals in Theorem 10, yet whose non-zero ideals (or accessible subrings) form a special class. Here is an example of a prime ring which is quite similar to the rings of Theorem 10 (iii) – in particular it satisfies the ideal isomorphism condition – but whose non-zero ideals do not form a special class.

Example 4. The ring $\mathbb{G}_{(3)}$, the localization of the Gaussian integers $\mathbb{G}$ at the prime 3, is additively torsion-free and reduced and is a principal ideal domain. Its proper ideals are the principal ideals (3^n) generated by the powers of 3. If $f : (3^m) \to (3^n)$ is an isomorphism, then

$$3^m f(3^m) = f(3^m 3^m) = f(3^m)f(3^m),$$

so, as $f(3^m) \neq 0$,we have $3^m = f(3^m) \in (3^n)$, whence $m \geq n$. But similarly $3^n = f^{-1}(3^n) \in (3^m)$, so $n \geq m$. Thus f is the identity map, the restriction of the identity automorphism of $\mathbb{G}_{(3)}$. But (e.g.) in $\mathbb{G}_{(3)}$, if $k \cdot 1 + \ell i \equiv 0 (mod 3)$ then $3|k$ and $3|\ell$. It follows that $\mathbb{G}_{(3)}/3\mathbb{G}_{(3)}$, additively a group of exponent 3, is not cyclic; in particular it is not isomorphic to any $\mathbb{Z}_n$, so the ideals of $\mathbb{G}_{(3)}$ don't form a special class.

There have been investigations of atoms in various other lattices of radicals (see, e.g., [36]). We shall not discuss these beyond noting some connections between supernilpotent atoms and special atoms. Every class $\mathcal{C}$ of rings is contained in a smallest supernilpotent radical class, namely $L_h(\mathcal{C}) = L(\widehat{\mathcal{C}} \cup \mathcal{B})$, where $\widehat{\mathcal{C}}$ is the hereditary closure of $\mathcal{C}$. For a class $\mathcal{C}$ which is hereditary and closed under prime homomorphic images, the smallest special radical class $L_s(\mathcal{C})$ containing it is

$$\{A : \text{every prime homomorphic image of } A \text{ has a non-zero ideal in } \mathcal{C}\}$$

[22]. In particular, for a ring A we have

$$L_h(A) = L(\mathcal{B} \cup \{R : R \text{ is an accessible subring of } A\}) \text{ and}$$

$$L_s(A) = \{R : \text{every prime homomorphic image of } R \text{ has a non-zero ideal which is a prime homomorphic image of an accessible subring of } A\}.$$

With these two characterizations before us, this is a convenient point at which to give a result which shows why all special atoms have the form $L_s(A)$ for a prime ring A as well as illuminating aspects of some of the results given so far. A ring is *antisimple* if it has no homomorphic image which is subdirectly irreducible with idempotent heart. The class $\mathcal{B}_\varphi$ of antisimple rings is a special radical class. All of this too originated in [1].

Theorem 11. *(1) Let $\mathcal{R}$ be an atom in the lattice of special radicals. Then $\mathcal{R} = L_s(A)$ for some prime ring A.*

(2) Let A be a prime ring for which $L_s(A)$ is an atom in the lattice of special radicals. The following conditions are equivalent.

(i) $L_s(A) = L_s(T)$ for some idempotent simple ring T.

(ii) A has an idempotent simple ideal.

(iii) $A \notin \mathcal{B}_\varphi$.

All of these statements are true if "supernilpotent" replaces "special" and $L_h()$ replaces $L_s()$ throughout.

Proof. (1) If $\mathcal{R}$ is a special atom, then $\mathcal{R} \neq \mathcal{B}$ so there is a ring $R \in \mathcal{R}$ with $R/\mathcal{B}(R) \neq 0$. Then $R/\mathcal{B}(R)$ has a prime homomorphic image A. We then have

$$\mathcal{B} \neq L_s(A) \subseteq L_s(R/\mathcal{B}(R)) \subseteq L_s(R) \subseteq \mathcal{R},$$

whence, as $\mathcal{R}$ is an atom, we have $L_s(A) = \mathcal{R}$.

(2) *(i)* $\Rightarrow$ *(ii)*. If $L_s(A) = L_s(T)$, where T is idempotent and simple, then $A \in L_s(T)$ so by our characterization, A has an ideal isomorphic to T.

$(ii) \Rightarrow (i)$. If A has a simple ideal V, then $V \in L_s(A)$, so $L_s(V) \subseteq L_s(A)$. But the latter is an atom, so the two radicals are equal.

$(ii) \Rightarrow (iii)$. If A has an idempotent simple ideal V, let J be an ideal of A such that $J \cap V = 0$ and J is maximal for this property. Then A/J is subdirectly irreducible with heart (isomorphic to) V.

