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Closure operators in modules and adjoint functors, II

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Abstract. In this work we study the relations between the closure operators of two module categories connected by two adjoint *contravariant* functors. The present article is a continuation of the paper [1] (Part I), where the same question is investigated in the case of two adjoint covariant functors.

An arbitrary bimodule ${}_RU_S$ defines a pair of adjoint contravariant functors $H_1 = Hom_R(\text{-}, U) : R\text{-Mod} \to \text{Mod-}S$ and $H_2 = Hom_S(\text{-}, U) : \text{Mod-}S \to R\text{-Mod}$ with two associated natural transformations $\Phi : \mathbb{1}_{R\text{-Mod}} \to H_2H_1$ and $\Psi : \mathbb{1}_{\text{Mod-}S} \to H_1H_2$. In this situation we study the connections between the closure operators of the categories R-Mod and Mod-S.

Mathematics subject classification: 16D90, 16S90, 16A40.
Keywords and phrases: category of modules, closure operator, adjoint functors, contravariant functor.

1 Introduction

The present paper is a continuation of the article [1] (Part I), where the closure operators of two module categories are investigated in the case of two adjoint covariant functors. In [1] two mappings are constructed between the closure operators of module categories R-Mod and S-Mod, proving some important properties of these mappings.

Using the similar methods, now we will study the relations between the closure operators of two module categories in the case of two adjoint *contravariant* functors. The purpose is to expose some possibilities of transition from the closure operators of one module category to the closure operators of another one and to establish some properties of these mappings.

To specify the investigated situation, we fix an arbitrary (R, S)-bimodule ${}_RU_S$, which defines the following pair of adjoint contravariant functors:

$$R\text{-Mod} \underset{H_2=Hom_S(\text{-},U)}{\overset{H_1=Hom_R(\text{-},U)}{\rightleftarrows}} \text{Mod-}S.$$

Adjointness means that for every pair of modules $X \in R\text{-Mod}$ and $Y \in \text{Mod-}S$ the natural isomorphism is defined:

$$Hom_R\big(X, Hom_S(Y, U)\big) \xrightarrow{\cong} Hom_S\big(Y, Hom_R(X, U)\big).$$

The functors (H_1, H_2) are accompanied by two natural transformations:

$$\Phi : \mathbb{1}_{R\text{-Mod}} \to H_2H_1, \quad \Psi : \mathbb{1}_{\text{Mod-}S} \to H_1H_2,$$

which satisfy the following relations:

$$H_1(\Phi_X) \cdot \Psi_{H_1(X)} = 1_{H_1(X)}, \quad H_2(\Psi_Y) \cdot \Phi_{H_2(Y)} = 1_{H_2(Y)}$$

for every $X \in R$-Mod and $Y \in$ Mod-S.

These relations uniquely define the considered situation. Moreover, every pair of adjoint contravariant functors between two module categories can be represented in such form (up to a functorial isomorphism). We remark that this situation was earlier considered (for example, in [2] and [3]) with the aim to study the relations between the *preradicals* of corresponding module categories.

We recall that a *closure operator* of R-Mod is a function C which associates to every pair $N \subseteq M$, where $N \in \mathbb{L}(M)$, a submodule of M, denoted by $C_M(N)$, which satisfies the conditions:

(c_1) $N \subseteq C_M(N)$ (*extension*);

(c_2) If $N_1, N_2 \in \mathbb{L}(M)$ and $N_1 \subseteq N_2$, then $C_M(N_1) \subseteq C_M(N_2)$ (*monotony*);

(c_3) For every R-morphism $f : M \to M'$ and $N \in \mathbb{L}(M)$ we have: $f\big(C_M(N)\big) \subseteq C_{M'}\big(f(N)\big)$ (*continuity*),

where $M \in R$-Mod and $\mathbb{L}(M)$ is the lattice of submodules of M ([1, 4, 5]).

The condition (c_3) is convenient to extend by C a morphism between the submodules $\overline{f} : N \to f(N)$, where $N \subseteq M$ and $f(N) \subseteq M'$, to the morphism between the C-closures of these submodules $(\overline{f}\,)' : C_M(N) \to C_{M'}\big(f(N)\big)$, which is also a restriction of $f : M \to M'$. Such procedure is often used in the proofs of the following propositions.

We denote by $\mathbb{CO}(R)$ and $\mathbb{CO}(S)$ the classes of closure operators of the categories R-Mod and Mod-S, respectively.

