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Quartic differential systems with an invariant straight line of maximal multiplicity

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Abstract. In this work we show that in the class of quartic differential systems the maximal algebraic multiplicity M_a of an invariant straight line is equal to 10.

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1 Introduction

We consider the real polynomial system of differential equations

$$\frac{dx}{dt} = P(x,y), \qquad \frac{dy}{dt} = Q(x,y), \tag{1}$$

and the vector field $\mathbb{X} = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}$ associated to system (1).

Denote $n = \max\{\deg(P), \deg(Q)\}$. If $n = 4$ then system (1) is called quartic.

A curve $f(x,y) = 0$, $f \in \mathbb{C}[x,y]$ is said to be an *invariant algebraic curve* of (1) if there exists a polynomial $K_f \in \mathbb{C}[x,y]$, $deg(K_f) \leq n-1$ such that the identity $\mathbb{X}(f) \equiv f(x,y)K_f(x,y)$ holds.

Definition 1. An invariant algebraic curve f of degree d for the vector field $\mathbb{X}$ is said to have *algebraic multiplicity* m if m is the greatest positive integer such that the m-th power of f divides $E_d(\mathbb{X})$, where

$$E_d(\mathbb{X}) = det\begin{pmatrix} \upsilon_1 & \upsilon_2 & ... & \upsilon_k \\ \mathbb{X}(\upsilon_1) & \mathbb{X}(\upsilon_2) & ... & \mathbb{X}(\upsilon_k) \\ ... & ... & ... & ... \\ \mathbb{X}^{k-1}(\upsilon_1) & \mathbb{X}^{k-1}(\upsilon_2) & ... & \mathbb{X}^{k-1}(\upsilon_k) \end{pmatrix},$$

and $\upsilon_1, \upsilon_2, ..., \upsilon_k$ is a basis of $\mathbb{C}_d[x,y]$ [3].

If $d = 1$ then $\upsilon_1 = 1$, $\upsilon_2 = x$, $\upsilon_3 = y$ and $E_1(\mathbb{X}) = P \cdot \mathbb{X}(Q) - Q \cdot \mathbb{X}(P)$.

At present, a great number of works are dedicated to the investigation of polynomial differential systems with invariant straight lines.

The problem of the estimation of the number of invariant straight lines which a polynomial differential system may have was considered in [1].

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An interesting relation between the number of invariant straight lines and the possible number of directions for them is established in [2].

In [4] the estimation $3n-2 \leq M_a(n) \leq 3n-1$ of maximal algebraic multiplicity $M_a(n)$ of an invariant straight line is given for the class of two-dimensional polynomial differential systems of degree $n \geq 2$ and it was shown that in the class of cubic differential systems the maximal multiplicity of an affine real straight line (of the line at infinity) is seven.

In this paper we show that in the class of quartic differential systems the maximal algebraic multiplicity M_a of an invariant straight line is equal to 10.

2 Maximal algebraic multiplicity of a real invariant straight line for the quartic differential systems

We consider the quartic differential system

$$\begin{aligned} \dot{x} &= P_0 + P_1(x,y) + P_2(x,y) + P_3(x,y) + P_4(x,y) \equiv P(x,y), \\ \dot{y} &= Q_0 + Q_1(x,y) + Q_2(x,y) + Q_3(x,y) + Q_4(x,y) \equiv Q(x,y), \end{aligned} \tag{2}$$

where P_k and $Q_k,\ k=0,1,2,3,4$, are homogeneous polynomials in x and y of degree k.

Suppose that

$$yP_4(x,y) - xQ_4(x,y) \not\equiv 0, \qquad \gcd(P,Q)=1, \tag{3}$$

i.e. at infinity the system (2) has at most five distinct singular points and the right-hand sides of (2) do not have the common divisors of degree greater than 0.