$(iii) \Rightarrow (i)$. Let A/I be subdirectly irreducible with idempotent heart H. Then as $L_s(A)$ is an atom, we have $L_s(A) = L_s(A/I) = L_s(H)$. □

Corollary 2. *Let $\mathcal{R}$ be an atom in the lattice of special radicals. Then $\mathcal{R}$ is not defined by a simple ring if and only if $\mathcal{B} \subsetneq \mathcal{R} \subseteq \mathcal{B}_\varphi$, so $\mathcal{R}$ is defined by a simple ring if and only if $\mathcal{R} \cap \mathcal{B}_\varphi = \mathcal{B}$.*

Remark 1. It follows from Theorem 11 that if B is a ring with a non-zero ideal $I \in \mathcal{B}$ such that B/I is a prime ring, then $L_s(B) = L_s(B/I)$ and $L_h(B) = L_h(B/I)$, so it's easy to manufacture examples of non-$*$-rings and so on which define special or supernilpotent atoms, cf. [37].

There is a connection between the two types of atoms.

Theorem 12. *(Puczyłowski and Roszkowska [36], Propositiom 12) Let A be a prime ring. If $L_h(A)$ is an atom in the lattice of supernilpotent radicals, then $L_s(A)$ is an atom in the lattice of special radicals.*

Theorem 12 has a partial converse.

Proposition 8. *(Puczyłowski and Roszkowska [36], Proposition 13) Let A be a prime ring such that $L_s(A)$ is an atom in the lattice of special radicals. If $\mathcal{R}(A) = A$ or* 0 *for every supernilpotent radical $\mathcal{R}$, then $L_h(A)$ is an atom in the lattice of supernilpotent radicals.*

In [36] it was asked whether there is a non-$*$-ring which generates a special radical. An example of such a ring is given in [30]: a certain semigroup ring over $\mathbb{Z}_2$ which is boolean and prime essential. But (cf. the proof of Theorem 11) such rings are plentiful, and the interesting question is whether there is a *prime* ring which is not a $*$-ring but generates a special atom. The example in [30] is also pertinent to the question of the validity of the full converse to Theorem 12. Certainly that ring generates a special atom but not a supernilpotent atom, but for all types of radicals (including the special and supernilpotent ones) the ring generates the same radical as $\mathbb{Z}_2$. The full converse of Theorem 12 would assert that all prime rings generating special atoms also generate supernilpotent atoms. By Proposition 8 any counterexample would have to be a prime non-$*$-ring.

It should not be supposed that under the conditions of Theorem 12 $L_s(A) = L_h(A)$. These two rings can in fact never be equal. We prove something a bit more general.

Theorem 13. *Let $\mathcal{C}$ be a class of rings such that for some cardinal number $\mathfrak{k}$ we have $|A| < \mathfrak{k}$ for each $A \in \mathcal{C}$. Then $L_h(\mathcal{C}) = L_s(\mathcal{C})$ if and only if $\mathcal{C} \subseteq \mathcal{B}$.*

Proof. We can assume that $\mathcal{C}$ is homomorphically closed and hereditary. If $\mathcal{C} \subseteq \mathcal{B}$ then trivially $L_h(\mathcal{C}) = \mathcal{B} = L_s(\mathcal{C})$.

If $\mathcal{C} \nsubseteq \mathcal{B}$ then $\mathcal{C}$ contains a non-zero semiprime ring A. Let E be a linearly ordered set with a smallest element but no largest element, such that $|[x,y]| = \mathfrak{k}$ for all intervals $[x,y]$ in E ($x < y$). Now E becomes a semigroup (semilattice) when we define $zw = max\{z,w\}$ for all $z,w \in E$. For the semigroup ring $A[E]$ we have the following.

(i) $A[E]$ is semiprime.

(ii) Every ideal has cardinality $\mathfrak{k}$.

(iii) Every prime homomorphic image is isomorphic to a prime homomorphic image of A.

For all this, as well as a demonstration that a suitable E exists, see [8], Lemmas 8 and 9; also [14], Lemma 1.

From (iii) we see that $A[E] \in L_s(\mathcal{C})$, while from (i) and (ii) $A[E]$ has no non-zero accessible ideal in $\mathcal{B}$ or in $\mathcal{C}$, so $A[E]$ is $L_h(\mathcal{C})$-semi-simple. $\square$

Corollary 3. *For a ring A, we have $L_h(A) = L_s(A)$ if and only if $A \in \mathcal{B}$.*

Corollary 4. *([24], Corollary 3.4.1) No atom in the lattice of supernilpotent radicals is special.*

The conclusion of Theorem 13 need not be true for a class of rings of unbounded cardinality.

Example 5. Let $\mathcal{C}_2$ be the class of commutative rings. Then

$$L(\mathcal{C}_2) = \{R : R/\mathcal{B}(R) \in \mathcal{C}_2\}$$

by Collary 3.7 of [38]. If $R \in L_s(\mathcal{C}_2)$ then $R/\mathcal{B}(R) \in L_s(\mathcal{C}_2)$ so every prime homomorphic image of $R/\mathcal{B}(R)$ has a non-zero ideal in $\mathcal{C}_2$.

But in general, if I is a non-zero commutative ideal of a prime ring A, then for all $a, b \in A$, $i, j \in I$ we have
$i(ab - ba)j = iabj - ibaj = ia \cdot bj - i \cdot baj = bj \cdot ia - baj \cdot i = b \cdot jia - ba \cdot ji = b(j \cdot ai) - baji = b(ai \cdot j) - baji = baij - baij = 0$.
Hence $I(ab - ba)I = 0$ for all $a, b \in A$. But A is prime and $I \neq 0$, so $I(ab - ba) = 0$ and then for the same reason $ab - ba = 0$ for all a, b, i.e. A is commutative.