Now we mention an important fact which distinguishes the case of functors (H_1, H_2) from the studied in [1] case (H, T). For the pair (H_1, H_2) the situation is completely symmetric, the functors H_1 and H_2 are of the same type and they have the similar properties. Therefore the mappings which ensure the transition from $\mathbb{CO}(R)$ to $\mathbb{CO}(S)$, and inversely, must be completely similar. We will use this fact in the further account, studying only the transition $\mathbb{CO}(R) \rightsquigarrow \mathbb{CO}(S)$, since the inverse passage can be defined exactly in the same manner, so it possesses the analogous properties.

2 The mapping $(-)^* : \mathbb{CO}(R) \to \mathbb{CO}(S)$

Throughout this work we will study the indicated above situation: a pair (H_1, H_2) of adjoint contravariant functors, defined by the bimodule ${}_RU_S$, with the associated natural transformations Φ and Ψ. Now we will define a mapping from $\mathbb{CO}(R)$ to $\mathbb{CO}(S)$ by the following rule.

Let $C \in \mathbb{CO}(R)$ and $n : N \xrightarrow{\subseteq} Y$ be an arbitrary inclusion of Mod-S. We apply the functor H_2 and using the operator C of R-Mod we obtain the following decomposition of the morphism $H_2(n)$:

$$
\begin{array}{ccc}
H_2(Y) & \xrightarrow{\quad H_2(n) \quad} & H_2(N) \\
\Big\downarrow \overline{H_2(n)} & \overset{\kappa_C^n}{\searrow} & \Big\uparrow \cup\!\mid i_C^n \\
Im\, H_2(n) & \underset{\subseteq}{\xrightarrow{\quad j_C^n \quad}} & C_{H_2(N)}\big(Im\, H_2(n)\big),
\end{array}
$$

i.e. $H_2(n) = i_C^n \cdot j_C^n \cdot \overline{H_2(n)}$, where $\overline{H_2(n)}$ is the restriction of $H_2(n)$ to its image, j_C^n and i_C^n are the inclusions, and $\kappa_C^n = j_C^n \cdot \overline{H_2(n)}$. Applying the functor H_1 and supplementing the diagram by Ψ, we have in Mod-S the following situation:

$$
\begin{array}{ccc}
N & \underset{\subseteq}{\xrightarrow{\quad n \quad}} & Y \\
\Big\downarrow \Psi_N & & \Big\downarrow \Psi_Y \\
H_1H_2(N) & \xrightarrow{\quad H_1H_2(n) \quad} & H_1H_2(Y) \\
\Big\downarrow H_1(i_C^n) & \overset{H_1(\kappa_C^n)}{\nearrow} & \Big\uparrow H_1\big(\overline{H_2(n)}\big) \\
H_1\big[C_{H_2(N)}\big(Im\, H_2(n)\big)\big] & \xrightarrow{H_1(j_C^n)} & H_1\big(Im\, H_2(n)\big).
\end{array}
$$

Definition. For every closure operator $C \in \mathbb{CO}(R)$ and for every inclusion $n : N \xrightarrow{\subseteq} Y$ of Mod-S we define the function C^* by the following rule

$$C_Y^*(N) \overset{\text{def}}{=\!=} \Psi_Y^{-1}\big(Im\, H_1(\kappa_C^n)\big). \tag{1}$$

Proposition 1. *The function C^* defined by the rule* (1) *is a closure operator of the category Mod-S for every closure operator $C \in \mathbb{CO}(R)$.*

Proof. We verify the conditions $(c_1) - (c_3)$ of the definition of closure operator (see Section 1).

(c_1) We show that $N \subseteq C_Y^*(N)$ for every inclusion $n : N \xrightarrow{\subseteq} Y$ of Mod-S. By naturality of Ψ we have $\Psi_Y \cdot n = H_1H_2(n) \cdot \Psi_N$, therefore:

$$\Psi_Y(N) = Im\,(\Psi_Y \cdot n) = Im\,\big(H_1H_2(n) \cdot \Psi_N\big)$$

$$= Im\,\big(H_1(\kappa_C^n) \cdot H_1(i_C^n) \cdot \Psi_N\big) \subseteq Im\, H_1(\kappa_C^n).$$