Let the system (2) have a real invariant straight line l. By an affine transformation we can make l to be described by the equation $x=0$. Then, the system (2) looks as:

$$\begin{aligned} \dot{x} &= x(a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2 + a_7x^3 + \\ &\quad + a_8x^2y + a_9xy^2 + a_{10}y^3), \\ \dot{y} &= b_0 + b_1x + b_2y + b_3x^2 + b_4xy + b_5y^2 + b_6x^3 + b_7x^2y + \\ &+ b_8xy^2 + b_9y^3 + b_{10}x^4 + b_{11}x^3y + b_{12}x^2y^2 + b_{13}xy^3 + b_{14}y^4. \end{aligned} \tag{4}$$

For (4) the determinant $E_1(\mathbb{X})$ is a polynomial in x and y of degree 11. We write it in the form:

$$\begin{aligned} E_1(\mathbb{X}) &= x(A_1(y) + A_2(y)x + A_3(y)x^2 + A_4(y)x^3 + A_5(y)x^4 + \\ &+ A_6(y)x^5 + A_7(y)x^6 + A_8(y)x^7 + A_9(y)x^8 + A_{10}(y)x^9 + A_{11}(y)x^{10}). \end{aligned} \tag{5}$$

The algebraic multiplicity $m_a(l)$ of the invariant straight line $x=0$ of the system (4) is greater than or equal to two if the identity $A_1(y) \equiv 0$ holds. Here we have $A_1(y) = -A_{11}(y) \cdot A_{12}(y)$, where

$A_{11}(y) = b_0 + b_2y + b_5y^2 + b_9y^3 + b_{14}y^4$,

$A_{12}(y) = a_1^2+a_3b_0-a_1b_2+2a_1a_3y+2a_6b_0y-2a_1b_5y+a_3^2y^2+2a_1a_6y^2+3a_{10}b_0y^2+a_6b_2y^2-a_3b_5y^2-3a_1b_9y^2+2a_1a_{10}y^3+2a_3a_6y^3-4a_1b_{14}y^3+2a_{10}b_2y^3-2a_3b_9y^3+2a_{10}a_3y^4+a_6^2y^4-3a_3b_{14}y^4+a_{10}b_5y^4-a_6b_9y^4+2a_{10}a_6y^5-2a_6b_{14}y^5+a_{10}^2y^6-a_{10}b_{14}y^6)$.

From conditions (3) the polynomial $A_{11}(y)$ is not identically equal to zero, therefore it is necessary that $A_{12}(y)$ be identically zero. The identity $A_{12}(y) \equiv 0$ holds if one of the following five sets of conditions is satisfied:

$$a_1 = a_3 = a_6 = a_{10} = 0; \tag{6}$$

$$a_3 = a_6 = a_{10} = b_2 - a_1 = b_5 = b_9 = b_{14} = 0,\ a_1 \neq 0; \tag{7}$$

$$a_6 = a_{10} = 0,\ b_0 = (-a_1^2 + a_1b_2)/a_3,\ b_5 - a_3 = b_9 = b_{14} = 0; \tag{8}$$

$$\begin{aligned}&a_{10} = 0,\ b_0 = a_1(b_5 - a_3)/a_6,\ b_2 = (-a_3^2 + a_1a_6 + a_3b_5)/a_6,\\ &b_9 - a_6 = b_{14} = 0;\end{aligned} \tag{9}$$

$$\begin{aligned}&b_0 = a_1(b_9 - a_6)/a_{10},\ b_2 = (a_1a_{10} - a_3a_6 + a_3b_9)/a_{10},\\ &b_5 = (a_{10}a_3 - a_6^2 + a_6b_9)/a_{10},\ b_{14} = a_{10}.\end{aligned} \tag{10}$$

In this way we have proved the following lemma.

Lemma 1. *For quartic differential system* $\{(4),\ (3)\}$ *the algebraic multiplicity of the invariant straight line* $x = 0$ *is greater than or equal to two if and only if at least one of the following five sets of conditions:* (6), (7), (8), (9), (10) *is satisfied.*