Thus, returning to our argument, each prime homomorphic image of the semiprime ring $R/\mathcal{B}(R)$ is commutative, so $R/\mathcal{B}(R)$ is in $\mathcal{C}_2$. This proves that $L_s(\mathcal{C}_2) \subseteq L(\mathcal{C}_2)$. But for *every* class $\mathcal{X}$ we have

$$L(\mathcal{X}) \subseteq L_h(\mathcal{X}) \subseteq L_s(\mathcal{X}),$$

so

$$L(\mathcal{C}_2) = L_h(\mathcal{C}_2) = L_s(\mathcal{C}_2).$$

Remark 2. It is part of folklore that if A is a prime ring and I a non-zero ideal satisfying any polynomial identity, then A satisfies that identity. See, e.g. [39] p. 309. There doesn't appear to be any printed proof of this result, but in any case the simple proof for commutativity seems worth presenting.

In his detailed examination of examples of special classes in [1], Andrunachievici considered classes of idempotent simple rings, showing, in §5 of Part II that a class $\mathcal{M}$ of simple rings is special if and only if each of its members has an identity. Thus $\mathcal{M}$ consists of rings with identity if and only if

(i) $U(\mathcal{M})$ is hereditary and

(ii) all $U(\mathcal{M})$-semi-simple rings are subdirect products of rings in $\mathcal{M}$.

It was natural then to ask what requirements (i) and (ii) individually impose on $\mathcal{M}$, and both questions were answered by Leavitt [40]. He showed that (ii) holds if and only if each ring in $\mathcal{M}$ has an identity. (This was some years before Beidar [7] showed that the hereditary property follows from the other properties in the characterization of special radicals from [1].) He also showed that for each prime p there exist idempotent simple rings of characteristic p without identity which can belong to a class $\mathcal{M}$ satisfying (i) if and only if the corresponding field $\mathbb{Z}_p$ is also in $\mathcal{M}$. Incidentally, the mysterious "certain rings ..." of Theorem 10 (ii) are the standard $\mathbb{Z}_p$-unital extensions of these non-unital simple rings of Leavitt.

If $\mathcal{R}$ is a radical class containing no non-zero nilpotent rings, then all rings in $\mathcal{R}$ are idempotent. If furthermore $\mathcal{R}$ is hereditary, all *ideals* of all rings in $\mathcal{R}$ are idempotent. Under these conditions $\mathcal{R}$ is said to be *subidempotent*. Subidempotent radicals are in a sense "opposite" to supernilpotent radicals (which contain all nilpotent rings). In §3 of Part I of [1], Andrunachievici defined a Galois correspondence between supernilpotent radicals and subidempotent radicals, associating with each radical $\mathcal{R}$ of either kind a radical $\mathcal{R}'$, namely the upper radical defined by the class of subdirectly irreducible rings with hearts in $\mathcal{R}$. Thus if $\mathcal{R}$ is supernilpotent, then so is $\mathcal{R}''$ and the latter is the upper radical defined by the subdirectly irreducible rings with $\mathcal{R}$-semi-simple hearts. In fact $\mathcal{R}''$ is special, and the class of subdirectly irreducibles with $\mathcal{R}$-semi-simple hearts is a special class. In general an upper radical defined by a special class of subdirectly irreducibles is called a *dual special* radical.

In [20] we introduced a type of special radical intermediate between special radicals in general and dual special radicals.

Let $\mathcal{R}$ be a supernilpotent radical with semi-simple class $\mathcal{S}$. A ring $B \in \mathcal{S}$ is $\mathcal{S}$-*subdirectly irreducible* if $B \neq 0$ and $\bigcap\{I \triangleleft B : I \neq 0 \& B/I \in \mathcal{S}\} \neq 0$. A supernilpotent radical class $\mathcal{R}$ with semi-simple class $\mathcal{S}$ is called *extraspecial* if each ring in $\mathcal{S}$ is a subdirect product of $\mathcal{S}$-subdirectly irreducible rings. The $\mathcal{S}$-subdirectly irreducible rings are precisely the essential extensions of rings in $\mathcal{S}$ whose proper homomorphic images are in $\mathcal{R}$. E.g. when $\mathcal{R} = \mathcal{B}$ they are the essential extensions of $*$-rings.

Theorem 14. *([20], 1.11 Theorem and 1.16 Theorem) Let $\mathcal{R}$ be a supernilpotent radical with semi-simple class $\mathcal{S}$ such that if $\{I_\lambda : \lambda \in \Lambda\}$ is a chain of ideals of a ring A with each $A/I_\lambda \in \mathcal{S}$ then $A/\bigcup_{\lambda\in\Lambda} I_\lambda \in \mathcal{S}$. Then $\mathcal{R}$ is extraspecial and $\mathcal{S}$ is a quasivariety.*

The semi-simple class of an extraspecial radical need not be a quasivariety: e.g. dual special radicals don't generally have that property. Sometimes they do, though. Semi- simple radical classes are the varieties generated by finite sets of finite fields [41] and such a class consists of all subdirect products of the fields it contains. The case of the class (variety) of boolean rings was discussed from this point of view in

§7 of Part II of [1], so *the first example of a semi-simple radical class is also due to Andrunachievici.*

Theorem 15. *([20], 1.21 Proposition and 2.7 Proposition) For any supernilpotent radical $\mathcal{R}$ with semi-simple class $\mathcal{S}$, the $\mathcal{S}$- subdirectly irreducible rings form a special class. Hence if $\mathcal{R}$ is extraspecial, it is special. Moreover, in this case the class of $\mathcal{S}$-subdirectly irreducible rings is the smallest special class with $\mathcal{R}$ as its upper radical.*

Remark 3. It is not known whether a special radical must be extraspecial if there is a smallest special class defining it as its upper radical.

