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Distances on Free Semigroups and Their Applications

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Abstract. In this article it is proved that for any quasimetric d on a set X with a base-point p_X there exists a maximal invariant extension $\hat{\rho}$ on the free monoid $F^a(X,\mathcal{V})$ in a non-Burnside quasi-variety $\mathcal{V}$ of topological monoids (Theorem 6.1). This fact permits to prove that for any non-Burnside quasi-variety $\mathcal{V}$ of topological monoids and any T_0-space X the free topological monoid $F(X,\mathcal{V})$ exists and is abstract free (Theorem 8.1). Corollary 10.2 affirms that $F(X,\mathcal{V})$, where $\mathcal{V}$ is a non-trivial complete non-Burnside quasi-variety of topological monoids, is a topological digital space if and only if X is a topological digital space.

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1 Introduction

By a space we understand a topological T_0-space X with a base-point p_X. We use the terminology from [19]. Let $\mathbb{N} = \{1, 2, ...\}$, $\omega = \{0, 1, 2, ...\}$ and $\mathbb{Z} = \{0, \pm1, \pm2, ...\}$ be the discrete semigroups with the additive operation $\{+\}$. By $cl_X H$ we denote the closure of a set H in a space X. $|A|$ is the cardinality of a set A.

A topological semigroup is a semigroup $(G, \cdot)$ endowed with a topology such that the multiplication $\cdot : G \times G \longrightarrow G$ is jointly continuous. A monoid is a simigroup with identity (unity).

If a group G with topology is a topological semigroup, then G is called a paratopological group [6].

In this paper we study properties of free topological monoids in a given quasi-variety of topological monoids $\mathcal{V}$. We apply the method of pseudo-quasimetrics. In particular, we prove that in any non-Burnside quasi-variety $\mathcal{V}$ of topological monoids the following assertions are true:

– any continuous pseudo-quasimetric d on a space X admits an extension $\hat{d}$ on the free monoid $F^a(X,\mathcal{V})$ such that $\hat{d}$ is the invariant pseudo-quasimetric on $F^a(X,\mathcal{V})$;

– any family of invariant pseudo-quasimetrics on a monoid G generates a topology relative to which G is a topological monoid;

– if the family $\mathcal{P}$ of pseudo-quasimetrics is additive and generates a T_0-topology on a set X, then the family $\{\hat{d} : d \in \mathcal{P}\}$ generate on $F^a(X,\mathcal{V})$ a topology relatively to which $F^a(X,\mathcal{V})$ is a topological monoid and a T_0-space;

– the T_0-space X is a subspace in the free topological monoid $F(X, \mathcal{V})$ and $p_X = e$ is the unity of the monoid $F(X, \mathcal{V})$.

The above results are connected with two problems posed by A. I. Malcev. Suppose that $\mathcal{V}$ is a class of topological universal algebras of the given signature with the following properties:

– there exists a topological algebra $G \in \mathcal{V}$ which contains a non-proper open subset U ($\emptyset \neq U \neq G$);

– if $(G, T_0) \in \mathcal{V}$ and T is a T_0-topology on G such that (G, T) is a topological algebra, then $(G, T) \in \mathcal{V}$;

– if H is a subalgebra of a topological algebra $G \in \mathcal{V}$, then $H \in \mathcal{V}$;

– the topological product of algebras from $\mathcal{V}$ is a topological algebra from $\mathcal{V}$.

In [10, 33] was proved: For each non-empty topological space X there exist two topological E-algebras $F(X, \mathcal{V}) \in \mathcal{V}$ and $F^o(X, \mathcal{V}) \in \mathcal{V}$ and a continuous mapping $v_X :\longrightarrow F^o(X, \mathcal{V})$ with the following properties:

1. The set $v_X(X)$ generates the algebra $F^o(X, \mathcal{V})$.

2. If $g : X \longrightarrow G \in \mathcal{V})$ is a continuous mapping, then there exists a unique continuous homomorphism $\bar{g} : F^o(X, \mathcal{V}) \longrightarrow G$ such that $g = \bar{g} \circ v_X$.

3. X is a subset of the E-algebra $F(X, \mathcal{V})$ and the set X generates the algebra $F(X, \mathcal{V})$.

4. If $g : X \longrightarrow G \in \mathcal{V})$ is a mapping, then there exists a unique continuous homomorphism $\bar{g} : F^o(X, \mathcal{V}) \longrightarrow G$ such that $g = \bar{g}|X$.

5. There exists a unique continuous homomorphism $w_X : F(X, \mathcal{V}) \longrightarrow F^o(X, \mathcal{V})$ such that $v_X = w_X|X$.

The algebra $F(X, \mathcal{V})$ is called the free E-algebra on the space X in the class $\mathcal{V}$ and the pair $(F^o(X, \mathcal{V}), v_X)$ is called the topological free E-algebra on the space X in the class $\mathcal{V}$. For any space X the free objects are unique.

A. I. Malcev [33] has posed the following problems:

First Malcev's Problem: Under which conditions the mapping v_X is an embedding?

Second Malcev's Problem: Under which conditions the homomorphism w_X is a continuous isomorphism?

For complete regular spaces X the Malcev's Problems were solved affirmatively by S. Swierczkowski [44], in the case of discrete signature E, end by M. M. Choban and S. S. Dumitrashcu for any signature [10, 18].

The theory of topological semigroups has multiple trends: compact semi-topological semigroups; compact semi-lattices; right-topological semigroups; Lie theory of semi-groups; free topological semigroups; weakly almost-periodic functions on a topological semigroup (a right-topological semigroup); topological dynamics; automata theory; etc (see [24, 34, 35, 43, 45]).

In [45] A. D. Wallace brings to the attention the following problems:

1W. *Which algebraic structures are admitted by what spaces?*

2W. *What compact connected Hausdorff spaces admit a continuous associative multiplication with identity?*

In connection with Problems 1W and 2W, W. D. Wallace [45] mentions the following remarkable theorem of E. Cartan: *If an n-sphere is a topological group, then n = 0, 1 or 3.* This fact was deeply improved by L. M. James [16,25,26]: *If an n-sphere is a topological groupoid with unit, then* $n \in \{0, 1, 3, 7\}$.

2 Distances on spaces

Let X be a non-empty set and $d : X \times X \to \mathbb{R}$ be a mapping such that for all $x, y \in X$ we have:

(i_m) $d(x, y) \geq 0$;

(ii_m) $d(x, x) = 0$.

Then (X, d) is called a *pseudo-distance space* and d is called a *pseudo-distance* on X.

If

(iii_m) $d(x, y) + d(y, x) = 0$ if and only if $x = y$,

then (X, d) is called a *distance space* and d is called a *distance* on X.

If

(iv_m) $d(x, y) = 0$ if and only if $x = y$,

then (X, d) is called a *strong distance space* and d is called a *strong distance* on X.

General problems of the distance spaces were studied in [3,5,7,9,20,36–41]. The notion of a distance space is more general than the notion of o-metric spaces in sense of A. V. Arhangel'skii [5] and S. I. Nedev [36]. A distance d is an o-metric if from $d(x, y) = 0$ it follows that $x = y$, i. e. d is a strong distance. These notions coincide in the class of T_1-spaces.

Let d be a pseudo-distance on X and $B(x, d, r) = \{y \in X : d(x, y) < r\}$ be the *ball* with the center x and radius $r > 0$. The set $U \subset X$ is called *d-open* if for any $x \in U$ there exists $r > 0$ such that $B(x, d, r) \subset U$. The family $\mathcal{T}(d)$ of all d-open subsets is the topology on X generated by d. A pseudo-distance space is a *sequential space*, i.e. a set $B \subseteq X$ is closed if and only if together with any sequence it contains all its limits [19].

Let (X, d) be a pseudo-distance space, $\{x_n : n \in \mathbb{N}\}$ be a sequence in X and $x \in X$. We say that the sequence $\{x_n : n \in \mathbb{N}\}$:

1) is *convergent* to x if and only if $lim_{n\to\infty} d(x, x_n) = 0$. We denote this by $x_n \to x$ or $x = lim_{n\to\infty} x_n$ (really, we may denote $x \in lim_{n\to\infty} x_n$);

2) is *convergent* if it converges to some point in X;

3) is *Cauchy* or *fundamental* if $lim_{n,m\to\infty} d(x_n, x_m) = 0$.

A pseudo-distance space (X, d) is *complete* if every Cauchy sequence in X converges to some point in X.

Lemma 2.1. *Let* (X, d) *and* (Y, ρ) *be pseudo-distance spaces,* $\varphi : X \longrightarrow Y$ *be a mapping and for each point* $x \in X$ *there exist two positive numbers* $c(x), k(x) > 0$ *such that* $\rho(\varphi(x), \varphi(y)) \leq k(x) \cdot d(x, y)$ *provided* $y \in X$ *and* $d(x, y) \leq c(x)$*. Then the mapping* φ *is continuous.*

Proof. Let $\{x_n \in X : n \in \mathbb{N}\}$ be a convergent to $x \in X$ sequence. Then $lim_{n\to\infty}d(x, x_n) = 0$, $lim_{n\to\infty}d(\varphi(x), \varphi(x_n)) = 0$ and $lim_{n\to\infty}\varphi(x_n) = \varphi(x)$. Hence the mapping φ is continuous. □

Let X be a non-empty set and d be a pseudo-distance on X. Then:

– (X, d) is called a *pseudo-symmetric space* and d is called a *pseudo-symmetric* on X if for all $x, y \in X$ we have

(v_m) $d(x, y) = d(y, x)$;

– (X, d) is called a *symmetric space* and d is called a *symmetric* on X if d is a distance and a pseudo-symmetric;

– (X, d) is called a *pseudo-quasimetric space* and d is called a *pseudo-quasimetric* on X if for all $x, y, z \in X$ we have

(vi_m) $d(x, z) \leq d(x, y) + d(y, z)$;

– (X, d) is called a *quasimetric space* and d is called a *quasimetric* on X if d is a distance and a pseudo-quasimetric;

– (X, d) is called a *pseudo-metric space* and d is called a *pseudo-metric* if d is a pseudo-symmetric and a pseudo-quasimetric simultaneously;

– (X, d) is called a *metric space* and d is called a *metric* if d is a symmetric and a quasimetric simultaneously.

Let G be a semigroup and d be a pseudo-distance on G. The pseudo-distance d is called:

– *left* (respectively, *right*) *invariant* if $d(xa, xb) \leq d(a, b)$ (respectively, $d(ax, bx) \leq d(a, b)$) for all $x, a, b \in G$;

– *invariant* if it simultaneously is both left and right invariant;

– *left* (respectively, *right*) *strongly invariant* if $d(xa, xb) = d(a, b)$ (respectively, $d(ax, bx) = d(a, b)$) for all $x, a, b \in G$;

– *strongly invariant* if $d(xa, xb) = d(a, b)$ and $d(ax, bx) = d(a, b)$ for all $x, a, b \in G$;

– *stable* if $d(xy, uv) \leq d(x, u) + d(y, v)$ for all $x, y, u, v \in G$ (see [11, 13]).

Proposition 2.1. *Let d be a pseudo-quasimetric on a semigroup G. The next assertions are equivalent:*

1. *d is invariant.*
2. *d is stable.*

Proof. Is obvious. □

Lemma 2.2. *Let d be a stable pseudo-quasimetric on a semigroup G. Then $(G, T(d))$ is a topological semigroup.*

Proof. In this case the balls $B(x, d, r)$ are d-open sets. Fix $x, y \in G$ and $\varepsilon > 0$. We consider that $0 < 2\delta \leq \varepsilon$. Then $B(x, d, \delta) \cdot B(y, d, \delta) \subseteq B(xy, d, \varepsilon)$. The proof is complete. □

Example 2.1. Let $\mathbb{R}$ be the group of reals and $\mathbb{R}^+$ be the semigroup of non-negative reals. Consider on $\mathbb{R}$ the pseudo-quasimetric $d(x, y) = min\{1, y - x\}$ if $x \leq y$ and

$d(x,y) = 1$ if $x > y$. Denote by S the monoid $\mathbb{R}$ with the topology $T(d)$ and by S^+ the monoid $\mathbb{R}^+$ with the topology $T(d)$. Then:

– S and S^+ are topological monoids;

– the topology $T(d)$ is generated by the open base consisting of the sets $[a,b) = \{x \in \mathbb{R} : a \leq x < b\}$, where $a, b \in \mathbb{R}$ and $a < b$;

– the space S is the Sorgenfrey line [4, 19];

– the spaces S and S^+ are homeomorphic;

– S is a hereditarily Lindelöf first-countable hereditarily separable non-metrizable space;

– the space S does not admit a structure of a topological group.