The algebraic multiplicity of the invariant straight line $x = 0$ is greater than or equal to three if the identity $A_2(y) \equiv 0$ holds. Putting each of the conditions (6)–(10) in the polynomial $A_2(y)$ we have respectively:

$$\begin{aligned}&A_2(y) = -A_{11}(y)\cdot(a_5b_0 - a_2b_2 + 2(a_9b_0 - a_2b_5)y + (a_9b_2 - a_5b_5-\\ &-3a_2b_9)y^2 - (4a_2b_{14} + 2a_5b_9)y^3 - (3a_5b_{14} + a_9b_9)y^4 - 2a_9b_{14}y^5);\end{aligned} \tag{11}$$

$$\begin{aligned}&A_2(y) = -2a_1a_2b_0 - a_5b_0^2 + a_1^2b_1 + a_1b_0b_4 - (2a_1^2a_2 + 4a_1a_5b_0+\\ &+2a_9b_0^2 - 2a_1^2b_4y - 2a_1b_0b_8)y - (3a_1^2a_5 + 6a_1a_9b_0 - 3a_1b_0b_{13}-\\ &-3a_1^2b_8)y^2 - (4a_1^2a_9 - 4a_1^2b_{13})y^3;\end{aligned} \tag{12}$$

$$\begin{aligned}&A_2(y) = -(a_1 + a_3y)(-3a_1^2a_2a_3 + a_1^3a_5 - 2a_1a_3^2b_1 + 4a_1a_2a_3b_2-\\ &-2a_1^2a_5b_2 + a_3^2b_1b_2 - a_2a_3b_2^2 + a_1a_5b_2^2 + a_1^2a_3b_4 - a_1a_3b_2b_4 + 2a_1\cdot\\ &(a_2a_3^2 - 2a_1a_3a_5 + a_1^2a_9 + 2a_3a_5b_2 - 2a_1a_9b_2 + a_9b_2^2 + a_1a_3b_8-\\ &-a_3^2b_4 - a_3b_2b_8)y + a_3(a_2a_3^2 + a_1a_3a_5 - 5a_1^2a_9 + 3a_1^2b_{13} + 2a_3a_5b_2+\\ &+4a_1a_9b_2 - 3a_1b_{13}b_2 + a_9b_2^2 - a_3^2b_4 - 2a_1a_3b_8 - a_3b_2b_8)y^2 + 2a_3^2\cdot\\ &(a_3a_5 - a_1b_{13} + 2a_9b_2 - b_{13}b_2 - a_3b_8)y^3 + 3a_3^3(a_9 - b_{13})y^4)/a_3^2\end{aligned} \tag{13}$$

$$\begin{aligned}A_2(y) = &-(a_1 + a_3y + a_6y^2)(-a_2a_3^3 + a_1a_3^2a_5 - 2a_1a_2a_3a_6 - a_3^2a_6b_1 - \\ &-a_1a_6^2b_1 + a_1a_3a_6b_4 + 2a_2a_3^2b_5 - 2a_1a_3a_5b_5 + 2a_1a_2a_6b_5 + a_3a_6b_1b_5 - \\ &-a_1a_6b_4b_5 - a_2a_3b_5^2 + a_1a_5b_5^2 - 2(a_2a_3^2a_6 + 2a_1a_3a_5a_6 - a_1a_2a_6^2 - \\ &-a_1a_3^2a_9 + a_3a_6^2b_1 + a_1a_6^2b_4 - 2a_2a_3a_6b_5 - 2a_1a_5a_6b_5 + 2a_1a_3a_9b_5 - \\ &-a_6^2b_1b_5 + a_2a_6b_5^2 - a_1a_9b_5^2 - a_1a_3a_6b_8 + a_1a_6b_5b_8)y + (3a_2a_3a_6^2 - \\ &-3a_3^2a_5a_6 + 3a_1a_5a_6^2 + a_3^3a_9 - 6a_1a_3a_6a_9 + a_6^3b_1 + 3a_1a_3a_6b_{13} - \\ &-2a_3a_6^2b_4 + 4a_3a_5a_6b_5 - 2a_2a_6^2b_5 - 2a_3^2a_9b_5 + 6a_1a_6a_9b_5 - a_5a_6b_5^2 - \\ &-3a_1a_6b_{13}b_5 + a_6^2b_4b_5 + a_3a_9b_5^2 + a_3^2a_6b_8 - 3a_1a_6^2b_8 - a_3a_6b_5b_8)y^2 - \\ &-2a_6(-a_3a_5a_6 + 2a_3^2a_9 - 2a_1a_6a_9 - a_3^2b_{13} + 2a_1a_6b_{13} - 2a_3a_9b_5 + \\ &+a_3b_{13}b_5 + a_3a_6b_8)y^3 + a_6^2(a_5a_6 + a_3a_9 - 2a_3b_{13} + 2a_9b_5 - b_{13}b_5 - \\ &-a_6b_8)y^4 + 2a_6^3(a_9 - b_{13})y^5)/a_6^2;\end{aligned} \tag{14}$$