Some rings considered earlier provide a family of extraspecial radicals.

Example 6. Let A be a prime ring of one of the types described in Theorem 10 (so that π_A consists of all non-zero ideals of A). Then $U(\pi_A)$ is extraspecial. To see this, observe that the rings in (*iii*) have only torsion proper homomorphic images, and these are in $U(\pi_A)$, while if A is such a ring and $I \triangleleft J \triangleleft A$, then $I \triangleleft A$, so J/I, as an ideal of A/I, is a torsion ring and hence is in $U(\pi_A)$. In the non-simple characteristic p examples, π_A has just two members, a simple ring and an essential extension of this. In this and the remaining cases (where A is simple) the radical concerned is a dual special radical. Thus in all cases π_A consists of $\mathcal{S}_A$-subdirectly irreducible rings, where $\mathcal{S}_A$ is the semi-simple class of $U(\pi_A)$.

If we take note of Theorem 15 it is clear that

$$\text{dual special} \Rightarrow \text{extraspecial} \Rightarrow \text{special.}$$

Neither implication is reversible. The generalized nil radical $\mathcal{N}_g$ is extraspecial, as its semi-simple class, the class of rings with no non-zero nilpotent elements, is the quasivariety defined by

$$x^2 = 0 \Rightarrow x = 0.$$

It is not dual special, however, since for instance the ring $\mathbb{Z}_{(2)} = \{\frac{2m}{2n+1} : m, n \in \mathbb{Z}\}$ is not a subdirect product of subdirectly irreducible rings without nilpotent elements.

The only example so far exhibited of a special radical which is not extraspecial is due to Beidar [42]. From Theorem 15 we see that a special radical which is the upper radical defined by two disjoint special classes cannot be extraspecial. Beidar provided an example of such a radical. Let $\mathbb{C}$ denote the field of complex numbers, $\mathbb{C}[X_1, X_2, \dots, X_n, \dots]$ the polynomial rings in countably many commuting indeterminates. Let $A = \mathbb{C}[X_1, X_2, \dots, X_n, \dots]$ and $B = A/I$, where I is the ideal generated by $X_1^2 + X_2^2 - 1$. We have

Theorem 16. *(Beidar [42]) The rings A and B generate disjoint special classes with the same upper radical. This radical is therefore special but not extraspecial.*

It is not known which, if any, of the "standard"' radicals are extraspecial. France-Jackson [43] has shown that the lattice of special radicals is atomic if and only if the prime radical is extraspecial. It is well known that the Jacobson radical is the upper radical defined by either of the distinct special classes of left primitive and right primitive rings. Sands has posed the interesting question: is the Jacobson radical the upper radical defined by the rings which are both left and right primitive? This question has an affirmative answer if the Jacobson radical is extraspecial.

Every special radical is the upper radical defined by at least one special class, e.g. the class of all semi-simple prime rings. Generally if we have a special class $\mathcal{K}$ we can find others with the same upper radical. If, say, $\mathbb{Z}$ and all the fields $\mathbb{Z}_p$ are in $\mathcal{K}$ we can discard $\mathbb{Z}$ and take the special class generated by what's left. In the other direction there may be a prime ring A which, while not in $\mathcal{K}$, is a subdirect product of rings which are. The special class generated by $\mathcal{K} \cup \{A\}$ will then have the same upper radical as $\mathcal{K}$. Some special radicals are, however, determined by a unique special class, which must be the class of all semi-simple prime rings.

Theorem 17. *(Vodyanyuk [44], Proposition 1, France-Jackson [45], Theorem 2) A special radical is the upper radical defined by a unique special class if and only if all prime $\mathcal{R}$-semi-simple rings are finite (and therefore finite fields or full matrix rings over such).*

Vodyanyuk goes on to characterize the corresponding radicals of algebras over an arbitrary commutative ring with identity.

Versions of special radicals have been introduced and studied for several other structures. The following list of pertinent references should be reasonably complete, but apologies in advance to anyone who's been left out!

Lattice-ordered rings: Shatalova, [46], [47]; Steinberg [48]; Shavgulidze [49], [50].

Rings with involution: Salavová [51], [52]; Booth and Groenewald [53]; Booth [54].

Graded rings: Balaba [55]. There is an extensive literature dealing with *individual* special radicals of graded rings.

Paragraded rings: Ilić-Georgijević and Vuković [56]. For paragraded structures see Krasner [57].

Operator groups: Ryabukhin [58]; Buys and Gerber [59]; Booth and Groenewald [60].