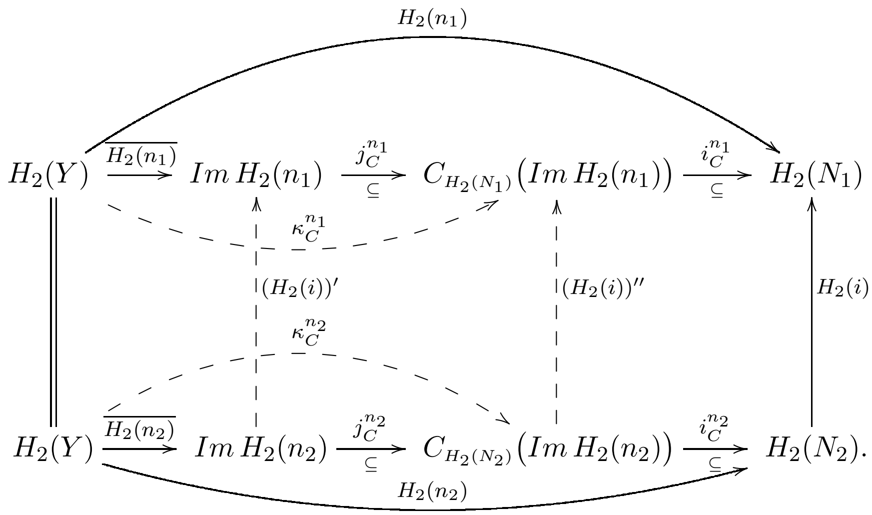
Hence:

$$N \subseteq N + \operatorname{Ker} \Psi_Y = \Psi_Y^{-1}\big(\Psi_Y(N)\big) \subseteq \Psi_Y^{-1}\big(Im\, H_1(\kappa_C^n)\big) \overset{\text{def}}{=\!=} C_Y^*(N).$$

(c_2) We prove the monotony of C^*: $N_1 \subseteq N_2 \Rightarrow C^*_Y(N_1) \subseteq C^*_Y(N_2)$. Let $N_1, N_2 \in \mathbb{L}(Y)$ and $N_1 \subseteq N_2$. We denote the corresponding inclusions as follows:

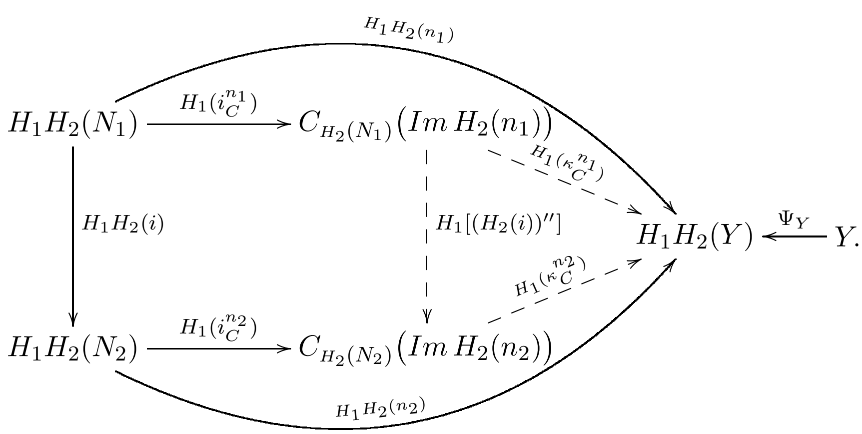
$$n_1 : N_1 \xrightarrow{\subseteq} Y, \quad n_2 : N_2 \xrightarrow{\subseteq} Y, \quad i : N_1 \xrightarrow{\subseteq} N_2.$$

We apply the functor H_2 and consider the decompositions by C of the morphisms $H_2(n_1)$ and $H_2(n_2)$ in R-Mod:



In this diagram $H_2(i)$ defines the morphism $\big(H_2(i)\big)'$, which implies by C the morphism $\big(H_2(i)\big)''$ such that the completed diagram is commutative.

Now applying H_1 we obtain in Mod-S the situation:



From the commutativity of this diagram it follows that $Im\, H_1(\kappa_C^{n_1}) \subseteq Im\, H_1(\kappa_C^{n_2})$, therefore $\Psi_Y^{-1}\big(Im\, H_1(\kappa_C^{n_1})\big) \subseteq \Psi_Y^{-1}\big(Im\, H_1(\kappa_C^{n_2})\big)$. By definition this means that $C^*_Y(N_1) \subseteq C^*_Y(N_2)$, i.e. the function C^* is monotone.

(c_3) Now we verify the continuity of the function C^*, i.e. the property $f\big(C^*_Y(N)\big) \subseteq C^*_{Y'}\big(f(N)\big)$ for every morphism $f : Y \to Y'$ of Mod-S. Let $n : N \xrightarrow{\subseteq} Y$ be an arbitrary inclusion of Mod-S. Denote by n' the inclusion $f(N) \xrightarrow{\subseteq} Y'$, i.e. we have in Mod-$S$ the situation:

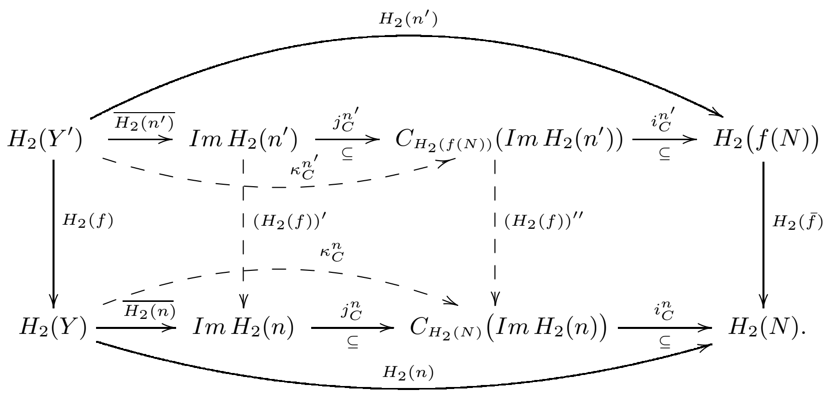
$$\begin{array}{ccc} f(N) & \xrightarrow[\subseteq]{n'} & Y' \\ \uparrow{\scriptstyle \bar f} & & \uparrow{\scriptstyle f} \\ N & \xrightarrow[\subseteq]{n} & Y, \end{array}$$

where $\bar f$ is the restriction of f to the submodule N.

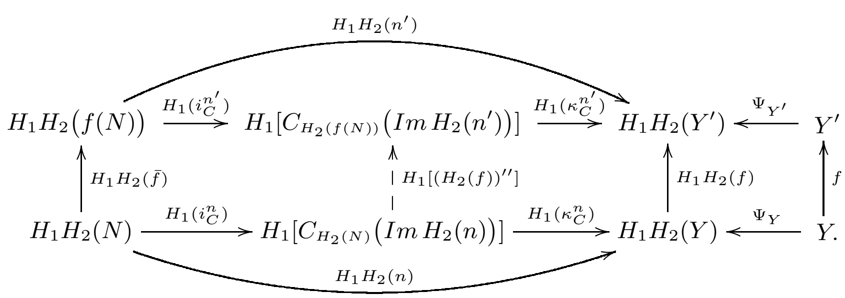
Applying H_2 we obtain in R-Mod the diagram:

$$\begin{array}{ccc} H_2(Y') & \xrightarrow{H_2(n')} & H_2\big(f(N)\big) \\ \downarrow{\scriptstyle H_2(f)} & & \downarrow{\scriptstyle H_2(\bar f)} \\ H_2(Y) & \xrightarrow{H_2(n)} & H_2(N). \end{array}$$

Now we consider the decompositions by C of the morphisms $H_2(n')$ and $H_2(n)$, supplementing the previous diagram as follows:



Here $H_2(f)$ implies the morphism $\big(H_2(f)\big)'$, with the help of which and by C we obtain the morphism $\big(H_2(f)\big)''$. Applying H_1 we have in Mod-S the following commutative diagram:



Now it is obvious that

$$H_1H_2(f)\big(Im\,H_1(\kappa_C^n)\big) = Im\,[H_1H_2(f)\cdot H_1(\kappa_C^n)] \subseteq Im\,H_1(\kappa_C^{n'}).$$

Since $\Psi_Y\big[\Psi_Y^{-1}\big(Im\,H_1(\kappa_C^n)\big)\big] = Im\,H_1(\kappa_C^n) \cap Im\,\Psi_Y \subseteq Im\,H_1(\kappa_C^n)$, we obtain:

$$H_1H_2(f)\big[\Psi_Y\big(\Psi_Y^{-1}(Im\,H_1(\kappa_C^n))\big)\big] \subseteq H_1H_2(f)\big[Im\,H_1(\kappa_C^n)\big] \subseteq Im\,H_1(\kappa_C^{n'}).$$

Using once again the naturality of Ψ:

$$\Psi_{Y'}\cdot f = H_1H_2(f)\cdot\Psi_Y,$$

and replacing the morphism $H_1H_2(f)\cdot\Psi_Y$ in the previous relation, we obtain:

$$(\Psi_{Y'}\cdot f)\big[\Psi_Y^{-1}\big(Im\,H_1(\kappa_C^n)\big)\big] \subseteq Im\,H_1(\kappa_C^{n'}).$$

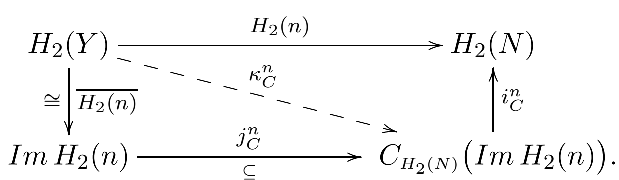
This means that $f\big[\Psi_Y^{-1}\big(Im\,H_1(\kappa_C^n)\big)\big] \subseteq \Psi_{Y'}^{-1}\big(Im\,H_1(\kappa_C^{n'})\big)$, therefore by definition of C^* we have $f\big(C_Y^*(N)\big) \subseteq C_{Y'}^*\big(f(N)\big)$. □

By the same method the inverse mapping $(-)^*\colon \mathbb{CO}(S) \to \mathbb{CO}(R)$ can be defined, changing the positions of the functors H_1 and H_2, and replacing Ψ by Φ. The total similarity of the construction delivers us from the necessity to define and prove the dual mapping.