$$\begin{aligned}A_2(y) = &-(a_1 + a_3y + a_6y^2 + a_{10}y^3)(-2a_1a_{10}a_2a_6 - a_2a_3a_6^2 + a_1a_5a_6^2 \\ &-a_1a_{10}^2b_1 - a_{10}a_3a_6b_1 + a_1a_{10}a_6b_4 + 2a_1a_{10}a_2b_9 + 2a_2a_3a_6b_9 - 2a_1a_5\cdot \\ &a_6b_9 + a_{10}a_3b_1b_9 - a_1a_{10}b_4b_9 - a_2a_3b_9^2 + a_1a_5b_9^2 + 2(a_1a_{10}^2a_2 - 2a_1a_{10}\cdot \\ &a_5a_6 - a_2a_6^3 + a_1a_6^2a_9 - a_{10}a_6^2b_1 - a_1a_{10}^2b_4 + a_1a_{10}a_6b_8 + 2a_1a_{10}a_5b_9 + \\ &+2a_2a_6^2b_9 - 2a_1a_6a_9b_9 + a_{10}a_6b_1b_9 - a_1a_{10}b_8b_9 - a_2a_6b_9^2 + a_1a_9b_9^2)y + \\ &+(a_{10}^2a_2a_3 + 3a_1a_{10}^2a_5 - 2a_{10}a_3a_5a_6 - a_{10}a_2a_6^2 - a_5a_6^3 - 6a_1a_{10}a_6a_9 + \\ &+a_3a_6^2a_9 - 2a_{10}^2a_6b_1 + 3a_1a_{10}a_6b_{13} - a_{10}^2a_3b_4 - a_{10}a_6^2b_4 - 3a_1a_{10}^2b_8 + \\ &+a_{10}a_3a_6b_8 + 2a_{10}a_3a_5b_9 + 4a_{10}a_2a_6b_9 + 2a_5a_6^2b_9 + 6a_1a_{10}a_9b_9 - 2a_3\cdot \\ &a_6a_9b_9 + 3a_{10}^2b_1b_9 - 3a_1a_{10}b_{13}b_9 + a_{10}a_6b_4b_9 - a_{10}a_3b_8b_9 - 3a_{10}a_2b_9^2 - \\ &-a_5a_6b_9^2 + a_3a_9b_9^2)y^2 + 2a_{10}(a_{10}a_3a_5 + 2a_{10}a_2a_6 - a_5a_6^2 + 2a_1a_{10}a_9 - \\ &-2a_3a_6a_9 + a_{10}^2b_1 - 2a_1a_{10}b_{13} + a_3a_6b_{13} - a_{10}a_6b_4 - a_{10}a_3b_8 - 2a_{10}\cdot \\ &a_2b_9 + 2a_5a_6b_9 + 2a_3a_9b_9 - a_3b_{13}b_9 + a_{10}b_4b_9 - a_5b_9^2)y^3 - a_{10}(a_{10}^2a_2 - \\ &-3a_{10}a_5a_6 - 3a_{10}a_3a_9 + 3a_6^2a_9 + 3a_{10}a_3b_{13} - a_6^2b_{13} - a_{10}^2b_4 + 2a_{10}\cdot \\ &a_6b_8 + 2a_{10}a_5b_9 - 4a_6a_9b_9 + a_6b_{13}b_9 - a_{10}b_8b_9 + a_9b_9^2)y^4 + 2a_{10}^2a_6\cdot \\ &(a_9 - b_{13})y^5 + a_{10}^3(a_9 - b_{13})y^6)/a_{10}^2\end{aligned} \tag{15}$$

Taking into account (3) the identity $A_2(y) \equiv 0$ gives, in each of the cases (11)–(15), the following series of conditions:

(11) $\Rightarrow$

$$a_2 = a_5 = a_9 = 0; \tag{16}$$

$$a_5 = a_9 = 0,\ a_2 \neq 0,\ b_2 = b_5 = b_9 = b_{14} = 0; \tag{17}$$

$$a_9 = 0,\ b_0 = a_2b_2/a_5,\ b_5 = b_9 = b_{14} = 0; \tag{18}$$

$$b_0 = a_2b_5/a_9,\ b_2 = a_5b_5/a_9,\ b_9 = b_{14} = 0; \tag{19}$$

(12) $\Rightarrow$

$$\begin{aligned}&a_2 = (-a_5b_0 + a_1b_4)/a_1,\ b_1 = b_0(-a_5b_0 + a_1b_4)/a_1^2,\\ &b_8 = (a_1a_5 + a_9b_0)/a_1,\ b_{13} = a_9;\end{aligned} \tag{20}$$

(13) $\Rightarrow$

$$b_2 = 2a_1,\ b_4 = (a_2a_3 + a_1a_5)/a_3,\ b_8 = (a_3a_5 + a_1a_9)/a_3,\ b_{13} = a_9; \tag{21}$$

$$\begin{aligned}&b_1 = a_2(b_2 - a_1)/a_3,\ b_4 = (a_2a_3 - a_1a_5 + a_5b_2)/a_3,\\ &b_8 = (a_3a_5 - a_1a_9 + a_9b_2)/a_3,\ b_{13} = a_9;\end{aligned} \tag{22}$$

(14) $\Rightarrow$

$$b_1 = a_2(b_5 - a_3)/a_6,\ b_4 = (-a_3a_5 + a_2a_6 + a_5b_5)/a_6,\\ b_8 = (a_5a_6 - a_3a_9 + a_9b_5)/a_6, b_{13} - a_9 = 0; \tag{23}$$

$$a_1 = ((b_5 - a_3)(2a_3 - b_5))/a_6,\ b_1 = (2a_3^2a_5 - 3a_2a_3a_6 + 2a_3a_6b_4 - \\ -3a_3a_5b_5 + 2a_2a_6b_5 - a_6b_4b_5 + a_5b_5^2)/a_6^2, \quad b_8 = (a_5a_6 - a_3a_9 + \\ +a_9b_5)/a_6,\ b_{13} = a_9; \tag{24}$$

(15) $\Rightarrow$

$$a_1 = (b_9 - a_6)(a_{10}a_3 + 2a_6^2 - 3a_6b_9 + b_9^2)/a_{10}^2,\\ b_1 = -(a_{10}^2a_3a_5 + a_{10}^2a_2a_6 + 2a_{10}a_5a_6^2 - a_{10}a_3a_6a_9 - 2a_6^3a_9 - \\ -a_{10}^2a_3b_8 - 2a_{10}a_6^2b_8 - a_{10}^2a_2b_9 - 3a_{10}a_5a_6b_9 + a_{10}a_3a_9b_9 + \\ +5a_6^2a_9b_9 + 3a_{10}a_6b_8b_9 + a_{10}a_5b_9^2 - 4a_6a_9b_9^2 - a_{10}b_8b_9^2 + a_9b_9^3)/a_{10}^3,\\ b_4 = (a_{10}^2a_2 - 3a_{10}a_5a_6 + 2a_6^2a_9 + 2a_{10}a_6b_8 + 2a_{10}a_5b_9 - 3a_6a_9b_9 - \\ -a_{10}b_8b_9 + a_9b_9^2)/a_{10}^2, \quad b_{13} = a_9; \tag{25}$$

$$b_1 = a_2(b_9 - a_6)/a_{10},\ b_4 = (a_{10}a_2 - a_5a_6 + a_5b_9)/a_{10},\\ b_8 = (a_{10}a_5 - a_6a_9 + a_9b_9)/a_{10},\ b_{13} = a_9; \tag{26}$$

Lemma 2. *For quartic differential system* $\{(4), (3)\}$ *the algebraic multiplicity of the invariant straight line* $x = 0$ *is greater than or equal to three if and only if at least one of the following eleven sets of conditions holds:*
1) (6), (16), *2)* (6), (17), *3)* (6), (18), *4)* (6), (19),
5) (7), (20), *6)* (8), (21), *7)* (8), (22), *8)* (9), (23),
9) (9), (24), *10)* (10), (25), *11)* (10), (26).