Nearrings: Kaarli [61]; Booth and Groenewald [62]; Birkenmeier, Heatherly and Lee [63]; Groenewald [64]. The situation with nearrings is complicated by the presence of several competing notions of "prime". Also some results are proved for arbitrary nearrings, some for zero-symmetric ones.

We saw above that a class of simple rings is a special class if and only if it consists of rings with identity. Re-phrasing this, we can say that the following conditions are equivalent for a class $\mathcal{M}$ of simple rings.

(i) $U(\mathcal{M})$ is hereditary and every $U(\mathcal{M})$-semi-simple ring is a subdirect product of rings in $\mathcal{M}$.

(ii) Every ring in $\mathcal{M}$ has an identity.

It follows from results of Suliński [65] that this equivalence persists for alternative rings. It also holds for right alternative rings ([66], Theorem 5.9) but not for power-associative rings ([66], Example 5.10). Suliński [65] asks whether (i) holds for Lie rings when $\mathcal{M}$ consists of complete simple rings, but it was shown by Andrunachievici and Ryabukhin [67] that it is not so.

3 The Lemma

The result which has come to be known as the Andrunachievici Lemma occurs in [1] (Part I, Lemma 4, p. 102). We promote it to a theorem for the occasion.

Theorem 18. *Let A be a ring, $I \triangleleft J \triangleleft A$, and let I^* be the ideal of A generated by I. Then $(I^*)^3 \subseteq I$.*

We give the short proof here so that we can later point to it to illustrate some of the problems that arise when attempts are made to generalize the result to other contexts.

Proof. We have
$$I^* = I + AI + IA + AIA$$
and $I^* \triangleleft J$, so
$$(I^*)^3 \subseteq JI^*J = JIJ + JA \cdot IJ + JI \cdot AJ + JA \cdot I \cdot AJ \subseteq JIJ \subseteq I.$$
□

Note that in non-associative rings the description of I^* is generally much more complicated, involving infinitely many terms.Note also that this proof we implicitly use both left and right distributivity of multiplication over addition; this will be pertinent to our later comments on nearrings.

This simple result is very useful in radical theory and in ring theory. For instance it provides simple proofs that

(i) if $I \triangleleft J \triangleleft A$ and J/I is semiprime, then $I \triangleleft A$;

(ii) if I is a minimal ideal then I is a(n idempotent) simple ring or $I^2 = 0$. In particular (ii) applies to hearts of subdirectly irreducible rings.

Not surprisingly, then, the possibility of generalizing The Lemma to other structures than (associative) rings has been extensively pursued. The question to ask in such contexts is "What is I^*/I like?" (We shall maintain the notation of Theorem 2.1, with $\triangleleft$ taking its appropriate meaning.)

Some contrast is provided by groups. In many ways the categories of groups and rings are alike, but in the case of groups, the question "What is I^*/I like?" has the answer "Nothing in particular.": every group can be an I^*/I.

Proposition 9. *Let G be any group, C_2 a cyclic group of order 2. Then in the wreath product $G \wr C_2$ we have, up to isomorphism,*
$$G \triangleleft G \times G \triangleleft G \wr C_2$$
and $G^* = G \times G$, so $G^*/G \cong G$.

This also shows that there is no non-trivial group H such that
$$A \triangleleft B \triangleleft C \& B/A \cong H \Rightarrow A \triangleleft C.$$

While we're dealing with groups, let us note also that a minimal normal subgroup need not be simple or abelian (as the ring case might lead one to suspect): consider the MacLain group (see [4], pp. 30-36, for example.)

We next consider another "associative" structure for which the possibility of an Andrunachievici Lemma has been explored. The Lemma can be formulated for zero-symmetric nearrings exactly as for rings, but is not always valid.

We refer to [68] for details concerning nearrings. We note that nearrings satisfy only one of the distributive laws, and which one to assume is a matter of taste and convenience. We shall not be doing many nearring calculations, but we'll assume left distributivity. We shall only consider *zero-symmetric* nearrings.

If a nearring is *distributive*, i.e. if it satisfies both distributive laws, then for $I \triangleleft J \triangleleft A$ we have $(I^*/I)^3 = 0$, and the proof is essentially identical to the ring one.

Kaarli [69] gave an example of a finite nearring for which The Lemma fails. This ring also has a minimal ideal U which is not simple and for which $U^2 \neq 0$. The problem of describing minimal ideals in nearrings seems to be difficult. Some further information is given in [70], [71] and [72].

Birkenmeier, Heatherly and Lee [73] studied a sort of localized version of The Lemma. They called an ideal J of A an $\mathcal{A}$-*ideal* if for each $I \triangleleft J$ we have $(I^*/I)^n = 0$ for some n. Then A itself is said to be an $\mathcal{A}$-*nearring* if all its ideals are $\mathcal{A}$-ideals. Thus as noted above, distributive nearrings are $\mathcal{A}$-nearrings. This can be strengthened considerably.

A nearring A is *distributively generated* (briefly *d.g.*) if it has a *multiplicative* subsemigroup S which generates A *additively* and satisfies the condition

$$(s+t)a = sa + ta \text{ (as well as } s(a+b) = sa + sb)$$

for all $s, t \in S$ and $a, b \in A$.