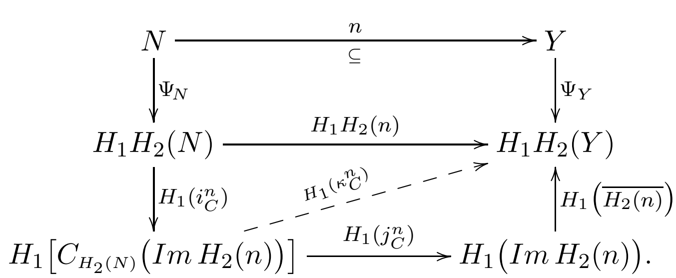
3 The "star" mapping in particular cases

We continue the study of the adjoint situation (H_1, H_2) and now we will verify the effect of the defined above mapping $C \rightsquigarrow C^*$ in the cases of extreme (trivial) closure operators of the classes $\mathbb{CO}(R)$ and $\mathbb{CO}(S)$.

1. Let $C = 1\!\!1_R$, where $1\!\!1_R$ is the greatest closure operator of $\mathbb{CO}(R)$, i.e. $C_X(M) = X$ for every inclusion $M \subseteq X$ of R-Mod. In this case for every inclusion $n : N \xrightarrow{\subseteq} Y$ of Mod-S we have the following decomposition of $H_2(n)$ by C in R-Mod:



Applying H_1 and supplementing the diagram, we obtain in Mod-S:

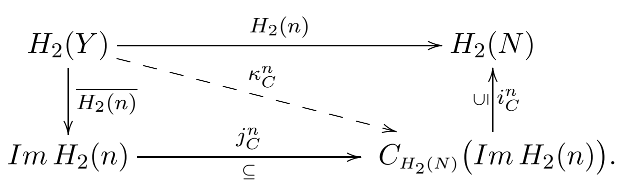


So in the case $C = 1\!\!1_R$ we have $H_1(\kappa_C^n) = H_1 H_2(n)$ and by definition $C_Y^*(N) = \Psi_Y^{-1}\big(Im\, H_1 H_2(n)\big)$. We denote by $D^{\min}$ the closure operator of Mod-S defined by the rule:

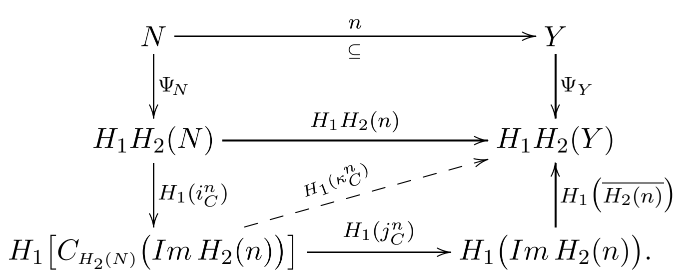
$$(D^{\min})_Y(N) \stackrel{\text{def}}{=\!=} \Psi_Y^{-1}\big(Im\, H_1 H_2(n)\big)$$

for every $N \subseteq Y$. By the previous remarks it follows that $(1\!\!1_R)^* = D^{\min}$. Since in general case (for every $C \in \mathbb{CO}(R)$) we have $Im\, H_1 H_2(n) \subseteq Im\, H_1(\kappa_C^n)$, it is clear that $D^{\min}$ is *the least* closure operator of the form C^* for some $C \in \mathbb{CO}(R)$.

2. Let $C = \mathbb{0}_R$, where $\mathbb{0}_R$ is the least closure operator of R-Mod, i.e. $C_X(M) = M$ for every $M \subseteq X$. Following the construction of C^*, we consider an inclusion $n : N \xrightarrow{\subseteq} Y$ of Mod-S and the decomposition by C of $H_2(n)$ in R-Mod:



Returning in Mod-S by H_1, we obtain:



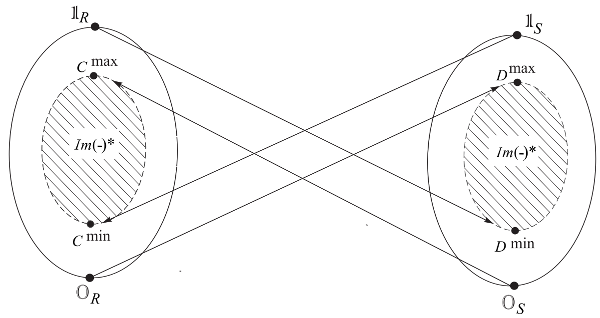
Therefore in the considered case $H_1(\kappa_C^n) = H_1\big(\overline{H_2(n)}\big)$ and so by definition $C_Y^*(N) = \Psi_Y^{-1}\big(Im\, H_1\big(\overline{H_2(n)}\big)\big)$. We denote by $D^{\max}$ the closure operator of Mod-S defined as follows:

$$(D^{\max})_Y(N) \stackrel{\text{def}}{=\!=} \Psi_Y^{-1}\big(Im\, H_1\big(\overline{H_2(n)}\big)\big).$$

Now by the facts mentioned above we have $(\mathbb{0}_R)^* = D^{\max}$. Since in general case the relation $Im\, H_1(\kappa_C^n) \subseteq Im\, H_1\big(\overline{H_2(n)}\big)$ is true, it is obvious that $D^{\max}$ is *the greatest* closure operator of Mod-S of the form C^*, where $C \in \mathbb{CO}(R)$.

Totalizing the remarks exposed above, now we can show the general situation on the images of the trivial closure operators:

$$\mathbb{CO}(R) \underset{(-)^*}{\overset{(-)^*}{\rightleftarrows}} \mathbb{CO}(S)$$



Proposition 2. $(\mathbb{1}_R)^* = D^{\min}, \quad (\mathbb{0}_R)^* = D^{\max};$
$(\mathbb{1}_S)^* = C^{\min}, \quad (\mathbb{0}_S)^* = C^{\max}.$ □

4 The "star" mapping and order relations

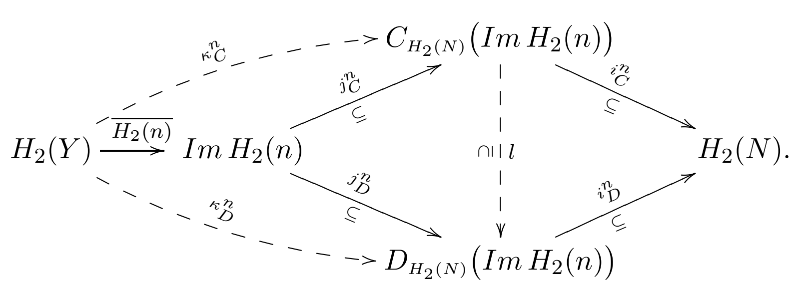
In this section we will study the behaviour of the mapping $C \rightsquigarrow C^*$ relative to the order relations in the classes $\mathbb{CO}(R)$ and $\mathbb{CO}(S)$. By definition for $C, D \in \mathbb{CO}(R)$:

$$C \leq D \Leftrightarrow C_X(M) \subseteq D_X(M)$$

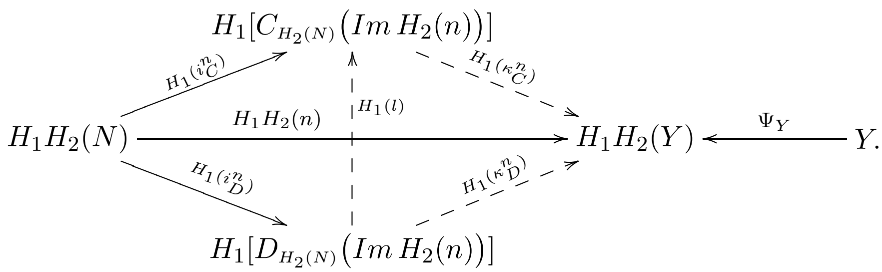
for every $M \subseteq X$ of R-Mod.

Proposition 3. *The mapping $C \rightsquigarrow C^*$ is antimonotone, i.e. it converts the order relations: $C \leq D \Rightarrow C^* \geq D^*$.*

Proof. Let $C, D \in \mathbb{CO}(R)$ and $C \leq D$. For every inclusion $n : N \xrightarrow{\subseteq} Y$ of Mod-S we have in R-Mod the following decompositions of $H_2(n)$ by C and D:



From the relation $C \leq D$ the inclusion l follows and applying H_1 we obtain in Mod-S the situation:



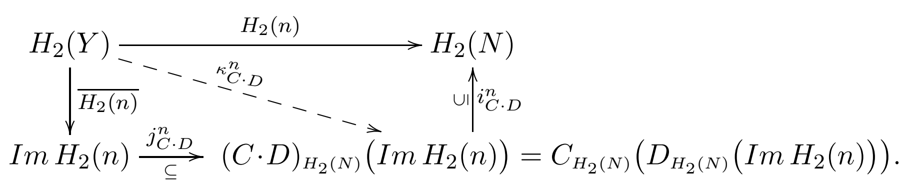
Since this diagram is commutative, we have the relation $Im\, H_1(\kappa^n_D) \subseteq Im\, H_1(\kappa^n_C)$, therefore $\Psi_Y^{-1}\big(Im\, H_1(\kappa^n_D)\big) \subseteq \Psi_Y^{-1}\big(Im\, H_1(\kappa^n_C)\big)$. By definition this means that $D^*_Y(N) \subseteq C^*_Y(N)$ for every $N \subseteq Y$, i.e. $D^* \leq C^*$. $\square$