The invariant straight line $x = 0$ has algebraic multiplicity $m_a \geq 4$ if in each of the cases 1)–11) of Lemma 2 the identity $A_3(y) \equiv 0$ holds. Taking into account (3) we have:

1) (6), (16) $\Rightarrow A_3(y) = -A_{11} \cdot (a_8b_0 - a_4b_2 - 2a_4b_5y - (a_8b_5 + 3a_4b_9)y^2 - 2(2a_4b_{14} + a_8b_9)y^3 - 3a_8b_{14}y^4) \equiv 0 \Rightarrow$

$$a_4 = a_8 = 0; \tag{27}$$

$$a_8 = 0,\ a_4 \neq 0,\ b_2 = b_5 = b_9 = b_{14} = 0; \tag{28}$$

$$b_0 = a_4b_2/a_8,\ b_5 = b_9 = b_{14} = 0; \tag{29}$$

2) (6), (17) $\Rightarrow A_3(y) = -b_0(2a_2^2 + a_8b_0 - a_2b_4 - 2a_2b_8y - 3a_2b_{13}y^2) \equiv 0 \Rightarrow$

$$a_8 = a_2(b_4 - 2a_2)/b_0,\ b_8 = b_{13} = 0; \tag{30}$$

3) (6), (18) $\Rightarrow A_3(y) = -b_2(a_2 + a_5y)(2a_2^2a_5 + a_5^2b_1 - a_4a_5b_2 + a_2a_8b_2 - a_2a_5b_4 + 2a_2a_5(2a_5 - b_8)y + a_5(2a_5^2 - 3a_2b_{13} - a_5b_8)y^2 - 2a_5^2b_{13}y^3)/a_5^2 \equiv 0 \Rightarrow$

$$b_1 = (-2a_2^2a_5 + a_4a_5b_2 - a_2a_8b_2 + a_2a_5b_4)/a_5^2,\ b_8 = 2a_5,\ b_{13} = 0; \tag{31}$$

4) (6), (19) $\Rightarrow A_3(y) = -b_5(a_2+a_5y+a_9y^2)(2a_2^2a_9+a_5a_9b_1-a_2a_9b_4-a_4a_5b_5+a_2a_8b_5+2a_9(2a_2a_5+a_9b_1-a_4b_5-a_2b_8)y+a_9(2a_5^2+4a_2a_9-3a_2b_{13}+a_9b_4-a_8b_5-a_5b_8)y^2+2a_5a_9(2a_9-b_{13})y^3+a_9^2(2a_9-b_{13})y^4)/a_9^2 \equiv 0 \Rightarrow$

$$b_1 = (-2a_2a_5+a_4b_5+a_2b_8)/a_9,\ b_4 = (-2a_5^2+2a_2a_9+a_8b_5+ +a_5b_8)/a_9,\ \ b_{13} = 2a_9; \tag{32}$$

5) (7), (20) $\Rightarrow A_3(y) = -3a_1a_4b_0-a_8b_0^2+2a_1^2b_3+a_1b_0b_7-a_1(3a_1a_4+5a_8b_0-2b_0b_{12}-3a_1b_7)y-4a_1^2(a_8-b_{12})y^2 \equiv 0 \Rightarrow$

$$b_3 = a_4b_0/a_1,\ b_7 = (a_1a_4+a_8b_0)/a_1,\ b_{12} = a_8; \tag{33}$$

6) (8), (21) $\Rightarrow A_3(y) = -(a_1^2a_2^2a_3+2a_1^3a_3a_4-a_1^3a_2a_5+a_1^4a_8-2a_1a_2a_3^2b_1+a_1^2a_3a_5b_1+a_3^3b_1^2-a_1^2a_3^2b_3-a_1^3a_3b_7+2a_1(3a_1a_3^2a_4-a_1a_2a_3a_5+3a_1^2a_3a_8-a_1^2a_2a_9+a_3^2a_5b_1+a_1a_3a_9b_1-a_1^2a_3b_{12}-a_3^3b_3-2a_1a_3^2b_7)y+a_3(6a_1a_3^2a_4-a_1a_2a_3a_5+12a_1^2a_3a_8-4a_1^2a_2a_9+a_3^2a_5b_1+4a_1a_3a_9b_1-7a_1^2a_3b_{12}-a_3^3b_3-5a_1a_3^2b_7)y^2+2a_3^2(a_3^2a_4+5a_1a_3a_8-a_1a_2a_9+a_3a_9b_1-4a_1a_3b_{12}-a_3^2b_7)y^3+3a_3^4(a_8-b_{12})y^4)/a_3^2 \equiv 0 \Rightarrow$