Theorem 19. *([73], Corollary 3.9) Every d.g. nearring A is an $\mathcal{A}$-nearring. Moreover, if $I \triangleleft J \triangleleft A$, then $(I^*)^4 \subseteq I$.*

In [73] a more general result is obtained using a generalization of the *d.g.* condition. It is also shown that nearrings satisfying certain identities, are $\mathcal{A}$-nearrings ([73], Proposition 4.1). There is also some information about minimal ideals:

Theorem 20. *([73], Proposition 5.3) If I is a minimal ideal of a nearring A and $I^2 \neq 0$ and if further I is an $\mathcal{A}$-ideal, then I is subdirectly irreducible and is either simple or Brown-McCoy radical. In particular if A is an $\mathcal{A}$-nearring, this conclusion applies to all minimal ideals.*

Now we'll look at two easy extensions of The Lemma to rings with an extra operation.

For rings with involution $\widehat{\ }$ (in which ideals are replaced by $\widehat{\ }$-invariant ideals), we have

Proposition 10. *([51], Lemma 2.12) If I is a $\widehat{\ }$-invariant ideal of J and J is a $\widehat{\ }$-invariant ideal of A let I^* be the $\widehat{\ }$-invariant ideal of A generated by I. Then $(I^*/I)^3 = 0$.*

For lattice-ordered rings (ℓ-rings) we denote by $< S >$ the ℓ-ideal generated by a subset S. We have the following variant of The Lemma.

Proposition 11. *([46], Lemma 2) If I is an ℓ-ideal of J and J is an ℓ-ideal of A, let I^* be the ordinary ring ideal of A generated by I. Then $< (I^*)^3 > \subseteq I$.*

Call an ℓ-ideal I *ℓ-idempotent* if $< I^2 >= I$. Then every ℓ-idempotent minimal ℓ-ideal is ℓ-simple. ([46], Remark 2, Note, p. 1086).

The story becomes much more complicated when we turn our attention to non-associative rings. The first case we consider will be that of alternative rings. We shall continue to use the notation $I \triangleleft J \triangleleft A$ and I^* as before.

In a sense *alternative rings* provide a fairly benign environment for radical theory. We can informally describe the situation by saying that "radical theory results for associative rings tend to hold also for alternative rings, but their proofs are much longer and/or messier". So it proves to be with The Lemma. Hentzel and Slater [74] showed that for alternative rings we have $\bigcap_{n\in\mathbb{Z}^+}(I^*)^n \cdot I^* \subseteq I$, with no restriction on torsion. This was somewhat improved by Pchelintsev [75] who showed that $(I^*)^{4\cdot 5^6} \subseteq I$, but only for algebras over a commutative ring containing $\frac{1}{6}$. Thus rings have to be effectively 6-torsion-free, and in particular for algebras over a field characteristics 2 and 3 are not covered. The latest result is due to Hentzel [76], who reduced the index but still had to place a restriction on torsion.

Theorem 21. *([76], Theorem 9) In any alternative ring A with $I \triangleleft J \triangleleft A$, I^* satisfies the condition*
$$x \in (I^*)^4 \Rightarrow (\exists n \in \mathbb{Z}^+)(3^n x \in I).$$

Corollary 5. *For alternative algebras over a ring containing $\frac{1}{3}$, we have $(I^*)^4 \subseteq I$ when $I \triangleleft J \triangleleft A$.*

This needs to be further explored; in particular we need to know what I^*/I is like in alternative rings over a field of characteristic 3. Characteristic 2 causes no problems though.

We get the hoped-for result on minimal ideals, and the theorem of Hentzel and Slater [74] cited above is good enough to give us a short proof.

Theorem 22. *In any alternative ring A, if K is a minimal ideal then $K^2 = 0$ or K is simple.*

Proof. Since A is alternative we have $K^2 \triangleleft A$, so $K^2 = 0$ or K. If $K^2 = K$ and $0 \neq I \triangleleft K$, let I^* denote the ideal of A generated by I as usual. Then $0 \neq I^* \subseteq K$, so by the minimality of K we have $I^* = K$. Also $(K/I)^n = K/I$ for each n. But now by the Theorem in [74] we have

$$0 = \bigcap_{n\in\mathbb{Z}^+}(I^*/I)^n \cdot (I^*/I) = \bigcap_{n\in\mathbb{Z}^+}(K/I)^n \cdot (K/I) = (K/I)^2 = K/I,$$

so $I = K$ and therefore K is simple. $\square$

It is instructive to examine the proof in [77] (pp. 169-170), due to Zhevlakov, that minimal ideals of alternative rings are simple or zerorings. This makes no use of anything like an Andrunachievici Lemma and is quite complicated.

For *(linear) Jordan algebras* over a commutative unital ring containing $\frac{1}{2}$, Slin'ko [78] proved a theorem which somewhat resembles The Lemma.

Theorem 23. *([78], Theorem 1) Let A be a Jordan algebra over a ring containing $\frac{1}{2}$. If $I \lhd J \lhd A$ and J/I has no non-zero nilpotent ideals, then $I \lhd A$.*

This is enough to prove

Theorem 24. *([78], Theorem 2) For Jordan algebras over a ring containing $\frac{1}{2}$, every radical class (hereditary or not) which contains all nilpotent algebras has a hereditary semi-simple class.*

Medvedev [79] showed that if A is a *finitely generated* Jordan algebra over a ring containing $\frac{1}{2}$ and $I \lhd J \lhd A$ then I^*/I is nilpotent, but constructed an example of a non-finitely generated algebra A with corresponding I, J for which I^*/I is not even solvable (and $I^2 = 0$).