Example 2.2. Let $\prec$ be a linear ordering on a monoid G. We put $d_l(x,x) = d_r(x,x) = 0$, $d_l(x,y) = d_r(y,x) = 0$ if $x \prec y$ and $d_l(x,y) = d_r(y,x) = 1$ if $y \prec x$. Then d_l and d_r are quasimetrics. We say that d_l and d_r are the quasimetrics generated by the linear ordering $\prec$. Assume now that $e \preceq x$ for any $x \in G$, where e is unity in G, and from $x \preceq u$, $y \preceq v$ it follows that $xy \preceq uv$. Then:

– the topologies $T(d_l)$ and $T(d_r)$ are T_0-topologies on G;

– $T(d_l)$ and $T(d_r)$ are not T_1-topologies;

– the quasimetrics d_l and d_r are stable on G;

– $(G, T(d_l))$ and $(G, T(d_r))$ are topological monoids.

3 Free topological monoids

A class $\mathcal{V}$ of topological monoids is called a quasi-variety of monoids if:

(F1) the class $\mathcal{V}$ is multiplicative;

(F2) if $G \in \mathcal{V}$ and A is a submonoid of G, then $A \in \mathcal{V}$;

(F3) every space $G \in \mathcal{V}$ is a T_0-space.

A class $\mathcal{V}$ of topological monoids is called a complete quasi-variety of monoids if it is a quasi-variety with the next property:

(F4) if $G \in \mathcal{V}$ and T is a T_0-topology on G such that (G,T) is a topological monoid, then $(G,T) \in \mathcal{V}$ too.

A quasi-variety $\mathcal{V}$ of topological monoids is non-trivial if $|G| \geq 2$ for some $G \in \mathcal{V}$.

Let X be a non-empty topological space and $\mathcal{V}$ be a quasi-variety of topological monoids. In the space X the basic point $p_X \in X$ is fixed, i.e. any space is pointed.

A free monoid of a space X in a class $\mathcal{V}$ is a topological monoid $F(X,\mathcal{V})$ with the properties:

– $X \subseteq F(X,\mathcal{V}) \in \mathcal{V}$ and p_X is the unity of $F(X,\mathcal{V})$;

– the set X generates the monoid $F(X,\mathcal{V})$;

– for any continuous mapping $f : X \longrightarrow G \in \mathcal{V}$, where $f(p_X) = e$, there exists a unique continuous homomorphism $\bar{f} : F(X,\mathcal{V}) \longrightarrow G$ such that $f = \bar{f}|X$.

An abstract free monoid of a space X in a class $\mathcal{V}$ is a topological monoid $F^a(X,\mathcal{V})$ with the properties:

– X is a subset of $F^a(X,\mathcal{V})$, $F^a(X,\mathcal{V}) \in \mathcal{V}$ and p_X is the unity of $F^a(X,\mathcal{V})$;

– the set X generates the monoid $F^a(X,\mathcal{V})$;

– for any mapping $f : X \longrightarrow G \in \mathcal{V}$, where $f(p_X) = e$, there exists a unique continuous homomorphism $\hat{f} : F^a(X, \mathcal{V}) \longrightarrow G$ such that $f = \hat{f}|X$.

In the proof of the next assertion we use the Kakutani's method [27].

Theorem 3.1. *Let $\mathcal{V}$ be a non-trivial quasi-variety of topological monoids. Then for each space X the following assertions are equivalent:*

1. There exists $G \in \mathcal{V}$ such that X is a subspace of G and p_X is the neutral element in G.

2. For the space X there exists the unique free topological monoid $F(X, \mathcal{V})$.

Proof. Implication $2 \to 1$ is obvious. Assume now that there exists $A \in \mathcal{V}$ such that X is a subspace of A and p_X is the neutral element in A. Let τ be an infinite cardinal number and $|X| \leq \tau$. Denote by $\mathcal{V}(\tau)$ the collection of all $G \in \mathcal{V}$ of the cardinality $\leq \tau$. Since we identify the topologically isomorphic topological monoids, the family $\mathcal{V}(\tau)$ is a set. Hence the collection $\{h_\mu : X \longrightarrow G_\mu : \mu \in M\}$ of all continuous mappings $f : X \longrightarrow G \in \mathcal{V}(\tau)$ with $f(p_X) = e \in G$ is a set too. Consider the diagonal product $h : X \longrightarrow G = \Pi\{G_\mu : \mu \in M\}$, where $h(x) = (h_\mu(x) : \mu \in M) \in G$ for every point $x \in X$. By construction, $h(p_X) = (e_\mu \in G_\mu : \mu \in M) = e \in G$ and h is a continuous mapping. Denote by $H(X)$ the submonoid of G generated by the set $Y = h(X)$ in G. For each $\eta \in M$ consider the projection $\pi_\eta : H(X) \longrightarrow G_\mu$, where $\pi_\eta(x_\mu : \mu \in M) = x_\eta$ for each point $(x_\mu : \mu \in M) \in H(X)$. Then $h_\eta = \pi_\eta \circ h$. Each projection π_η is a homomorphism.

Since $|Y| \leq |X| \leq \tau$, we have $|H(X)| \leq \tau$ and $H(X) \in \mathcal{V}(\tau)$.

For some $\lambda \in M$ we have that G_λ is a submonoid of A and $h_\lambda : X \longrightarrow G_\lambda$ is an embedding of X in G_λ and $e_\lambda = p_X$ is the unity of the monoid G_λ. We have $h_\lambda(x)$ $= x$ for each $x \in X$. Since $h_\lambda = p_\lambda \circ h$ is an embedding, h is an embedding too. Hence, we can assume that $X = h(X) = Y$ is a subspace of $H(X)$ and $h(x) = x$ for each point $x \in X$.

Fix a continuous mapping $f : X \longrightarrow G \in \mathcal{V}$, where $f(p_X) = e \in G$. There exists $\eta \in M$ such that G_η is the submonoid of G generated by $f(X)$ and $f(x)$ $h_\eta(x)$ for each $x \in X$. Then $p_\eta(x) = \pi_\eta(h(x)) = f(x)$ for each $x \in X$. Since X generated $H(X)$, the homomorphism $\bar{f}$ is unique. Thus we can assume that $\pi_\eta = \bar{f}$ and $H(X)$ is the free topological monoid of the space X in the class $\mathcal{V}$. The existence of the free topological monoid of the space X is proved.

Let $F(X, \mathcal{V})$ and $F_1(X, \mathcal{V})$ be two free topological monoids of the space X. There existtwo continuous homomorphisms $h : F_1(X, \mathcal{V}) \longrightarrow F(X, \mathcal{V})$ and $g : F(X, \mathcal{V}) \longrightarrow$ $F_1(X, \mathcal{V})$ such that $h(x) = g(x) = x$ for each $x \in X$. Consider the homomorphism $\varphi = h \circ g : F(X, \mathcal{V}) \longrightarrow F(X, \mathcal{V})$. That homomorphism is unique and is generated by the embedding of X in $F(X, \mathcal{V})$. Hence φ is the identical mapping and $h = g^{-1}$. Thus h and g are topological isomorphisms and the uniqueness of the free topological monoid of the space X is proved. □

Corollary 3.1. *Let $\mathcal{V}$ be a non-trivial quasi-variety of topological monoids. Then for each space X there exists the unique abstract free monoid $F^a(X, \mathcal{V})$.*

Let $\mathcal{V}$ be a non-trivial quasi-variety of topological monoids.

Problem 3.1. *Let $\mathcal{V}$ be a non-trivial quasi-variety of topological monoids. Under which conditions for a space X there exists the free topological monoid $F(X, \mathcal{V})$?*

Fix a space X for which there exists the free topological monoid $F(X, \mathcal{V})$. Then there exists a unique continuous homomorphism $\pi_X : F^a(X, \mathcal{V}) \longrightarrow F(X, \mathcal{V})$ such that $\pi_X(x) = x$ for each $x \in X$. The monoid $F(X, \mathcal{V})$ is called abstract free if π_X is a continuous isomorphism.

Problem 3.2. *Let $\mathcal{V}$ be a non-trivial quasi-variety of topological monoids. Under which conditions for a space X there exists the free topological monoid $F(X, \mathcal{V})$, which is abstract free?*

The Problems 3.1 and 3.2 are important in the theory of universal algebras with topologies (see [10–13, 17, 33]). These problems for varieties of topological algebras were posed by A. I. Malcev [33].

We say that a space$X$ is zero-dimensional and denote $indX = 0$ if X has a base whose elements are open-and-closed [19].

Theorem 3.2. *Let $\mathcal{V}$ be a non-trivial quasi-variety of topological monoids and there exists $H \in \mathcal{V}$ and point $b \in H$ such that $e \neq b$, and $E = \{e, b\}$ is a discrete subspace of H. Then for each zero-dimensional space X there exists the unique free topological monoid $F(X, \mathcal{V})$.*

Proof. Let $\{(U_\mu, V_\mu) : \mu M\}$ be a family of open-and-closed subsets of the space X with a fixed point p_X such that:

– $X = U_\mu \cup V_\mu$ and $U_\mu \cap V_\mu = \emptyset$ for each $\mu \in M$;

– if the set U is open in X, $x \in U$ and $x \neq p_X$, then there exists $\mu \in M$ such that $x \in V_\mu \subseteq U$;

– if the set U is open in X and $p_X \in U$, then there exists $\mu \in M$ such that $p_X \in U_\mu \subseteq U$.

We put $h_\mu(U_\mu) = \{e\}$ and $h_\mu(V_\mu) = \{b\}$. Then $h_\mu : X \longrightarrow H$ is a continuous mapping and the diagonal product $h : X \longrightarrow H^M$, where $h(x) = (h_\mu(x) : \mu \in M)$ for each point $x \in X$, is an embedding of X into $G = H^M$ and $h(p_X)$ is the unity of G. Theorem 3.1 completes the proof. □

The condition of the existence of a topological monoid H with a discrete space E is essential in the above theorem.

Example 3.1. Let H be the topological monoid ω with the topology $\{\emptyset, H\} \cup \{U_n = \{i \in \omega : i \leq n\} : n \in \omega\}$. The set $\{0\}$ is open and dense in H. Let $\mathcal{V}(H)$ be the quasi-variety of topological monoids generated by H. Any element of $\mathcal{V}(H)$ is a topological submonoid of the topological monoid H^M for some non-empty set M. In any $G \in \mathcal{V}(H)$ the unity $\{e\}$ is a dense subset. We have the following cases:

Case 1. If X is a space with the fixed point p_X and the set $\{p_X\}$ is closed in X (for instance, X is a T_1-space), then for X the free topological monoid $F(X, \mathcal{V}(H))$ does not exist.

Case 2. Let X be the space H with the fixed point $p_X = 0$. By virtue of Theorem 3.1, the free topological monoid $F(X, \mathcal{V}(H))$ of the space X exists.

Case 3. Let X be the space H with the fixed point $p_X \neq 0$. If $f : X \longrightarrow H$ is a continuous mapping and $f(p_X) = 0$ then $f(x) = 0$ for each $x \leq p_X$. Hence the free topological monoid $F(X, \mathcal{V}(H))$ of the space X =H with the fixed point $p_X \neq 0$ does not exist.

4 Construction of the abstract free monoid

Fix a non-trivial quasi-variety $\mathcal{V}$ of topological monoids. Consider a space X for which we can assume that $X \subseteq F^a(X, \mathcal{V})$ as a subset and $p_X = e$ is the unity (neutral element) in $F^a(X, \mathcal{V})$. In this case $e \in X \subseteq F^a(X, \mathcal{V})$. The set $A = X \setminus \{e\}$ is called an alphabet. If $n \geq 1$ and $x_1, x_2, ..., x_n \in X$, then the symbol $x_1x_2...x_n$ is called a word of the length n in the alphabet A. The word e is the empty word. Any word $x_1x_2...x_n$, where $x_1, x_2, ..., x_n \in X$, represents a unique element $x_1x_2...x_n$ $= x_1 \cdot x_2 \cdot ... \cdot x_n \in F^a(X, \mathcal{V})$. A given element $b \in F^a(X, \mathcal{V})$ is represented by many words. There exists a word of the minimal length which represents the given element b. The length n of this word is called the length of the element b and we put $l(b) = n$. If the element b is represented by the words $x_1x_2...x_n$, $y_1y_2...y_m$ of the minimal length, then $n = m$ and $\{x_1, x_2, ..., x_n\} = \{y_1, y_2, ..., y_m\}$. In this case we say that the word $x_1x_2...x_n$ is irreducible and that $Sup(b) = \{x_1, x_2, ..., x_n\}$ is the support of the element b. If the element b is represented by the words $x_1x_2...x_n$, $y_1y_2...y_n$ of the minimal length, then there exists a bijection $h : \{1, 2, ..., n\} \longrightarrow \{1, 2, ..., n\}$ such that $x_i = y_{h(i)}$ for each $i \leq n$. Obviously, $Sup(e) = \{e\}$ and $e \notin Sup(b)$ if $b \neq e$. If $e \in Y \subseteq X$, $b \in F^a(X, \mathcal{V})$ and $Sup(b) \subseteq Y$, then $b \in F^a(Y, \mathcal{V})$. In particular, $F^a(Y, \mathcal{V})$ is the submonoid of $F^a(X, \mathcal{V})$ generated by the set Y.