$$b_1 = a_1a_2/a_3,\ b_3 = (a_1a_3^2a_4-a_1a_2a_3a_5+a_1^2a_2a_9+a_3^2a_5b_1-a_1a_3\cdot a_9b_1)/a_3^3,\ b_7 = (a_3^2a_4+a_1a_3a_8-a_1a_2a_9+a_3a_9b_1)/a_3^2,\ b_{12} = a_8; \tag{34}$$

7) (8), (22) $\Rightarrow A_3(y) = -(a_1+a_3y)(-4a_1^2a_3a_4+a_1^3a_8+5a_1a_3a_4b_2-2a_1^2a_8b_2-a_3a_4b_2^2+a_1a_8b_2^2-3a_1a_3^2b_3+a_3^2b_2b_3+a_1^2a_3b_7-a_1a_3b_2b_7-a_3(-2a_1a_3a_4+5a_1^2a_8-2a_1^2b_{12}-a_3a_4b_2-5a_1a_8b_2+2a_1b_{12}b_2+a_3^2b_3+3a_1a_3b_7)y+a_3^2(2a_3a_4+a_1a_8-3a_1b_{12}+3a_8b_2-b_{12}b_2-2a_3b_7)y^2+3a_3^3(a_8-b_{12})y^3)/a_3^2 \equiv 0 \Rightarrow$

$$b_3 = a_4(b_2-a_1)/a_3,\ b_7 = (a_3a_4-a_1a_8+a_8b_2)/a_3,\ b_{12} = a_8; \tag{35}$$

8) (9), (23) $\Rightarrow A_3(y) = (a_1+a_3y+a_6y^2)(a_3^3a_4+3a_1a_3a_4a_6-a_1a_3^2a_8+a_3^2a_6b_3+2a_1a_6^2b_3-2a_3^2a_4b_5-3a_1a_4a_6b_5+2a_1a_3a_8b_5-a_3a_6b_3b_5+a_3a_4b_5^2-a_1a_8b_5^2-a_1a_3a_6b_7+a_1a_6b_5b_7+a_6(3a_3^2a_4-3a_1a_4a_6+5a_1a_3a_8-2a_1a_3b_{12}+3a_3a_6b_3-5a_3a_4b_5-5a_1a_8b_5+2a_1b_{12}b_5-2a_6b_3b_5+2a_4b_5^2+3a_1a_6b_7)y+a_6(-3a_3a_4a_6+4a_3^2a_8-4a_1a_6a_8-a_3^2b_{12}+4a_1a_6b_{12}+a_4a_6b_5-5a_3a_8b_5+a_3b_{12}b_5+a_8b_5^2+3a_3a_6b_7-a_6b_5b_7)y^2-a_6^2(a_4a_6+2a_3a_8-3a_3b_{12}+a_8b_5-a_6b_7)y^3-2a_6^3(a_8-b_{12})y^4)/a_6^2 \equiv 0 \Rightarrow$

$$b_3 = a_4(b_5-a_3)/a_6,\ b_7 = (a_4a_6-a_3a_8+a_8b_5)/a_6,\ b_{12} = a_8; \tag{36}$$

$$a_1 = a_3^2/(4a_6),\ b_5 = 3a_3/2,\ b_7 = (a_4a_6-a_3a_8+a_8b_5)/a_6,\ b_{12} = a_8; \tag{37}$$