Thus *there is no Andrunachievici Lemma for Jordan algebras.* There are more technical related results in [80] and [81].

We saw above that Slin'ko's Andrunachievicesque result (Theorem 23) shows that semi-simple classes of Jordan algebras are hereditary if all nilpotent algebras are radical. Subsequently Nikitin [82] has shown that the condition on nilpotent algebras is unnecessary.

Theorem 25. *([82], Theorem 3) All semi-simple classes of Jordan algebras over a ring containing $\frac{1}{2}$ are hereditary.*

Despite the absence of an Andrunachievici Lemma, the problem of characterizing minimal ideals of Jordan algebras has been solved. We have the following result of Skosyrskii [83], also proved (differently) by Medvedev [79].

Theorem 26. *([83], Corollary 3.1,[79], unnumbered theorem.) If I is a minimal ideal of a Jordan algebra, then I is simple or $I^2 = 0$.*

In the more general setting of *quadratic* Jordan algebras (without divisibility requirements) minimal ideals have also been described; see [84], 2.4 Corollary,[85], 3.1 Theorem.

In an arbitrary universal class of (not necessarily associative) rings it is not possible to say very much about the hearts of subdirectly irreducible rings; in particular they need not be simple or zerorings. If K is the heart of a subdirectly irreducible ring B, and if $K^2 \lhd B$, then as $K^2 \subseteq K$ we can say that $K^2 = 0$ or K. But K^2 *may not be an ideal.* This (among other pathological properties) is illustrated by the following example.

Example 7. Let A be a $\mathbb{Z}_2$-algebra with basis $\{u, v, w\}$ and multiplication given by the following table.

$\cdot$	u	v	w
u	0	w	v
v	w	0	w
w	v	w	0

Then A is subdirectly irreducible, its heart is $H =< v, w >$, the subspace spanned by $\{v, w\}$ and $H^2 =< v >$. Now A has the following properties.

- $H^2 \neq 0$ and H is not simple; in fact
- $H^2 \neq H$.
- $H \triangleleft A$ but $H^2 \not\triangleleft A$.

We have seen several examples of universal classes where the first of these properties is impossible. In associative and alternative rings the third is impossible. For Jordan algebras the square of an ideal need not be an ideal (though the cube must be) but the second property is not possible for linear Jordan algebras (Theorem 26).

In [86] a variety $\mathcal{W}$ of (not necessarily associative) rings used as a universal class is called an *Andrunachievici variety* of index n if for $I \triangleleft J \triangleleft A$ in $\mathcal{W}$ we have (in our standard notation) $(I^*/I)^{(n)} = 0$ and n (independent of I, J, A) is the smallest such integer. Note that the factor ring I^*/I is required to be *solvable*, rather than *nilpotent*; the two properties are equivalent for associative rings but not for alternative rings.

A variety $\mathcal{W}$ is called an *s-variety* if for every ideal M of every ring R in $\mathcal{W}$, the power M^s is also an ideal, and s is the smallest such integer. Here M^s is the set of finite sums of s-fold products of elements of M with arbitrary bracketings. Associative and alternative rings form 2-varieties, as do Lie rings and $(-1, 1)$-rings, those right-alternative rings satisfying the identity

$$(a, b, c) + (b, c, a) + (c, a, b) = 0,$$

where (a, b, c) is the *associator*: $(ab)c - a(bc)$ etc. Linear Jordan algebras over a ring containing $\frac{1}{2}$ form a 3-variety.

Theorem 27. *In a universal class which is an Andrunachievici variety of index n and an s-variety, let A have a minimal ideal J. Then $J^s = 0$ or J is (idempotent and) simple.*

Proof. Since $J^s \triangleleft A$ and $J^s \subseteq J$, we have $J^s = 0$ or J. In the latter case we have

$$J \supseteq J^J \supseteq J^s = J,$$

so $J^2 = J$. If now $0 \neq I \triangleleft J$ then I^* (the ideal of A generated by I as usual) is contained in J and therefore $I^* = J$. But then

$$J/I = (J/I)^2 = (J/I)^{(2)} = (J/I)^{(3)} = \cdots = (J/I)^{(n)} = (I^*/I)^{(n)} = 0,$$

so $I = J$ and J is simple. $\square$

The Lemma is not generally used to prove that semi-simple classes are hereditary, as we have the (ADS) condition, but its variant in an Andrunachievici variety has been so used.

Theorem 28. *([86] Theorem 3.2, Theorem 3.4 and Theorem 3.7) If a universal class $\mathcal{W}$ is an Andrunachievici variety and $\mathcal{R}$ is a radical class in $\mathcal{W}$ which contains all zerorings, then the semi-simple class of $\mathcal{R}$ is hereditary, but if $\mathcal{R}$ contains no zerorings (except 0), its semi-simple class need not be hereditary. If $\mathcal{W}$ is both an Andrunachievici variety and an s-variety for some s, then all radical classes in $\mathcal{W}$ which contain all zerorings or none have hereditary semi-simp0le classes.*

Corollary 6. *In a universal class of algebras over a field which is both an Andrunachievici variety and an s-variety for some s, all semi-simple classes are hereditary.*

Radical theory in Andrunachievici varieties which are s-varieties was further studied by Ánh, Loi and Wiegandt [87].