For any two elements $a, b \in F^a(Y, \mathcal{V})$ we put $Sup(a, b) = Sup(a) \cup Sup(b) \cup \{e\}$. In particular, $Sup(a, a) = Sup(a) \cup \{e\}$.

Remark 4.1. Let $b \in F^a(X, \mathcal{V})$ and $b \neq e$. Then $x \in Sup(b)$ if and only if $x \neq e$ and $b \notin F^a(X \setminus \{x\}, \mathcal{V})$.

Remark 4.2. Let $b = x_1x_2...x_n \in F^a(X, \mathcal{V})$. Then we have $Sup(b) \subseteq Sup(b, b) \subseteq \{e, x_1, x_2, ..., x_n\}$.

Remark 4.3. If $\mathcal{V}$ is the variety of all topological monoids, then any $b \in F^a(X, \mathcal{V})$ is represented by some word of the minimal length. If the monoids from $\mathcal{V}$ are commutative and p_X, a, b are distinct elements of X, then ab and ba are distinct words, but $ab = ba$ in $F^a(Y, \mathcal{V})$.

5 On the non-Burnside quasi-varieties

A quasi-variety $\mathcal{V}$ of topological monoids is called a Burnside quasi-variety if there exist two minimal numbers $p = p(\mathcal{V}), q = q(\mathcal{V}) \in \omega$ such that $0 \leq q < p$ and $x^p = x^q$ for all $x, y \in G \in \mathcal{V}$. In this case any $G \in \mathcal{V}$ is a (p, q)-periodic monoid of the exponent (p, q). If $q = 0$, then any monoid $G \in \mathcal{V}$ is a periodic monoid of the exponent p and $x^p = e$ for each $x \in G \in \mathcal{V}$.

The trivial quasi-variety is considered Burnside of the exponent $(0, 1)$.

Example 5.1. Fix $0 \leq q < p$ and an element $b \neq e$. We put $b^0 = e$, $b^1 = b$ and $b^{n+1} = b^n \cdot b = b \cdot b^n$ for each $n \in \mathbb{N}$. We consider that $b^p = b^q$ and all elements $\{b^i : i < p\}$ are distinct. Then $G_{(p,q)} = \{b^n : n \in \mathbb{N}\} = \{b^i : i < p\}$ is a monoid and $|G_{(p,q)}| = p$. Denote by $\mathcal{W}_{(p,q)}$ the complete variety of topological monoids generated by the discrete monoid $G_{(p,q)}$, i.e. is the minimal class of topological monoids with the properties:

– the class $\mathcal{W}_{(p,q)}$ is a complete quasi-variety of topological monoids;

– $G_{(p,q)} \in \mathcal{W}_{(p,q)}$

– if $f : A \to B$ is a continuous homomorphism of a topological monoid A onto a a topological monoid B, $A \in \mathcal{W}_{(p,q)}$ and B is a T_0-space, then $B \in \mathcal{W}_{(p,q)}$.

Then $\mathcal{W}_{(p,q)}$ is a variety of topological commutative monoids of the exponent (p, q).

Example 5.2. Let $\mathcal{W}_\omega$ is the complete quasi-variety generated by the discrete monoid $\omega = \{0, 1, 2, ...\}$ with the additive operation. The class $\mathcal{W}_\omega$ is a non-Burnside quasi-variet of commutative topological monoids.

Theorem 5.1. *Let $\mathcal{V}$ be a non-trivial Burnside quasi-variety of the exponent $p \geq 2$. Then:*

1. Each topological monoid $G \in \mathcal{V}$ is a topological group.

2. If d is a stable pseudo-quasimetric on $G \in \mathcal{V}$, then d is a pseudo-metric on G and $d(x, y) = d(y, x) = d(xz, yz) = d(zx, zy) = d(y^{-1}, x^{-1}) \leq (p-1)d(y, x)$ for all $x, y, z, \in G \in \mathcal{V}$.

3. If $p = 2$ and d is a stable pseudo-quasimetric on $G \in \mathcal{V}$, then d is a pseudo-metric on G.

Proof. Let $x \in G \in \mathcal{V}$ and $p(x) = min\{q \in \mathbb{N} : x^q = e\}$. If $p(x) \geq 2$, then $x^{p(x)} = e$. Thus we can assume that $x^{p(x)-1} = x^{-1}$. Thus G is a group. If d is a stable pseudo-quasimetric on G, then $d(x, y) = d(xz, yz) = d(zx, zy) = d(y^{-1}xx^{-1}, y^{-1}yx^{-1}) = d(y^{-1}, x^{-1})$ for all $x, y, z, \in G$. If $p = 2$, then $x = x^{-1}$. Assertion 2 is proved. Assertion 3 follows from Assertion 2.

Let $G \in \mathcal{V}$ be a paratopological group. A topological group is a paratopological group with a continuous inverse operation $x \to x^{-1}$. Since the inverse operation $x \to x^{p-1} = x^{-1}$ is continuous, Assertion 1 is proved. The proof is complete. □

Theorem 5.2. *Let $\mathcal{V}$ be a non-trivial quasi-variety of topological monoids. Then the following assertions are equivalent:*

1. $\mathcal{V}$ is a non-Burnside quasi-variety.

2. On ω there exists a topology T for which $(\omega, T) \in \mathcal{V}$.

Proof. Implication $2 \to 1$ is obvious. Assume that $\mathcal{V}$ is a non-Burnside quasi-variety. Let $\{(p_n, q_n) : n \in \mathbb{N}\}$ is the collection of all pairs $(p, q) \in \omega \times \omega$ such that $q < p$. For each $n \in \mathbb{N}$ there exist $G_n \in \mathcal{V}$ and $a_n \in G_n$ such that all elements $a_n^0 = e, a_n^1, a_n^2, ..., a_n^{p_n-1}$ are distinct and $a_n^{p_n} = a_n^{q_n}$. We put $G = \Pi\{G_n : n \in \mathbb{N}\}$ and $a = (a_n : n \in \mathbb{N})$. Then $a \in G \in \mathcal{V}$. We put $H = \{a^n : n \in \omega\}$. Then $H \in \mathcal{V}$ is a submonoid of the monoid G. The mapping $n \to a^n$ is a isomorphism of ω onto H. Implication $1 \to 2$ and the theorem are proved. □

Corollary 5.1. *Let $\mathcal{V}$ be a non-Burnside quasi-variety, X be a space, $b = x_1x_2...x_n \in F^a(X,\mathcal{V})$, $l(b) = m$ and $Sup(b) = \{y_1, y_2, ..., y_s\}$. Then:*

1. If $b = e$, then $s = 1$, $m = 0$ and $x_i = y_1 = e$ for each $i \leq n$.

2. Let $b \neq e$. Then $n \geq m \geq s \geq 1$ and $\{y_1, y_2, ..., y_s\} \subseteq \{x_1, x_2, ..., x_n\} \subseteq \{e\} \cup \{y_1, y_2, ..., y_s\}$, i.e. for each $i \leq n$ we have $x_i \in Sup(b, b)$. Moreover, if $A = \{i \leq n : x_i \neq e\}$, then there exists a mapping $h : A \longrightarrow \{1, 2, ..., s\}$ such that $h(A) = \{1, 2, ..., m\}$, $A = \{i_1, i_2, ..., i_m\}$, $x_i = y_{h(i)}$ for each $i \in A$ and $x = [x_{i_1}x_{i_2}...x_{i_m}]$ is an irreducible word.

3. $Sup(b) \subseteq \{x_1, x_2, ..., x_n\} \subseteq Sup(b, b)$.

Corollary 5.2. *Let $\mathcal{V}$ be a non-Burnside quasi-variety, X be a space and $b = x_1x_2...x_m = y_1, y_2, ..., y_m \in F^a(X,\mathcal{V})$ and $x_i \neq e$ for each $i \leq m$. Then there exists a one-to-one mapping $h : \{1, 2, ..., m\} \longrightarrow \{1, 2, ..., m\}$ such that $x_i = y_{h(i)}$ for each $i \leq m$.*

Remark 5.1. Assertions of Corollary 5.1 are not true for Burnside quasi-varieties. Consider the quasi-variety $\mathcal{W}_{(0,2)}$ of topological monoids (groups) with the identity $x^2 = e$. Let $X = \{e, a, b, c\}$ be a discrete space with four distinct points. Then $z = a = cabeeaecba = bba = acc \in F^a(X, \mathcal{W}_{(0,2)})$ and $Sup(z) = \{a\}$.

The following theorem solves Problem 3.1 for complete non-Burnside quasi-varieties of topological monoids.

Theorem 5.3. *Let $\mathcal{V}$ be a complete non-Burnside quasi-variety of topological monoids. Then for each T_0-space X there exists the free topological monoid $F(X,\mathcal{V})$.*

Proof. By virtue of Theorem 5.2 the discrete monoid ω is an element of $\mathcal{V}$. Denote by ω_l the monoid ω with the topology $T_l = \{\emptyset, \omega\} \cup \{V_n = \{i \in \omega : i \leq n\} : n \in \omega\}$ and by ω_r the monoid ω with the topology $T_r = \{\emptyset, \omega\} \cup \{W_n = \{i \in \omega : i \geq n\} : n \in \omega\}$. Obviously, the topological monoids ω_l and ω_r are elements of $\mathcal{V}$.

Consider a space X with the fixed point p_X. Let U be an open subset of the space X. We construct a topological monoid $G_U \in \mathcal{V}$ with the unity e_U and a continuous mapping $h_U : X \longrightarrow G_U$ such that $h_U(p_X) = e_U$ and $U = h_U^{-1}(h_U(U))$. For that we consider two cases.

Case 1. $p_X \in U$.

In this case we put $G_U = \omega_l$, $h_U(U) = \{0\}$ and $h_U(X \setminus U) = \{1\}$.

Case 2. $p_X \notin U$.

In this case we put $G_U = \omega_r$, $h_U(U) = \{1\}$ and $h_U(X \setminus U) = \{0\}$.

Now consider the diagonal product $h : X \longrightarrow G = \Pi\{G_U : U \textit{ is open subset of } X\}$, where $h(x) = (h_U(x) : U \textit{ is open subset of } X)$ for each $x \in X$. By construction, $G \in \mathcal{V}$, h is an embedding of X in G and $h(p_X) = e$ is the neutral element in G. Theorem 3.1 completes the proof. ∎

The following theorem solves Problem 3.1 for complete non-trivial quasi-varieties of topological monoids.

Theorem 5.4. *Let $\mathcal{V}$ be a complete non-trivial quasi-variety of topological monoids. Then for each completely regular space X there exists the free topological monoid $F(X, \mathcal{V})$.*

Proof. In [10] it was proved that any topological monoid $G \in \mathcal{V}$ is a submonoid of some arcwise connected topological monoid from $\mathcal{V}$. Hence there exists a topological monoid $H \in \mathcal{V}$ such that the closed interval $[0,1]$ is a subspace of H and $e = 0$ is the neutral element in H.

Let βX be the Stone-Čech compactification of the given completely regular space with the fixed point p_X. Let $\{(U_\mu, F_\mu) : \mu \in M\}$ be the collection of all pairs (U, F), where U is an open subset of the space βX, F is a closed subset of the space βX and $F \subseteq U$ and $p_X \in F$ provided $p_X \in U$. We construct a topological monoid $G_\mu = H \in \mathcal{V}$ with the unity e_μ and a continuous mapping $h_\mu : X \longrightarrow G_\mu$ such that $h_\mu(p_X) = e_\mu$ and $h_\mu(F_\mu) \cap h_\mu(X \setminus U_\mu) = \emptyset$. For that we consider two cases.

Case 1. $p_X \in U_\mu$.

In this case we fix a continuous mapping $h : X \longrightarrow [0,1] \subseteq H = G_\mu$ such that $h_\mu(F_\mu) = \{0\}$ and $h_\mu(X \setminus U_\mu) = \{1\}$.

Case 2. $p_X \notin U_\mu$.

In this case we fix a continuous mapping $h : X \longrightarrow [0,1] \subseteq H = G_\mu$ such that $h_\mu(F_\mu) = \{1\}$ and $h_\mu(X \setminus U_\mu) = \{0\}$.

Now consider the diagonal product $h : X \longrightarrow G = \Pi\{G_\mu : \mu \in M\}$, where $h(x) = (h_\mu(x) : \mu \in M)$ for each $x \in X$. By construction, $G \in \mathcal{V}$, h is an embedding of X in G and $h(p_X) = e$ is the neutral element in G. Theorem 3.1 completes the proof. □

The following corollary follows from Theorems 5.1 and 5.3.