5 The "star" mapping and the product of operators

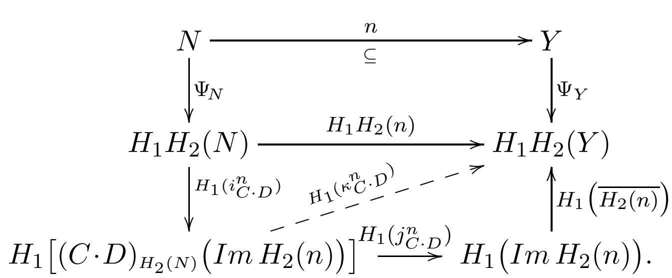
One of the principal operations in the class of closure operators is the multiplication: if $C, D \in \mathbb{CO}(R)$, then the product $C \cdot D$ is defined by the rule $(C \cdot D)_X(M) \stackrel{\text{def}}{=\!=} C_X\big(D_X(M)\big)$ for every $M \subseteq X$ of R-Mod. In continuation we will verify the concordance of the mapping $C \rightsquigarrow C^*$ with this operation.

Proposition 4. *For every closure operators $C, D \in \mathbb{CO}(R)$ the relation $(C \cdot D)^* \leq C^* \cdot D^*$ is true.*

Proof. Let $C, D \in \mathbb{CO}(R)$ and $n : N \xrightarrow{\subseteq} Y$ be an inclusion of Mod-S. Firstly we calculate the left side of indicated relation. We follow the construction of "star" mapping for the operator $C \cdot D$. In R-Mod we have the decomposition of $H_2(n)$:

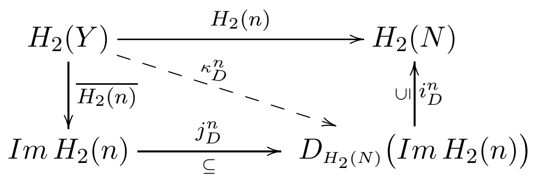


By H_2 we obtain in R-Mod the situation:

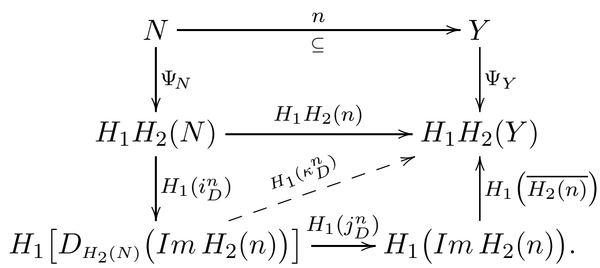


By definition we have: $(C \cdot D)^*_Y(N) \stackrel{\text{def}}{=\!=} \Psi_Y^{-1}\big(Im\, H_1(\kappa^n_{C\cdot D})\big)$.

Passing to the right side of the relation of proposition, now we will show the submodule $D^*_Y(N)$ for $n : N \stackrel{\subseteq}{\longrightarrow} Y$. In R-Mod we have the decomposition of $H_2(n)$ by D:

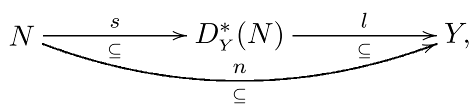


Using H_1 we obtain in Mod-S:

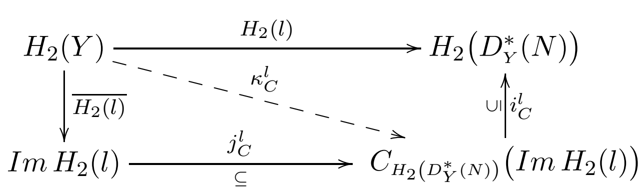


By definition we have: $D^*_Y(N) \stackrel{\text{def}}{=\!=} \Psi_Y^{-1}\big(Im\, H_1(\kappa^n_D)\big)$.