The usual way of proving that semi-simple classes of associative rings are hereditary is by the use of the (ADS) condition:

$$\textit{If } J \triangleleft A \textit{ then } \mathcal{R}(J) \triangleleft A \textit{ for all radicals } \mathcal{R}.$$

(See [3], p. 40.) This condition holds also for alternative rings and groups.

It might be amusing, therefore, to deduce (ADS) from The Lemma, and this we shall now do. In the sequel, A^+ denotes the additive group of a ring A, and S^0 denotes the zeroring on an abelian group G; all rings are associative from now on except for a few brief mentions where we explicitly refer to rings of other kinds. We first state a result which plays a crucial role in our proof.

Proposition 12. *([3], Lemma 3.19, p.17, [4], 5.6 Corollary, p. 84.) If A is a nilpotent ring and $\mathcal{R}$ is a radical class, then $A \in \mathcal{R}$ if and only if $(A^+)^0 \in \mathcal{R}$.*

The proof of this result in [4] is valid for non-associative rings also.

We note a "special case" of (ADS) which is valid in all universal classes and the use of which does not jeopardize our proof. If $\mathcal{R}$ is a radical class, A a zeroring and $J \triangleleft A$, then $\mathcal{R}(J) \triangleleft A$ since all subrings of A are ideals. From this one deduces that the class of zerorings in a semi-simple class is always hereditary. This fact is used in the proof of Proposition 12.

Theorem 29. *The Andrunachievici Lemma implies (ADS).*

Proof. Let $\mathcal{R}$ be a radical class. For $I \triangleleft A$ we have $\mathcal{R}(I) \triangleleft I \triangleleft A$. To streamline notation, let $M = \mathcal{R}(I)$. As before we let M^* be the ideal of A generated by M, so that $(M^*/M)^3 = 0$. Now

$$M^* = M + AM + MA + AMA,$$

so

$$(M^*/M)^+ = ((M + AM + MA + AMA)/M)^+$$

$$= ((M + AM)/M)^+ + ((M + MA)/M)^+ + ((M + AMA)/M)^+.$$

For $a \in A$, define $f_a : M/M^2 \to (((M + AM)/M)$ by setting $f_a(m + M^2) = am + M$ for each $m \in M$. If $m + M^2 = n + M^2$, i.e. $m - n \in M^2$, then

$$a(m - n) \in AM^2 = (AM)M \subseteq M^*M \subseteq IM \subseteq M,$$

so $am + M = an + M$ and hence f_a is well defined. Clearly f_a preserves addition. For all $m, k \in M$ and $a, b \in A$ we have $am \cdot bk = amb \cdot k \in M^*M \subseteq M$, so $(M + AM)/M$ is a zeroring and hence f_a is a ring homomorphism, since M/M^2 is a zeroring. Adding all the f_a we get a surjective group homomorphism, and hence a surjective ring homomorphism

$$\sum_{a \in A} f_a : M/M^2 \to (M + AM)/M. \tag{1}$$

Similarly there is a surjective ring homomorphism

$$M/M^2 \to (M + MA)/M. \tag{2}$$

Now for each $a, b \in A$ define

$$g_{ab} : (M/M^3)^{+0} \to ((M + AMA)/M)^{+0}$$

by the rule

$$g_{ab}(m + M^3) = amb + M \text{ for all } m \text{ in } M.$$

If $m + M^3 = n + M^3$, then

$$a(m - n)b \in AM^3A = AM \cdot M \cdot MA \subseteq M^*MM^* \subseteq M$$

and hence $amb + M = anb + M$ and g_{ab} is well defined. As g_{ab} is clearly a group homomorphism and we are dealing with zerorings, it is a ring homomorphism, and as in the other cases we can sum the g_{ab} to get a surjective ring homomorphism

$$\sum_{a,b \in A} g_{ab} : (M/M^3)^{+0} \to ((M + AMA)/M)^{+0}. \tag{3}$$

Now $M/M^3 \in \mathcal{R}$ (as $M \in \mathcal{R}$) and hence by Proposition 12, so is $(M/M^3)^{+0}$. Also $M/M^2 \in \mathcal{R}$. But then (1), (2) and (3) imply that

$$(M^*/M)^{+0} = ((M + AM)/M)^{+0} + ((M + MA)/M)^{+0} + ((M + AMA)/M)^{+0} \in \mathcal{R}.$$

As $(M^*/M)^3 = 0$ (The Lemma!), Proposition 12 says that $M^*/M \in \mathcal{R}$. But $M = \mathcal{R}(I) \in \mathcal{R}$, so $M^* \in \mathcal{R}$. Since $M^* \subseteq I$ we have $M^* \triangleleft I$, whence $M^* \subseteq \mathcal{R}(I) = M$. Thus

$$\mathcal{R}(I) = M = M^* \triangleleft A$$

and this is (ADS). □

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