Corollary 5.3. *Let $\mathcal{V}$ be a complete non-trivial Burnside quasi-variety of the exponent $p \geq 2$. Then for a space X there exists the free monoid $F(X, \mathcal{V})$ if and only if the space X is Tychonoff.*

Completeness of quasi-variety $\mathcal{V}$ is essential in the conditions of the above two theorems.

Example 5.3. Let H be a discrete monoid and $\mathcal{V}(H)$ the quasi-variety of topological monoids generated by H. Any element of $\mathcal{V}(H)$ is a topological submonoid of the topological monoid H^M for some non-empty set M. Hence, for a space X there exists the free monoid $F(X, \mathcal{V})$ if and only if the space X is Tychonoff and $indX = 0$.

Example 5.4. Let ω_r be the monoid ω with the topology $T_r = \{\emptyset, \omega\} \cup \{W_n = \{i \geq n : n \in \omega\}\}$ and $\mathcal{V}(\omega_r)$ be the quasi-variety of topological monoids generated by ω_r. Any element of $\mathcal{V}(\omega_r)$ is a topological submonoid of the topological monoid ω_r^M for some non-empty set M. For a space X there exists the free monoid $F(X, \mathcal{V})$ if and only if the space X is a T_0-space and the set $\{p_X\}$ is closed in X. Denote by Z an infinite space with a fixed point p_Z and the topology $\{\emptyset, Z\} \cup \{U \subseteq Z : p_Z \in U\}$. The subset $\{p_Z\}$ is open and dense in Z. Moreover, if $f : Z \longrightarrow \omega_r$ is a continuous mapping and $f(p_Z) = 0$, then $f(Z) = \{0\}$. Thus the free topological monoid for the space Z in the quasi-variety $\mathcal{V}(\omega_r)$ does not exist.

6 Extension of pseudo-quasimetrics

Lemma 6.1. *Let d_1, d_2 be two pseudo-quasimetrics on a monoid G. Then:*

1. $d(x, y) = sup\{d_1(x, y), d_2(x, y)\}$ is a pseudo-quasimetric on G.

2. If the pseudo-quasimetrics d_1, d_2 are invariant on G, then the pseudo-quasimetric d is invariant on G too.

Proof. Fix $x, y, z, v \in G$. Then $d(x, z) = sup\{d_1(x, z), d_2(x, z)\} \leq sup\{d_1(x, y) + d_1(y, z), d_2(x, y) + d_2(y, z)\} \leq sup\{d_1(x, y), d_2(x, y)\} + sup\{d_1(y, z), d_2(y, z)\} = d(x, y) + d(x, z)$. Hence d is a pseudo-quasimetric on G.

Assume that the pseudo-quasimetrics d_1, d_2 are invariant on G. We observe that $d(zxv, zyv) = sup\{d_1(zxv, zyv), d_2(zxv, zyv)\} \leq sup\{d_1(x, y), d_2(x, y)\} = d(x, y)$. Thus the pseudo-quasimetric d is invariant too. □

Fix a non-trivial complete quasi-variety $\mathcal{V}$ of topological monoids. Consider a non-empty set X with a fixed point $e \in X$. We assume that $e \in X \subseteq F^a(X, \mathcal{V})$ and e is the identity of the monoid $F^a(X, \mathcal{V})$. Let ρ be a pseudo-quasimetric on the set X. Denote by $Q(\rho)$ the set of all stable pseudo-quasimetrics d on $F^a(X, \mathcal{V})$ for which $d(x, y) \leq \rho(x, y)$ for all $x, y \in X$. The set $Q(\rho)$ is non-empty, since it contains the trivial pseudo-quasimetric $d(x, y) = 0$ for all $x, y \in F^a(X, \mathcal{V})$. For all $a, b \in F^a(X, \mathcal{V})$ we put $\hat{\rho}(a, b) = sup\{d(a, b) : d \in Q(\rho)\}$. We say that $\hat{\rho}$ is the maximal stable extension of ρ on $F^a(X, \mathcal{V})$.

Property 6.1. $\hat{\rho} \in Q(\rho)$.

Proof. Obviously $d(x, y) \leq \rho(x, y)$ for $x, y \in X$. Let $d \in Q(\rho)$. Fix two points $a, b \in F^a(X, \mathcal{V})$. There exists $n \in \mathbb{N}$ and $x_1, y_1, x_2, y_2, ..., x_n, y_n \in X$ such that $a = x_1x_2...x_n$ and $b = y_1y_2...y_n$. Then $d(a, b) \leq \Sigma\{d(x_i, y_i) : i \leq n\} \leq \Sigma\{\rho(x_i, y_i) : i \leq n\}$. Hence $\rho(a, b) \leq sup\{\Sigma\{d(x_i, y_i) : i \leq n\} : d \in Q(\rho)\} \leq \Sigma\{\rho(x_i, y_i) : i \leq n\} < +\infty$. Therefore, by virtue of Lemma 6.1, $\hat{\rho}$ is a stable pseudo-quasimetric from the set $Q(\rho)$. □

For any $r > 0$ we put $d_r(a, a) = 0$ and $d_r(a, b) = r$ for all distinct points $a, b \in F^a(X, \mathcal{V})$. Then d_r is an invariant metric on $F^a(X, \mathcal{V})$.

Property 6.2. Let $r > 0$ and $\rho(x, y) \geq r$ for all distinct points $x, y \in X$. Then $\hat{\rho}$ is a quasimetric on $F^a(X, \mathcal{V})$, $d_r \in Q(\rho)$ and $\hat{\rho}(a, b) \geq r$ for all distinct points $a, b \in F^a(X, \mathcal{V})$.

Proof. It is obvious. □

For any $a, b \in F^a(X, \mathcal{V})$ we put $\bar{\rho} = inf\{\Sigma\{\rho(x_i, y_i) : i \leq n\} : n \in \mathbb{N}, x_1, y_1, x_2, y_2, ..., x_n, y_n \in X, a = x_1x_2...x_n, b = y_1y_2...y_n\}$ and $\rho^*(a, b) = inf\{\bar{\rho}(a, z_1) + ... + \bar{\rho}(z_i, z_{i+1}) + ... + \bar{\rho}(z_n, b) : n \in \mathbb{N}, z_1, z_2, ..., z_n \in F^a(X, \mathcal{V})\}$.

Property 6.3. *$\bar{\rho}$ is a pseudo-distance on $F^a(X,\mathcal{V})$ and $\bar{\rho}(x,y) \leq \rho(x,y)$ for all $x,y \in X$.*

Proof. Obviously, $\bar{\rho}$ is a pseudo-distance. If $a,b \in X$, then $a = ae = a$, $b = be = b$ and $\bar{\rho}(a,b) = inf\{\Sigma\{\rho(x_i,y_i) : i \leq n\} : n \in \mathbb{N}, x_1, y_1, x_2, y_2, ..., x_n, y_n \in X, a = x_1x_2...x_n, b = y_1y_2...y_n\} \leq \rho(a,b)$. □

Property 6.4. *Let $\mathcal{V}$ be a non-Burnside quasi-variety. Then $\bar{\rho}(x,y) = \rho(x,y)$ for all $x,y \in X$.*

Proof. Assume that $n \in \mathbb{N}$, $x_1, y_1, x_2, y_2, ..., x_n, y_n \in X$, $x = x_1x_2...x_n$ and $y = y_1y_2...y_n$. There exist $i,j \leq n$ for which $x = x_i$ and $y = y_j$. We have two possible cases.

Case 1. $i = j$.

In this case, as was mention in Corollary 5.1, $x_k = y_k = e$ for each $k \neq i$. Thus $\Sigma\{\rho(x_i,y_i) : i \leq n\} = \rho(x_i,y_i) = \rho(x,y)$.

Case 2. $i \neq j$.

In this case, as was mention in Corollary 5.1, we have $x_j = y_i = e$. Hence $\Sigma\{\rho(x_i,y_i) : i \leq n\} \geq \rho(x_i,y_i) + \rho(x_j,y_j) = \rho(x,e) + \rho(e,y) \geq \rho(x,y)$. The proof is complete. □

Property 6.5. *The pseudo-distance $\bar{\rho}$ is stable on $F^a(X,\mathcal{V})$.*

Proof. Fix $a,b,c \in F^a(X,\mathcal{V})$ and $\varepsilon > 0$. Let $c = z_1z_2...z_m$. There exist $n \in \mathbb{N}$ and the words $a = x_1x_2...x_n$, $b = y_1y_2...y_n$ such that $\bar{\rho}(a,b) \leq \Sigma\{\rho(x_i,y_i) : i \leq n\} < \rho(a,b) + \varepsilon$. Then $\bar{\rho}(ac,bc) = \bar{\rho}(x_1x_2...x_nz_1z_2...z_m, y_1y_2...y_nz_1z_2...z_m) \leq \Sigma\{\rho(x_i,y_i) : i \leq n\} < \bar{\rho}(a,b) + \varepsilon$. Hence $\bar{\rho}(ac,bc) \leq \bar{\rho}(a,b)$. The proof of inequality $\bar{\rho}(ca,cb) \leq \bar{\rho}(a,b)$ is similar. Proposition 2.1 completes the proof. □

Property 6.6. *The pseudo-distance ρ^* is a stable pseudo-quasimetric on $F^a(X,\mathcal{V})$* and $\rho^* \in Q(\rho)$.

Proof. Follows from Properties 6.2 and 6.4. □

In the following properties we assume that $\mathcal{V}$ is a non-Burnside quasi-variety.

Property 6.7. *If ρ is a quasimetric on X, then $\bar{\rho}$ is a distance on $F^a(X,\mathcal{V})$.*

Proof. Assume that ρ is a quasimetric on X and $\bar{\rho}$ is not a distance on $F^a(X,\mathcal{V})$. There exist two distinct points $b,c \in F^a(X,\mathcal{V})$ such that $\bar{\rho}(b,c) = \bar{\rho}(c,b) = 0$. Suppose that $n \geq 2$ and $l(b) + l(c) \leq n$. Then $\bar{\rho}(b,c) = inf\{\Sigma\{\rho(x_i,y_i) : i \leq m\} : m \in \mathbb{N}, m \leq 4n^2, x_1, x_2, ..., x_m \in Sup(b,b), y_1, y_2, ..., y_m \in Sup(c,c), b = x_1x_2...x_m, c = [y_1y_2...y_m]\}$.

Since $\bar{\rho}(b,c) = 0$, there exist $m \in \mathbb{N}$, $x_1, x_2, ..., x_m \in Sup(\{b\}) \cup \{e\}$, and $y_1, y_2, ..., y_m \in Sup(\{c\}) \cup \{e\}$ such that $b = x_1x_2...x_m$, $c = y_1y_2...y_m$ and $\bar{\rho}(b,c) = \Sigma\{\rho(x_i,y_i) : i \leq m\} = 0$. Since $\bar{\rho}(c,b) = 0$, there exist $k \in \mathbb{N}$, $c_1, c_2, ..., c_k \in Sup(\{c\}) \cup \{e\}$, $b_1, b_2, ..., b_k \in Sup(\{b\}) \cup \{e\}$ such that $b = b_1b_2...b_k$, $c = c_1c_2...c_k$ and $\bar{\rho}(c,b) = \Sigma\{\rho(c_j,b_j) : j \leq k\} = 0$.

Fix $i_1 \leq m$. Then $\rho(x_{i_1}, y_{i_1}) = 0$. There exists j_1 such that $c_{j_1} = y_{i_1}$. Then $\rho(c_{j_1}, b_{j_1}) = 0$. There exists i_2 such that $x_{i_2} = b_{j_1}$. Then $\rho(x_{i_2}, y_{i_2}) = 0$ and so on. As a result, we obtain a sequence x_{i_1}, $y_{i_1} = c_{j_1}$, $b_{j_1} = x_{i_2}$, $y_{i_2} = c_{j_2}$, ..., x_{i_p}, $y_{i_p} = c_{j_p}$, $b_{j_p} = x_{i_{p+1}}$, $y_{i_{p+1}} = c_{j_{p+1}}$, ... such that $\rho(x_{i_p}, y_{i_p}) = \rho(c_{j_p}, b_{j_p}) = 0$ for any $p \in \mathbb{N}$. Since x_{i_1}, x_{i_2}, ... , x_{i_p}, ... are elements of a finite set $Sup(b,b) = Sup(b) \cup \{e\}$, there exist two numbers $p, q \in \mathbb{N}$ such that $q < p$ and $x_{i_q} = x_{i_p}$. Hence $\rho(x_{i_q}, y_{i_q}) = 0$ and $0 \leq \rho(y_{i_q}, x_{i_q}) = \rho(y_{i_q}, x_{i_p}) \leq \rho(y_{i_q}, c_{j_q}) + \rho(c_{j_q}, b_{j_q}) + \rho(x_{i_{q+}}, y_{i_{q+1}}) + ... + \rho(c_{j_{p-1}}, b_{p_{p-1}}) + \rho(b_{j_{p-1}}, x_{i_p}) = 0$, a contradiction. The proof is complete. □

Property 6.7 is not true for Burnside quasi-varieties.