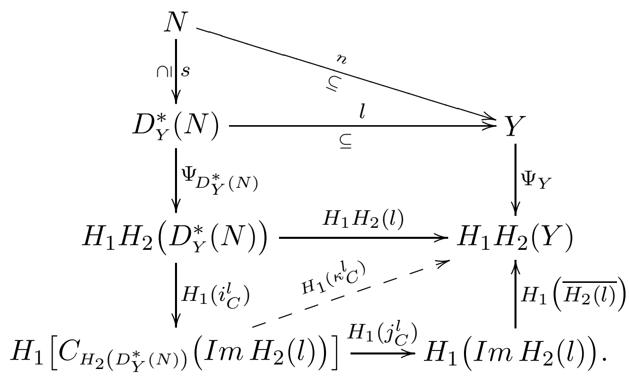
Now we consider the inclusions:



where $l \cdot s = n$. To obtain the submodule $C^*_Y\big(D^*_Y(N)\big)$ we repeat the construction of definition, using the inclusion $l : D^*_Y(N) \stackrel{\subseteq}{\longrightarrow} Y$ and the operator C. In R-Mod we have the decomposition of $H_2(l)$ by C:

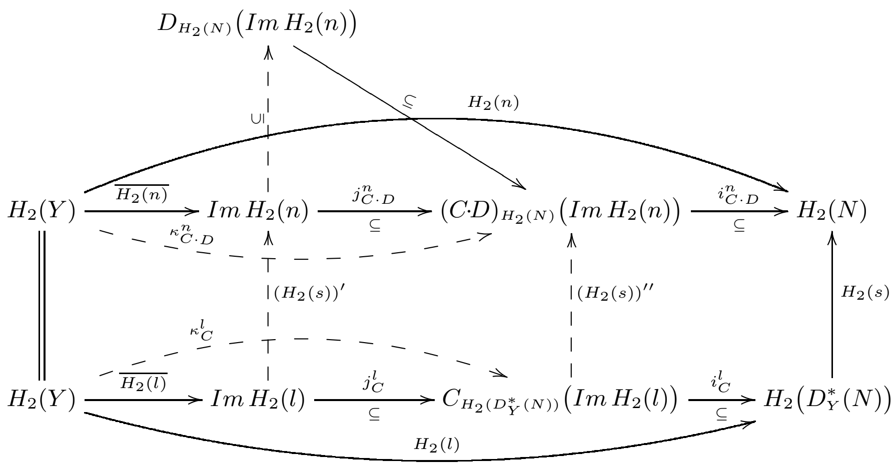


Applying H_1 we obtain in Mod-S the situation:



In this case by definition we have: $C_Y^*\big(D_Y^*(N)\big) \overset{\text{def}}{=\!=} \Psi_Y^{-1}\big(Im\,H_1(\kappa_C^l)\big)$.

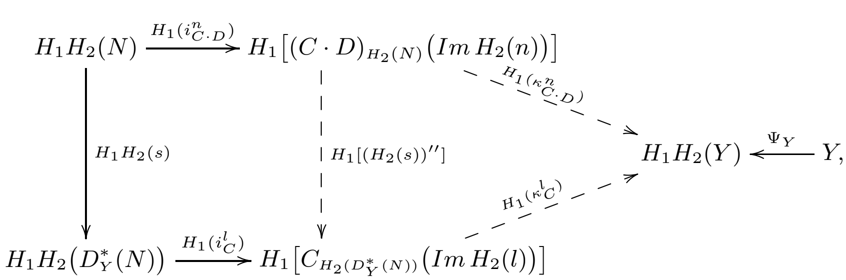
To establish the relation between $(C\cdot D)_Y^*(N)$ and $C_Y^*\big(D_Y^*(N)\big)$ we will utilize the equality $l\cdot s = n$, which implies in R-Mod the situation:



Here $H_2(s)$ implies the morphism:

$$(H_2(s))' : Im\,H_2(l) \longrightarrow Im\,H_2(n) \xrightarrow{\subseteq} D_{H_2(N)}\big(Im\,H_2(n)\big),$$

which by the operator C can be extended to the morphism $\big(H_2(s)\big)''$ such that $\kappa_{C\cdot D}^n = \big(H_2(s)\big)''\cdot\ \kappa_C^l$. Therefore in Mod-$S$ we have the diagram:



where $H_1(\kappa^n_{C\cdot D}) = H_1(\kappa^l_C) \cdot H_1[(H_2(s))'']$. Hence $Im\, H_1(\kappa^n_{C\cdot D}) \subseteq Im\, H_1(\kappa^l_C)$, and so $\Psi_Y^{-1}\big(Im\, H_1(\kappa^n_{C\cdot D})\big) \subseteq \Psi_Y^{-1}\big(Im\, H_1(\kappa^l_C)\big)$.

By definition this means that $(C\cdot D)^*_Y(N) \subseteq C^*_Y\big(D^*_Y(N)\big)$ for every $N \subseteq Y$, i.e. $(C\cdot D)^* \leq C^* \cdot D^*$. □

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