Example 6.1. Let $n \in \mathbb{N}$ and $n \geq 2$. Consider the quasi-variety $\mathcal{W}$ of topological monoids (groups) with the identities $x^n = e$. Let $\prec$ be a linear ordering on a set X, $|X| \geq 2$, and $e \preceq x$ for each $x \in X$. We put $\rho(x,x) = 0$ for each $x \in X$ and for distinct $x, y \in X$ with $x \prec y$ we put $\rho(x,y) = 1$ and $\rho(y,x) = 0$. Then ρ is a quasimetric on X. Fix $a, b \in X$ with $a \preceq b$. Then $\bar{\rho}(b,a) = 0$ and $\bar{\rho}(a,b) = \bar{\rho}(b^n a, be^n) \leq \rho(b,a) + (n-1)\rho(b,e) + \rho(a,e) = 0$.

Fix now $a, b \in F^a(X, \mathcal{W})$. There exists $m \in \mathbb{N}$ and $x_1, y_1, x_2, y_2, ..., x_m, y_m \in X$ such that $a = x_1 x_2 ... x_m$ and $b = y_1 y_2 ... y_m$. By virtue of Property 6.5, we have $0 \leq \bar{\rho}(a,b) = \bar{\rho}(x_1 x_2 ... x_m, y_1 y_2 ... y_m) \leq \Sigma\{\bar{\rho}(x_i, y_i) : i \leq m\} = 0$. Hence $\bar{\rho}(x,y) = 0$ for all $x, y \in F^a(X, \mathcal{W})$. Therefore $\hat{\rho}(x,y) = 0$ for all $x, y \in F^a(X, \mathcal{W})$.

Example 6.2. Let $p, q \in \mathbb{N}$ and $1 \leq q < p = q + k$. Consider the non-trivial quasi-variety $\mathcal{W}$ of topological monoids with the identity $x^q = x^p$. Fix a set X with three distinct elements $\{e, a, b\}$. Let $\prec$ be a linear ordering on a set X and $e \prec a \prec b$. We put $\rho(x,x) = 0$ for each $x \in X$ and for distinct $x, y \in X$ with $x \prec y$ we put $\rho(x,y)$ $= 1$ and $\rho(y,x) = 0$. Then ρ is a quasimetric on X. We have $\rho(x,x) = 0$ for each $x \in X$, $\rho(e,a) = \rho(e,b) = \rho(a,b) = 1$ and $\rho(b,a) = \rho(a,e) = \rho(b,e) = 0$.

We put $u = b^q \in F^a(X, \mathcal{W})$ and $v = a^q b^q \in F^a(X, \mathcal{W})$. There exist two numbers for which $q + k(p-q) = 2q + m$. By construction, $\hat{\rho}(v,u) = \hat{\rho}(a^q b^q, e^q b^q) \leq q(\rho(a,e) + \rho(b,b)) = 0$ and $\hat{\rho}(u,v) = \bar{\rho}(b^q, a^q b^q) = \bar{\rho}(b^{q+k(p-q)}, a^q b^q e^m) = \bar{\rho}(b^q b^q b^m, a^q b^q e^m) = q\rho(b,a) + q\rho(b,b) + m\rho(b,e) = 0$. Hence $\bar{\rho}(x,y) + \bar{\rho}(v,u) = 0$. Therefore $\hat{\rho}(u,v) + \hat{\rho}(v,u) = 0$.

Example 6.3. Consider the quasi-variety $\mathcal{V} = \mathcal{W}_{(0,2)}$ of topological monoids with the identity $x^2 = e$. Let $X = \{e, a, b\}$, $\rho(x,x) = 0$ for each $x \in X$, $\rho(a,b) = \rho(e,a)$ $= \rho(b,e) = 0$, $\rho(b,a) = \rho(a,e) = \rho(e,b) = 1$. We have $F^a(X, \mathcal{V}) = \{e, a, b, ab\}$ and $ab = ba$. In this case ρ is not a quasimetric and $\bar{\rho}(b,a) = \bar{\rho}(be, ea) = 0 < \rho(b,a) = 1$, $\bar{\rho}(a,b) = \rho(a,b) = 0$, $\bar{\rho}(a,ab) = \bar{\rho}(ea,bb) = 0$, $\bar{\rho}(ab,a) = \bar{\rho}(ab,ae) = 0$, $\bar{\rho}(ab,b) = \bar{\rho}(ab,be) = 0$, $\bar{\rho}(b,ab) = \bar{\rho}(eb,ab) = 0$, $\bar{\rho}(e,b) = \bar{\rho}(bb,be) = 0$, $\bar{\rho}(ab,e) = \bar{\rho}(ab,bb) = 0$, $\bar{\rho}(e,ab) = \bar{\rho}(ebb,aeb) = 0$, $\bar{\rho}(e,b) = \bar{\rho}(ebb,eeb) = 0$. Hence $\bar{\rho} = \hat{\rho}$ is the trivial pseudo-metric on $F^a(X, \mathcal{V})$.

Property 6.7 is not true for distances which are not quasimetrics.

Example 6.4. Consider a non-trivial quasi-variety $\mathcal{V}$ of topological monoids. Let $X = \{e, a, b\}$, $\rho(x,x) = 0$ for each $x \in X$, $\rho(a,b) = \rho(e,a) = \rho(b,e) = 0$, $\rho(b,a) =$

$\rho(a,e) = \rho(e,b) = 1$. In this case $\bar{\rho}(b,a) = \bar{\rho}(be,ea) = 0 < \rho(b,a) = 1$ and $\bar{\rho}(a,b) = \rho(a,b) = 0$.

Property 6.8. *Let $a,b \in F^a(X,\mathcal{V})$ be two distinct points in $F^a(X,\mathcal{V})$ and $r(a,b) = min\{\rho(x,y) : x \in Sup(a,a), y \in Sup(b,b), x \neq y\}$. Then $\hat{\rho}(a,b) = \rho^*(a,b) \geq r(a,b)$.*

Proof. Assume that $r(a,b) - \rho^*(a,b) = 3\delta > 0$. There exist $n \in \mathbb{N}$ and $z_1, z_2, ..., z_n \in F^a(X,\mathcal{V})$ such that $\rho^*(a,b) \leq \bar{\rho}(a,z_1)$ +...+ $\bar{\rho}(z_i,z_{i+1})$ + ... + $\bar{\rho}(z_n,b) < \rho^*(a,b) + \delta$. Let $z_0 = a$ and $z_{n+1} = b$. For each $i \in \{0,1,2,...,n\}$ there exist the representations $z_i = u_{(i,1)}u_{(i,2)}...u_{(i,m_i)}$ and $z_{i+1} = v_{(i,1)}v_{(i,2)}...v_{(i,m_i)}$ such that $\{u_{(i,1)}, u_{(i,2)}, ..., u_{(i,m_i)}\} \subseteq Sup(z_i,z_i)$, $\{v_{(i,1)}, v_{(i,2)}, ..., v_{(i,m_i)}\} \subseteq Sup(z_{i+1},z_{i+1})$ and $\bar{\rho}(z_i,z_{i+1}) \leq \Sigma\{\rho(u_{(i,j)}, v_{(i,j)} : j \leq m_i\} \leq \bar{\rho}(z_i,z_{i+1}) \leq \delta/(n+1)$. Without lost of generality, we can assume that there exists $m \in \mathbb{N}$ such that $m_i = m$ for each $i \in \{0,1,2,...,n\}$. For each $i \in \{0,1,2,...,n\}$ there exists a one-to-one mapping $h_i : \{1,2,...,m\} \longrightarrow \{1,2,...,m\}$ such that $v_{(i,j)} = u_{(i+1,h_i(j))}$ for each $j \leq m$. Then the chain $j_0 = j$, $j_1 = h_1(j)$, $j_2 = h_2(j_1)$, ..., $j_n = h_n(j_{n-1})$ and the number r_j $= \rho(u_{(0,j_0)}, v_{(0,j_0)}) + \rho(u_{(1,j_1)}, v_{(1,j_1)}) + ...+ \rho(u_{(n,j_n)}, v_{(n,j_n)}) \geq \rho(u_{(0,j_0)}, v_{(n,j_n)})$ are determined for any $j \leq m$. We put $h(j) = j_n$. Then $h : \{1,2,...,m\} \longrightarrow \{1,2,...,m\}$ is a one-to-one mapping as the composition of the mappings $h_1, h_2, ..., h_n$. We obtain that $\rho^*(a,b) + 3\delta \leq \bar{\rho}(a,z_1), ..., \bar{\rho}(z_i, z_{i+1}$ + ... + $\bar{\rho}(z_n,b) \geq \bar{\rho}(a,b)$ $r(a,b)$. The proof is complete. □

The following properties follow from Property 6.8.

Property 6.9. *If ρ is a quasimetric on X, then ρ^* and $\hat{\rho}$ are quasimetrics on $F^a(X,\mathcal{V})$.*

Property 6.10. *If ρ is a strong quasimetric on X, then ρ^* and $\hat{\rho}$ are strong quasimetrics on $F^a(X,\mathcal{V})$.*

Proved properties lead us to the following general result:

Theorem 6.1. *Let ρ be a pseudo-quasimetric on X, Y be a subspace of X and $e \in Y$. Denote by $M(Y) = F^a(Y,\mathcal{V})$ the submodule of the module $F^a(X,\mathcal{V})$ generated by the set Y and by d_Y the extension $\widehat{\rho|Y}$ on $M(Y)$ of the pseudo-quasimetric ρ_Y on Y, where $\rho_Y(y,z) = \rho(y,z)$ for all $y,z \in Y$. Then:*

1. $d_Y(a,b) = \hat{\rho}(a,b)$ for all $a,b \in M(Y)$.

2. If ρ is a (strong) quasimetric on Y, then $\hat{\rho}$ is a (strong) quasimetric on $M(Y)$.

3. If ρ is a metric on Y, then $\hat{\rho}$ is a metric on $M(Y)$.

4. If $a,b \in F^a(Y,\mathcal{V})$ are distinct points and ρ is a quasimetric on $Sup(a,b)$, then $\hat{\rho}(a,b) + \hat{\rho}(b,a) > 0$.

5. If $a,b \in F^a(Y,\mathcal{V})$ are distinct points and ρ is a strong quasimetric on $Sup(a,b)$, then $\hat{\rho}(a,b) > 0$ and $\hat{\rho}(b,a) > 0$.

6. For any $a,b \in F^a(Y,\mathcal{V})$ there exist $n \in \mathbb{N}$, $x_1, x_2, ..., x_n \in Sup(a,a)$ and $y_1, y_2, ..., y_n \in Sup(b,b)$ such that $a = x_1x_2...x_n$, $b = y_1y_2...y_n$, $n \leq l(a) + l(b)$ and $\bar{\rho}(a,b) = \Sigma\{\rho(x_i,y_i) : i \leq n\}$.

7. $\hat{\rho} = \bar{\rho} = \rho^*$.

The following assertion is obvious.

Proposition 6.1. *Let ρ be a pseudo-quasimetric on X and $\mathcal{V}$ be a non-Burnside quasi-variety of topological monoids. For any $a{=}a_1a_2...a_n \in F^a(X,\mathcal{V})$ we put $a^{\leftarrow} = a_n...a_2a_1$. Then $a^{\leftarrow} \in F^a(X,\mathcal{V})$, $\rho^*(a,b){=}\rho(a^{\leftarrow},b^{\leftarrow})$ and $(ab)^{\leftarrow}{=}b^{\leftarrow}a^{\leftarrow}$ for all $a,b \in F^a(X,\mathcal{V})$.*

Remark 6.1. Invariant pseudo-metrics on free groups were constructed by M. I. Graev [21]. Stable metrics on free algebras were considered in [11]. Invariant quasi-metrics on free groups were constructed in [17] and [42].

Remark 6.2. Let A be a non-empty set and $\mathcal{V}$ be the non-Burnside quasi-variety of all topological monoids. Consider that $\varepsilon \notin A$ and $X = A \cup \{\varepsilon\}$. Let $\rho(x,x) = 0$ and $\rho(x,y) = 1$ for all distinct points $x,y \in X$. Then $L(A) = F(X,\mathcal{V})$ is the family of all strings on the alphabet A. In this case there exists the maximal invariant extension $\hat{\rho}$ of ρ on $L(A)$. The metric $\hat{\rho}$ was studied in [14,15]. It was proved that the metric $\hat{\rho}$ coincides with the V. I. Levenshtein metric on $L(A)$ [32].

7 Strongly invariant quasimetrics

Fix the non-Burnside quasi-variety of topological monoids $\mathcal{V}$ and a space X with basepoint p_X.

Consider on X some linear ordering for which $p_X \preceq x$ for any $x \in X$. On X consider the following distances ρ_l, ρ_r, ρ_s, where $\rho_l(x,x) = \rho_r(x,x) = 0$ for any $x \in X$; if $x,y \in X$ and $x \prec y$, then $\rho_l(x,y) = 1$, $\rho_l(y,x) = 0$, $\rho_r(x,y) = 0$, $\rho_r(y,x) = 1$, $\rho_s(x,y) = \rho_l(x,y) + \rho_r(x,y)$. By construction, ρ_l and ρ_r are quasimetrics and ρ_s is a metric on X. Then $\rho_l^*(x,y)$ and $\rho_r^*(x,y)$ are invariant discrete quasimetrics on $F(X,\mathcal{V})$ and ρ_s^* is a discrete invariant metric on $F(X,\mathcal{V})$. We consider this metric below.

A distance d on a semigroup G is strongly invariant if $d(xz,yz) = d(zx,zy) = d(x,y)$ for all $x,y,z \in G$.

On a group any invariant pseudo-quasimetric is strongly invariant. For monoids that fact is not true.

Example 7.1. Consider a semigroup $H = \{e,a,b\}$, where $ex = xe = x$ for each $x \in H$ and $xy = a$ provided $e \notin \{x,y\} \subset H$. The discrete metric d on H such that $d(x,y) = 0$ for $x = y$ and $d(x,y) = 1$ for $x \neq y$ is invariant on H and is not strongly invariant, since $0 = d(a,a) = d(ab,bb) = d(ba,bb) < d(a,b) = 1$. Let $\mathcal{W}(H)$ be the complete variety of topological monoids generated by the monoid H. For every monoid $G \in \mathcal{W}(H)$ there exists a unique point $a_G \in G$ such that $xy = a_G$ provided that $e \notin \{x,y\}$. Let X be a space with the basepoint p_X, $|X| \geq 2$ and ρ be a metric on X such that $\rho(x,y) = 1$ for all distinct points $x,y \in X$. Then ρ^* is an invariant metric on $F(X,\mathcal{W}(H))$ and $\rho^*(x,y) \geq 1$ for all distinct points $x,y \in F(X,\mathcal{W}(H))$. Let $c \in X \subseteq F(X,\mathcal{W}(H))$ and $c \neq p_X = e$. Then $c^2 \in F(X,\mathcal{W}(H))$ and $c^2 \neq c$. We have that $c^n = c^3 = c^2$ for any $n \geq 3$. Hence $1 \leq \rho^*(c,c^2)$ and $0 = \rho^*(c^2,c^2) = \rho^*(c^2,c^3) = \rho^*(c \cdot c, c^2 \cdot c) < \rho^*(c,c^2)$. In $F(X,\mathcal{W}(H))$ there exists a point $a \neq e$ such that $xy = a$ provided $e \notin \{x,y\}$. Hence the metric ρ^* is not strongly invariant on $F(X, \mathcal{W}(H))$. We observe that $\mathcal{W}(H)$ is a Burnside variety of the exponent (3,2). The above considerations permit to state that on the free monoid $F(X, \mathcal{W}(H))$ any invariant quasimetric is not strongly invariant.

For any pseudo-distance d S. Nedev [36] considered the adjoint pseudo-distance d^a defined by $d^a(x, y) = d(y, x)$.

Two properties $\mathcal{P}_1$ and $\mathcal{P}_2$ are called adjoint properties if the pseudo-distance d on a space X has property $\mathcal{P}_1$ if and only if the adjoint pseudo-distance d^a on a space X has property $\mathcal{P}_2$. If $\mathcal{P}_1 = \mathcal{P}_2$ and the properties $\mathcal{P}_1$ and $\mathcal{P}_2$ are adjoint, then we say that the property $\mathcal{P}_1$ is auto-adjoint.

Remark 7.1. The auto-adjoint properties are the conditions for pseudo-distance to be invariant or strongly invariant on a semigroup G.

The proof of the following assertion is simple.

Proposition 7.1. *Let $\mathcal{V}$ be a non-trivial quasi-variety of topological monoids, ρ be a pseudo-distance on a space X with basepoint p_X. If $d = \rho^a$, then $d^* = \rho^{*a}$, i.e. $\rho^{a*} = \rho^{*a}$.*

The quasi-variety of topological monoids $\mathcal{V}$ is rigid if for any space X, any word $a \in F(X, \mathcal{V})$, any point $c \in X \setminus \{p_x\}$ and any representation $ac = x_1x_2...x_n$, where $x_1, x_2, ..., x_n \in X$, there exists $m \leq n$ such that $x_m = c$ and $a = x_1x_2...x_{m-1}$. In this case $x_i = p_X = e$ for each $i > m$.

The variety of all topological monoids is rigid.

Theorem 7.1. *Let $\mathcal{V}$ be a non-Burnside rigid quasi-variety of topological monoids, ρ be a quasimetric on a space X with basepoint p_X and $\rho(x, p_X) = \rho(y, p_X)$ for all $x, y \in X \setminus \{p_X\}$, or $\rho(p_X, x) = \rho(p_X, y)$ for all $x, y \in X \setminus \{p_X\}$. Then $\rho^*(ac, bc) = \rho^*(ca, cb) = \rho^*(a, b)$ for all $a, b, c \in F(X, \mathcal{V})$.*

Proof. Assume that $\rho(p_X, x) = \rho(p_X, y)$ for all $x, y \in X \setminus \{p_X\}$. It is sufficient to prove the assertion of the theorem for $c \in X$. Assume that $\rho^*(ac, bc) = r < \rho^*(a, b)$, where $a, b \in F(X, \mathcal{V})$ and $c \in A$. Then, by definition, there exist the representations $ac = x_1x_2 \cdots x_n$ and $bc = y_1y_2 \cdots y_n$ such that $\rho^*(ac, bc) = \Sigma\{d(x_i, y_i) : i \leq p\}$.

From the definition of rigidity, there exist $p, q \leq n$ such that $x_p = y_q = c$, $a = x_1x_2...x_{p-1}$, $b = y_1y_2...y_{q-1}$ and $x_i = y_j = p_X$ with $p < i \leq n$ and $q < j \leq n$. We can assume that $n = max\{p, q\}$.

Case 1. $n = p = q$.

In this case $a = x_1x_2 \cdots x_{n-1}$, $b = y_1y_2 \cdots y_{n-1}$ and $\rho^*(a, b) \leq \Sigma\{d(x_i, y_i) : i \leq n - 1\} = \Sigma\{d(x_i, y_i) : i \leq n\} = \rho^*(ac, bc) < \rho^*(a, b)$, a contradiction.

Case 2. $q < p = n$.

Then $y_n = p_X$, $x_n = y_q = c$, $a = x_1x_2 \cdots x_{n-1}$, $b = y_1y_2...y_{q-1} = y'_1y'_2...y'_{n-1}$, where $y'_j = y_j$ for $j < q$ and $y'_j = p_X$ for $j \geq q$. Since $\rho(x_q, p_X) \leq \rho(x_q, c) + \rho(c, p_X)$, we have $\rho^*(a, b) \leq \Sigma\{d(x_i, y'_i) : i \leq n - 1\} \leq \Sigma\{d(x_i, y_i) : i \leq n\} = \rho^*(ac, bc) < \rho^*(a, b)$, a contradiction.

Case 3: $p < q = n$.

Then $x_n = p_X$, $y_n = x_p = c$, $a = x_1x_2\cdots x_{p-1} = x'_1x'_2...x'_{n-1}$, $b = y_1y_2...y_{n-1}$, where $x'_i = x_i$ for $i < p$ and $x'_i = p_X$ for $i \geq p$. Since $\rho(p_X, y_p) \leq \rho(p_X, c)$, we have $\rho^*(a, b) \leq \Sigma\{d(x'_i, y_i) : i \leq n-1\} \leq \Sigma\{d(x_i, y_i) : i \leq n\} = \rho^*(ac, bc) < \rho^*(a, b)$, a contradiction.

Therefore, we proved that $\rho^*(ac, bc) = \rho^*(a, b)$ for all $a, b, c \in F(X, \mathcal{V})$. By virtue of Proposition 6.1, we have $\rho^*(ca, cb) = \rho^*(a^{\leftarrow}c^{\leftarrow}, b^{\leftarrow}c^{\leftarrow}) = \rho^*(a^{\leftarrow}, b^{\leftarrow}) = \rho^*(a, b)$ for all $a, b, c \in F(X, \mathcal{V})$.

Since the properties "$\rho(x, p_X) = \rho(y, p_X)$ for all $x, y \in X \setminus \{p_X\}$" and "$\rho(p_X, x) = \rho(p_X, y)$ for all $x, y \in X \setminus \{p_X\}$" are adjoint, the proof is complete. □

Corollary 7.1. *Let $\mathcal{V}$ be the non-Burnside rigid quasi-variety of topological monoids, the space X is linear ordered such that $p_X \preceq x$ for any $x \in X$. If $\rho \in \{\rho_l, \rho_r, \rho_s\}$, then ρ^* is a strongly invariant quasimetric on $F(X, \mathcal{V})$.*

The following question is open.

Problem 7.1. *Does Theorem 7.1 hold for any non-Burnside quasi-variety of topological monoids?*

8 Free monoids of T_0-spaces

Suppose that X is a topological space. Let x and y be points in X. We say that x and y can be separated by a function if there exists a continuous function $f : X \to [0, 1]$ into the unit interval such that $f(x) = 0$ and $f(y) = 1$.

A functionally Hausdorff space is a space in which any two distinct points can be separated by a continuous function.

The pseudo-distance d is continuous on a space X if any d-open subset $U \in \mathcal{T}(d)$ is open in X.

Lemma 8.1. *Let Y be a non-empty finite subspace of a T_0-space X. Then on X there exists a continuous pseudo-quasimetric d_Y such that d_Y on Y generates the topology of the subspace Y.*

Proof. There exists a finite minimal family $\{U_1, U_2, ..., U_n\}$ of open subsets of X such that $T = \{U_1 \cap Y, U_2 \cap Y, ..., U_n \cap Y\}$ is the topology of the subspace Y. For each $i \leq n$ we put $d_i(x, y) = 1$ for $x \in U_i$, $y \in X \setminus U_i$ and $d_i(x, y) = 0$ for $x \in X \setminus U_i$ or $y \in U_i$. Then d_i is a continuous pseudo-quasimetric on X and $\mathcal{T}(d_i) = \{\emptyset, U_i, X\}$. Hence $d_Y(x, y) = max\{d_i(x, y) : i \leq n\}$ is the desired pseudo-quasimetric on X. □

The following theorem improves Theorem 5.3 and solves Problem 3.2 for complete non-Burnside quasi-varieties of topological monoids.

Theorem 8.1. *Let $\mathcal{V}$ be a non-trivial complete non-Burnside quasi-variety of topological monoids. Then:*

1. For each T_0-space X on the free monoid $F^a(X, \mathcal{V})$ there exists a T_0-topology $\mathcal{T}(qm)$ such that:

– $(F^a(X,\mathcal{V}),\mathcal{T}(qm)) \in \mathcal{V}$;

– X is a subspace of the space $(F^a(X,\mathcal{V}),\mathcal{T}(qm))$;

– the topology $\mathcal{T}(qm)$ is generated by the family of all invariant pseudo-quasimetrics on $F^a(X,\mathcal{V})$ which are continuous on X.

2. For each T_0-space X the free topological monoid $F(X,\mathcal{V})$ exists and is abstract free.

3. A space X is a T_1-space if and only if spaces $F(X,\mathcal{V})$ and $(F^a(X,\mathcal{V}),\mathcal{T}(qm))$ are T_1-spaces.

4. A space X is functionally Hausdorff if and only if the spaces $F(X,\mathcal{V})$ and $(F^a(X,\mathcal{V}),\mathcal{T}(qm))$ are functionally Hausdorff.

Proof. Fix a T_0-space X. Let $Q(X)$ be the family of all continuous pseudo-quasimetrics on X and $IQ(X)$ be the family of all invariant pseudo-quasimetrics on $(F^a(X,\mathcal{V}))$ which are continuous on X. Then $\mathcal{T}(qm)$ is the topology on $(F^a(X,\mathcal{V}))$ generated by the pseudo-quasimetrics $IQ(X)$.

Claim 1. *X is a subspace of the space $(F^a(X,\mathcal{V}),\mathcal{T}(qm))$.*

By virtue of Theorem 6.1, for each $\rho \in Q(X)$ we have $\hat{\rho} \in IQ(X)$ and $\rho(x,y) = \hat{\rho}(x,y)$ for all $x,y \in X$. Hence the pseudometrics $Q(X)$ and $IQ(X)$ generate on X the same topology. By virtue of Lemma 8.1, the topology of the space X is generated by the family of all continuous pseudo-quasimetrics $Q(X)$. Hence X is a subspace of the space $(F^a(X,\mathcal{V}),\mathcal{T}(qm))$.

Claim 2. *$(F^a(X,\mathcal{V}),\mathcal{T}(qm))$ is a T_0-space.*

Fix two distinct points $a,b \in F^a(X,\mathcal{V})$. Let Y be a finite subspace of X such that $p_X \in Y$ and $a,b \in F^a(Y,\mathcal{V}) \subseteq F^a(X,\mathcal{V})$. By virtue of Lemma 8.1, on X there exists a continuous pseudo-quasimetric d_Y which is a quasimetric on Y. From the assertion 4 of Theorem 6.1 it follows that $\hat{d_Y}$ is a quasimetric on $F^a(Y,\mathcal{V})$. Hence $\hat{d_Y}(a,b) + \hat{d_Y}(b,a) > 0$. Therefore $(F^a(X,\mathcal{V}),\mathcal{T}(qm))$ is a T_0-space.

Claim 3. *The topology $\mathcal{T}(qm)$ is generated by the family of all invariant pseudo-quasimetrics $F^a(X,\mathcal{V})$ which are continuous on X.*

That assertion follows from the definition of the topology $\mathcal{T}(qm)$.

Claim 4. *$(F^a(X,\mathcal{V}),\mathcal{T}(qm)) \in \mathcal{V}$.*

Since the topology $\mathcal{T}(qm)$ is generated by the invariant pseudo-quasimetrics, $(F^a(X,\mathcal{V}),\mathcal{T}(qm))$ is a a topological monoid. Hence the assertion of Claim 4 follows from Claim 2 and completeness of the quasi-variety $\mathcal{V}$.

Claim 5. *For the T_0-space X the free topological monoid $F(X,\mathcal{V})$ is abstract free.*

Let G be the topological monoid $(F^a(X,\mathcal{V}),\mathcal{T}(qm))$. There exists a continuous homomorphism $h : F(X,\mathcal{V}) \longrightarrow G$ such that $h(x) = x$ for each $x \in X$. Since G is abstract free relatively to X, h is a continuous isomorphism. Claim 5 is proved.

Claim 6. *A space X is a T_1-space if and only if the spaces $F(X,\mathcal{V})$ and $(F^a(X,\mathcal{V}),\mathcal{T}(qm))$ are T_1-spaces.*

If $F(X,\mathcal{V})$ is a T_1-space, then X is a T_1-space as a subspace of T_1-space. If $(F^a(X,\mathcal{V}),\mathcal{T}(qm))$ is a T_1-space, then $F(X,\mathcal{V})$ is a T_1-space, since $F(X,\mathcal{V})$ admits a continuous isomorphism onto $(F^a(X,\mathcal{V}),\mathcal{T}(qm))$.

Assume now that X is a T_1-space. Fix two distinct points $a, b \in F^a(X, \mathcal{V})$. Let Y be a finite subspace of X such that $p_X \in Y$ and $a, b \in F^a(Y, \mathcal{V}) \subseteq F^a(X, \mathcal{V})$. By virtue of Lemma 8.1, on X there exists a continuous pseudo-quasimetric d_Y which is a discrete metric on Y. Then $\hat{d_Y}$ is a discrete metric on $F^a(Y, \mathcal{V})$ and $F^a(Y, \mathcal{V})$ is a discrete subspace of $(F^a(X, \mathcal{V}), \mathcal{T}(qm))$. Hence $\{a, b\}$ is a discrete subspace and $(F^a(X, \mathcal{V}), \mathcal{T}(qm))$ is a T_1-space. Claim 6 is proved.

Claim 7. *Let Y be a finite subspace of the functionally Hausdorff space X and $p_X \in Y$. Then there exists $d \in IQ(X)$ such that d is a pseudo-metric and $d(a, b) \geq 1$ for all distinct points $a, b \in F^a(Y, \mathcal{V})$.*

Let $\{(x_i, y_i) : i \leq n\}$ be the family of all ordered pairs $x, y \in Y$ such that $x \neq y$. For any $i \leq n$ fix a continuous function $f_i : X \to [0, 1]$ such that $h_i(x_i) = 0$ and $h_i(y_i) = 1$. Then $r_Y(x, y) = min\{1, \Sigma\{|f_i(x) - f_i(y)| : i \leq n\}\}$ is a continuous pseudo-metric on X and $r_Y(x, y) = 1$ for any two distinct points $x, y \in Y$. Then $\hat{r_Y}$ is the desired pseudo-metric from $IQ(X)$.

Claim 8. *The space X is functionally Hausdorff if and only if the spaces $F(X, \mathcal{V})$ and $(F^a(X, \mathcal{V}), \mathcal{T}(qm))$ are functionally Hausdorff.*

If $F(X, \mathcal{V})$ is a functionally Hausdorff space, then X is a T_1-space as a subspace of a functionally Hausdorff space. If $(F^a(X, \mathcal{V}), \mathcal{T}(qm))$ is a functionally Hausdorff space, then $F(X, \mathcal{V})$ is a functionally Hausdorff space, since $F(X, \mathcal{V})$ admits a continuous isomorphism onto $(F^a(X, \mathcal{V}), \mathcal{T}(qm))$.

Assume now that X is a functionally Hausdorff space. Fix two distinct points $a, b \in F^a(X, \mathcal{V})$. Assume that $Y = Sup(a, b) = \{x_1, x_2, ..., x_n\}$, where $x_i \neq x_j$ for $i \neq j$. Since X is functionally Hausdorff space, there exists a construction function $f : X \to [0, 1]$ such that $f(x_i) \neq f(x_j)$ for $i \neq j$. Consider the continuous pseudo-metric $\rho(x, y) = |f(x) - f(y)|$, $x, y \in X$. We have $\rho(x_i, y_i) \neq 0$ for $i \neq j$. Hence ρ is a metric on Y. Then ρ^* is a continuous pseudo-metric on $F^a(X, \mathcal{V})$, and ρ^* is a metric on $F^a(Y, \mathcal{V})$. Hence $\rho^*(a, b) \neq 0$. The function $g(x) = \rho^*(a, x)$ is continuous on $(F^a(X, \mathcal{V}), \mathcal{T}(qm))$, $g(a) = 0$ and $g(b) \neq 0$. The function f is continuous on the space $(F^a(X, \mathcal{V}), \mathcal{T}(qm))$, $f(a) = 0$ and $f(b) = 1$. Hence $(F^a(X, \mathcal{V}), \mathcal{T}(qm))$ is a functionally Hausdorff space. The Claim 8 and Theorem 8.1 are proved. □

Corollary 8.1. *Let $\mathcal{V}$ be a complete non-trivial quasi-variety of topological monoids. Then for each completely regular space X:*

– on the free monoid $F^a(X, \mathcal{V})$ there exists a completely regular topology $\mathcal{T}(m)$ generated by a family of invariant pseudo-metrics such that $(F^a(X, \mathcal{V}), \mathcal{T}(m)) \in \mathcal{V}$, X is a subspace of the space $(F^a(X, \mathcal{V}), \mathcal{T}(m))$;

– the free topological monoid $F(X, \mathcal{V})$ exists, it is a functionally Hausdorff space and abstract free.

The following question is open.

Problem 8.1. *Let $\mathcal{V}$ be a non-trivial quasi-variety of topological monoids. Under which conditions for a space X the free topological monoid $F(X, \mathcal{V})$ is a Hausdorff space, or a regular space, or a completely regular space?*

Remark 8.1. Let X be a T_0-space and $\mathcal{V}$ be a non-trivial complete non-Burnside quasi-variety of topological monoids. Then on $F(X, \mathcal{V})$ there exist:

– the free topology $\mathcal{T}(f)$ such that $(F(x,\mathcal{V}),\mathcal{T}(f))$ is the free monoid of the space X in the quasi-variety $\mathcal{V}$;

– the topology $\mathcal{T}(qm)$ generated by the invariant continuous pseudo-quasimetrics on $(F(x,\mathcal{V}),\mathcal{T}(f))$;

– the topology $\mathcal{T}(m)$ generated by the invariant continuous pseudo-metrics on $(F(x,\mathcal{V}),\mathcal{T}(f))$.

These topologies satisfy the following properties:

P1. $\mathcal{T}(m) \subset \mathcal{T}(qm) \subset \mathcal{T}(f)$.

P2. $(F(x,\mathcal{V}),\mathcal{T}(m)),(F(x,\mathcal{V}),\mathcal{T}(f)) \in \mathcal{V}$.

P3. $(F(x,\mathcal{V}),\mathcal{T}(m)) \in \mathcal{V}$ if and only if X is a functionally Hausdorff space.

If the point p_X is isolated in X and $\mathcal{V}$ is the variety of all topological monoids, then on $F(X,\mathcal{V})$ we have $\mathcal{T}(qm) = \mathcal{T}(f)$. The invariant pseudo-metrics on topological groups were examined by G. Birkhoff [8] and Sh. Kakutani [28,29] (see [6,21,22]). There exists a locally compact topological group G with countable base without invariant metrics (see [22,28]). Since in G the involution $x \to x^{-1}$ is a homeomorphism, the topology of G is not generated by some family of invariant pseudo-quasimetrics.

The following question is open.

Problem 8.2. *Let $\mathcal{V}$ be a non-trivial quasi-variety of topological monoids. Under which conditions on $F(X,\mathcal{V})$ we have that $\mathcal{T}(qm) = \mathcal{T}(f)$?*

9 Free semi-topological monoids of T_0-spaces

A semi-topological semigroup is a semigroup with topology in which all translations $x \to ax$, $x \to xa$ are continuous.

A class $\mathcal{W}$ of semi-topological monoids is called a quasi-variety of monoids if:

(F1) the class $\mathcal{W}$ is multiplicative;

(F2) if $G \in \mathcal{W}$ and A is a submonoid of G, then $A \in \mathcal{V}$;

(F3) every space $G \in \mathcal{W}$ is a T_0-space.

A class $\mathcal{W}$ of semi-topological monoids is called a complete quasi-variety of monoids if it is a quasi-variety with the next property:

(F4) if $G \in \mathcal{V}$ and T is a T_0-topology on G such that (G,T) is a semi-topological monoid, then $(G,T) \in \mathcal{V}$ too.

A quasi-variety $\mathcal{V}$ of topological monoids is non-trivial if $|G| \geq 2$ for some $G \in \mathcal{V}$.

Let X be a non-empty topological space with a basepoint p_X and $\mathcal{W}$ be a quasi-variety of topological monoids.

A free monoid of a space X in a class $\mathcal{W}$ is a semi-topological monoid $F(X,\mathcal{W})$ with the properties:

– $X \subseteq F(X,\mathcal{V}) \in \mathcal{W}$ and p_X is the unity of $F(X,\mathcal{V})$;

– the set X generates the monoid $F(X,\mathcal{V})$;

– for any continuous mapping $f : X \longrightarrow G \in \mathcal{V}$, where $f(p_X) = e$, there exists a unique continuous homomorphism $\bar{f} : F(X,\mathcal{V}) \longrightarrow G$ such that $f = \bar{f}|X$.

The abstract free monoid $F^a(X,\mathcal{W})$ of a space X in a class $\mathcal{W}$ is defined for quasi-varieties of topological monoids.

Theorem 9.1. *Let $\mathcal{W}$ be a non-trivial quasi-variety of semi-topological monoids. Then for each space X the following assertions are equivalent:*

1. There exists $G \in \mathcal{W}$ such that X is a subspace of G and p_X is the neutral element in G.

2. For the space X there exists the unique free topological monoid $F(X, \mathcal{W})$.

Proof. Is similar to the proof of Theorem 3.1. □

Corollary 9.1. *Let $\mathcal{W}$ be a non-trivial quasi-variety of semi-topological monoids. Then for each space X there exists the unique abstract free monoid $F^a(X, \mathcal{W})$.*

Let $\mathcal{W}$ be a non-trivial quasi-variety of semi-topological monoids.

We put $\mathcal{W}_t = \{G \in \mathcal{W} : G \textit{ is a topological monoid}\}$. Obviously, $\mathcal{W}_t$ is a quasi-variety of topological monoids.

Fix a space X for which there exists the free semi-topological monoid $F(X, \mathcal{W})$. Then there exists a unique continuous homomorphism $\lambda_X : F^a(X, \mathcal{V}) \longrightarrow F(X, \mathcal{V})$ such that $\lambda_X(x) = x$ for each $x \in X$. The monoid $F(X, \mathcal{W})$ is called abstract free if λ_X is a continuous isomorphism.

Theorem 9.2. *Let $\mathcal{W}$ be a non-trivial non-Burnside quasi-variety of semi-topological monoids. Then for each space X the following assertions are equivalent:*

1. The class $\mathcal{W}_t$ is a non-trivial non-Burnside quasi-variety of topological monoids.

2. For each space X we have $F^a(X, \mathcal{W}) = F^a(X, \mathcal{W}_t)$.

3. For each T_0-space X on the free monoid $F^a(X, \mathcal{W})$ there exists a T_0-topology $\mathcal{T}(qm)$ such that:

– $(F^a(X, \mathcal{V}), \mathcal{T}(qm)) \in \mathcal{W}_t \subseteq \mathcal{W}$;

– X is a subspace of the space $(F^a(X, \mathcal{W}), \mathcal{T}(qm))$;

– the topology $\mathcal{T}(qm)$ is generated by the family of all invariant pseudo-quasimetrics on $F^a(X, \mathcal{V})$ which are continuous on X.

4. For each T_0-space X there exists the free topological monoid $F(X, \mathcal{W})$ and it is abstract free. Also, there exists a continuous isomorphism $\mu_X : F(X, \mathcal{W}) \longrightarrow F(X, \mathcal{W}_t)$ such that $\mu_X(x) = x$ for each $x \in X$.

5. A space X is a T_1-space if and only if spaces $F(X, \mathcal{W})$ and $(F^a(X, \mathcal{W}), \mathcal{T}(qm))$ are T_1-spaces.

6. A space X is functionally Hausdorff if and only if the spaces $F(X, \mathcal{W})$ and $(F^a(X, \mathcal{W}), \mathcal{T}(qm))$ are functionally Hausdorff.

Proof. Assertion 1 is obvious. For any space X denote by X_t the set X with the discrete topology. Then $G_t \in \mathcal{W}_t$ for each $G \in \mathcal{W}$. Fix a T_0-space X. The space $F^a(X, \mathcal{W})$ is discrete. Hence $F^a(X, \mathcal{W}) \in \mathcal{W}_t$ and Assertion 2 is proved.

Assertion 3 follows from Assertion 2 and Theorem 8.1.

Assertions 4 - 6 follow from Assertion 3 and Theorem 8.1. □

Condition of completeness is essential.

Example 9.1. Let B be the semigroup ω with the topology $T(B) = \{\emptyset, B\} \cup \{B \backslash F : F \text{ is a finite subset of } B\}$. Then B is a semi-topological monoid and B is not a topological monoid. Denote now by $W(B)$ the quasi-variety generated by B. Then the elements of $W(B)$ are the submonoids of the monoids of the form B^M. Thus any non-trivial monoid $G \in W(B)$ is not a topological monoid. Therefore the class $W(B)_t$ is trivial.

10 On topological digital spaces

A space X is called an Alexandroff space if it is a T_0-space and the intersection of any family of open sets is open [2].

Alexandroff spaces were first introduced in 1937 by P. S. Alexandroff [2] (see also [1]) under the name discrete spaces, where he provided the characterizations in terms of sets and neighbourhoods.

If (X, T) is an Alexandroff space, then we say that T is a T_0-discrete topology.

We observe the importance of distances with natural values. We affirm that this fact is important from topological point of view as well.

Theorem 10.1. *On a space X there exists a quasimetric with the natural values if and only if X is an Alexandroff space.*

Proof. Let X be an Alexandroff space. For any $x \in X$ denote by M_x the intersection of all open sets which contains x. Then M_x is the minimal open set which contains the point $x \in X$. Observe that if $x, y \in X$, $x \neq y$, and $y \in M_x$, then $M_y \subset M_x$ and $x \notin M_y$. Consider the distance $\rho(x, y)$, where $\rho(x, x) = 0$ for any $x \in X$, $\rho(x, y) = 0$ if $y \in M_x$, and $\rho(x, y) = 1$ if $y \notin M_x$. We affirm that ρ is a quasimetric with natural values. By construction, $\rho(x, y) \in \{0, 1\}$ and ρ has natural values. Let $x, y, z \in X$. If $\rho(x, y) = \rho(y, z) = 0$, then $y \in M_x$ and $z \in M_y \subset M_x$. Hence $\rho(x, z) = 0$. In this case $\rho(x, y) + \rho(y, z) = \rho(x, z)$. If $\rho(x, y) + \rho(y, z) \geq 1$, then $\rho(x, z) \leq 1$ and $\rho(x, y) + \rho(y, z) \geq \rho(x, z)$. Therefore ρ is a quasimetric.

If d is a quasimetric on X with natural values, then $M_x = \{y \in X : d(x, y) < 1\}$ is the minimal open set which contains the point $x \in X$. Therefore $(X, T(d))$ is an Alexandroff space, and this concludes the proof of Theorem 10.1. □

General criteria of quasi-metrizability of spaces were proved in [36].

Let $\preceq$ be a partial ordering on a set X. For any point $x \in X$ we put $M(x, \preceq) = \{y \in X : x \preceq y\}$. Then $\{M(x, \preceq) : x \in X\}$ is a base of the T_0-discrete topology $T(\preceq)$ on X.

Let T be a T_0-topology on a set X. For any points $x, y \in X$ we put $x \preceq_T y$ if and only if $x \in cl_X\{y\}$. Then $\preceq_T$ is a partial ordering on X. By construction, $\preceq = \preceq_{T(\preceq)}$, $T \subset T(\preceq_T)$ and $T = T(\preceq_T)$ if and only if T is T_0-discrete topology (see [2]).

For any T_0-topology T on X we put $aT = T(\preceq_T)$. If $M(x) = \cap\{U \in T : x \in U\}$, then $\{M(x) : x \in X\}$ is the minimal base of the topology aT. We say that aT is the Alexandroff modification of the topology T.

The following assertion is obvious.

Proposition 10.1. *Let T be a T_0-topology on a set X. Then aT is the unique T_0-discrete topology on the space X such that $\preceq_T = \preceq_{aT}$. Moreover, $\preceq_T = \preceq_{T'}$ for any intermediary topology $T \subset T' \subset aT$.*

Theorem 10.2. *Let (G, T) be a topological semigroup. Then (G, aT) is a topological semigroup too.*

Proof. We put $M(x) = \cap\{U \in T : x \in U\}$. Then $\{M(x) : x \in X\}$ is the base of the topology aT and $M(x) \cdot M(y) \subset M(x \cdot y)$. The proof is complete. □

Corollary 10.1. *Let $\mathcal{V}$ be a non-trivial complete non-Burnside quasi-variety of topological monoids. Then for each space X the following assertions are equivalent:*

1. *$F(X, \mathcal{V})$ is an Alexandroff space.*
2. *On a space $F(X, \mathcal{V})$ there exists a quasimetric with the natural values.*
3. *X is an Alexandroff space.*

Proposition 10.2. *Let G be a topological semigroup and X be a connected subspace of G. If X algebraically generates the semigroup G, then G is a connected space.*

Proof. For each $n \in \mathbb{N}$ we put $G_n(X) = \{x_1 \cdot x_2 \cdot ... \cdot x_n : x_1, x_2, ...x_n \in X\}$. By construction, the subspace $G_n(X)$ of G is connected as a continuous image of the connected space X^n and $G_n(X) \subset G_{n+1}(X)$. Hence $G = \cup\{G_n(X) : n \in \mathbb{N}\}$ is a connected space. The proof is complete. □

A digital space is a pair (D, α), where D is a non-empty set and α is a binary, symmetric relation on D such that for any two elements $x, y \in D$ there is a finite sequence $\{x_0, x_1, ..., x_n\}$ of elements in D such that $x = x_0$, $y = x_n$ and $(x_j, x_j + 1) \in \alpha$ for $j \in 0, 1, ..., n-1\}$.

The topological methods may be applied in the study of reflexive or anti-reflexive binary structures. We develop that point of view for reflexive digital structures.

Let ρ be a distance on the non-empty set D. We consider that $(x, y) \in \alpha_\rho$ if and only if $\rho(x, y) \cdot \rho(y, x) = 0$. We say that α_ρ is the binary relation generated by the distance ρ.

A binary relation α on the set D is compatible with the topology T on D if T is a T_0-topology and $(x, y) \in \alpha$ if and only if $x \in cl_{(X,T)}\{y\}$ or $x \in cl_{(X,T)}\{y\}$.

Proposition 10.3. *If a binary relation α on the set D is compatible with the topology T on D, then the binary relation α is compatible by the T_0-discrete topology aT.*

Proof. For any $x \in D$ denote $M_x = \cap\{U \in T : x \in U\}$. Let T_a be the topology on D generated by the open base $\{M_x : x \in D\}$. Then M_x is the minimal open set from T_a which contains the point $x \in X$. It is obvious that $x \in cl_{(X,T)}\{y\}$ if and only if $x \in cl_{(X,aT)}\{y\}$. The proof is complete. □

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Proposition 10.4. *Let a symmetric binary relation α on the non-empty set D is compatible with the T_0-discrete topology T on D. The following assertions are equivalent:*

1. (D, α) is a digital space.

2. (D, T) is a connected space.

3. There exists a discrete quasimetric ρ on D such that $\alpha = \alpha_\rho$ and the space $(D, T(\rho))$ is connected.

Proof. Implication $1 \to 2$ follows from Proposition 10.3. Implication $2 \to 3 \to 2$ follows from Theorem 10.1.

Assume that (D, T) is a connected Alexandroff space.

For any $x \in D$ denote by $M_1(x)$ the intersection of all open sets which contains x. Let $M_{n+1}(x) = \cup\{M_1(y) : M_1(y) \cap M_n(x) \neq \emptyset\}$ and $M_\omega(x) = \cup\{M_n(x) : n \in \mathbb{N}\}$.

By construction, if $y \in M_1(x)$, then $(x, y) \in \alpha$. Hence, if $y \in M_n(x)$, then there is a sequence $\{x_0, x_1, ..., x_n\}$ of elements in D such that $x = x_0$, $y = x_n$ and $(x_j, x_j + 1) \in \alpha$ for $j \in \{0, 1, ..., n-1\}$.

Fix $x \in D$. We affirm that the set $M_\omega(x)$ is closed. If the set $M_\omega(x)$ is not closed, then there exists a point $y \in cl_X M_\omega(x) \setminus M_\omega(x)$. Hence $M_1(y) \cap M_\omega(x) \neq \emptyset$. In this case $M_1(y) \cap M_n(x) \neq \emptyset$ for some $n \in \mathbb{N}$ and $y \in M_{n+1}(x) \neq \emptyset$, a contradiction. Thus the set $M_\omega(x)$ is non-empty and open-and-closed. Since (X, T) is a connected space, we have $M_\omega(x) = X$. Therefore (D, α) is a digital space. Implication $2 \to 1$ is proved. The proof is complete. □

If the digital structure α on a set D is compatible with a T_0-discrete topology T on D, then we say that (D, T) is a topological digital space and put $(D, \alpha) \equiv (D, T)$. Otherwise the digital space (D, α) is not topological. Hence a topological space X is a topological digital space if and only if X is a connected Alexandroff space (see [23, 30, 31]).

From Corollary 10.1 and Propositions 10.2 and 10.4 follows:

Corollary 10.2. *Let $\mathcal{V}$ be a non-trivial complete non-Burnside quasi-variety of topological monoids. Then for each space X the following assertions are equivalent:*

1. $F(X, \mathcal{V})$ is a topological digital space.

2. X is a topological digital space.

There exists a non-topologically digital spaces (D, α) (see [23]). For example, let $D = \{a, b, c, d, e\}$ and $\alpha = \{(a, a), (a, b), (b, a), (b, b), (b, c), (c, b), (c, c), (c, d), (d, c), (d, d), (d, e), (e, d), (e, e), (e, a), (a, e)\}$. Then the digital space (D, α) is not topological.

If D is a non-empty set and $\alpha = D \times D$, then (D, α) is a digital space such that for any linear ordering $\preceq$ on D we have $\alpha = b(\preceq)$ and binary relation α is compatible with the topology $T((\preceq)$. We observe that a topology is compatible with a unique binary structure and a binary structure may be compatible with a set of arbitrary cardinality of topologies.

Now let α be an anti-reflexive digital structure on G. Let ρ be a distance on the non-empty set D. We consider that $(x, y) \in \alpha_\rho$ if and only if $x \neq y$ and

$\rho(x, y) \cdot \rho(y, x) = 0$. We say that α_ρ is the binary relation generated by the distance ρ. A binary anti-reflexive relation α on the set D is compatible with the topology T on D if T is a T_0-topology and $(x, y) \in \alpha$ if and only if $x \neq y$ and $x \in cl_{(X,T)}\{y\}$ or $x \in cl_{(X,T)}\{y\}$. For anti-reflexive digital structures similar assertions hold as in the reflexive case.

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