9) (9), (24) $\Rightarrow A_3(y) = -(2a_3-b_5+a_6y)(6a_3^4a_5^2-5a_3^4a_4a_6-11a_2a_3^3a_5a_6+5a_2^2a_3^2a_6^2+2a_3^5a_8-3a_3^3a_6^2b_3+10a_3^3a_5a_6b_4-9a_2a_3^2a_6^2b_4+4a_3^2a_6^2b_4^2-19a_3^3a_5^2b_5+18a_3^3a_4a_6b_5+24a_2a_3^2a_5a_6b_5-6a_2^2a_3a_6^2b_5-9a_3^4a_8b_5+8a_3^2a_6^2b_3b_5-21a_3^2a_5a_6b_4b_5+10a_2a_3a_6^2b_4b_5-4a_3a_6^2b_4^2b_5+22a_3^2a_5^2b_5^2-24a_3^2a_4a_6b_5^2-17a_2a_3a_5a_6b_5^2+2a_2^2a_6^2b_5^2+16a_3^3a_8b_5^2-7a_3a_6^2b_3b_5^2+14a_3a_5a_6b_4b_5^2-3a_2a_6^2b_4b_5^2+a_6^2b_4^2b_5^2-11a_3a_5^2b_5^3+14a_3a_4a_6b_5^3+4a_2a_5a_6b_5^3-14a_3^2a_8b_5^3+2a_6^2b_3b_5^3-3a_5a_6b_4b_5^3+2a_5^2b_5^4-3a_4a_6b_5^4+6a_3a_8b_5^4-a_8b_5^5+2a_3^4a_6b_7-7a_3^3a_6b_5b_7+9a_3^2a_6b_5^2b_7-5a_3a_6b_5^3b_7+a_6b_5^4b_7-2(-7a_3^3a_4a_6^2+a_2a_3^2a_5a_6^2-a_2^2a_3a_6^3+6a_3^4a_6a_8-2a_3^4a_5a_9+$

$2a_2a_3^3a_6a_9 - 2a_3^4a_6b_{12} - 3a_3^2a_6^3b_3 - 2a_3^2a_5a_6^2b_4 + 3a_2a_3a_6^3b_4 - 2a_3^3a_6a_9b_4 - 2a_3a_6^3b_4^2 + 18a_3^2a_4a_6^2b_5 - a_2a_3a_5a_6^2b_5 - 21a_3^3a_6a_8b_5 + 7a_3^3a_5a_9b_5 - 5a_2a_3^2a_6a_9b_5 + 7a_3^3a_6b_{12}b_5 + 5a_3a_6^3b_3b_5 + 3a_3a_5a_6^2b_4b_5 - a_2a_6^3b_4b_5 + 5a_3^2a_6a_9b_4b_5 + a_6^3b_4^2b_5 - 15a_3a_4a_6^2b_5^2 + 27a_3^2a_6a_8b_5^2 - 9a_3^2a_5a_9b_5^2 + 4a_2a_3a_6a_9b_5^2 - 9a_3^2a_6b_{12}b_5^2 - 2a_6^3b_3b_5^2 - a_5a_6^2b_4b_5^2 - 4a_3a_6a_9b_4b_5^2 + 4a_4a_6^2b_5^3 - 15a_3a_6a_8b_5^3 + 5a_3a_5a_9b_5^3 - a_2a_6a_9b_5^3 + 5a_3a_6b_{12}b_5^3 + a_6a_9b_4b_5^3 + 3a_6a_8b_5^4 - a_5a_9b_5^4 - a_6b_{12}b_5^4 + 4a_3^3a_6^2b_7 - 10a_3^2a_6^2b_5b_7 + 8a_3a_6^2b_5^2b_7 - 2a_6^2b_5^3b_7)y - a_6(-3a_3^2a_5^2a_6 + 12a_3^2a_4a_6^2 + 5a_2a_3a_5a_6^2 - 2a_2^2a_6^3 - 22a_3^3a_6a_8 + 7a_3^3a_5a_9 - 7a_2a_3^2a_6a_9 + 13a_3^3a_6b_{12} + 3a_3a_6^3b_3 - 4a_3a_5a_6^2b_4 + 3a_2a_6^3b_4 + 7a_3^2a_6a_9b_4 - a_6^3b_4^2 + 5a_3a_5^2a_6b_5 - 18a_3a_4a_6^2b_5 - 4a_2a_5a_6^2b_5 + 54a_3^2a_6a_8b_5 - 17a_3^2a_5a_9b_5 + 10a_2a_3a_6a_9b_5 - 32a_3^2a_6b_{12}b_5 - 2a_6^3b_3b_5 + 3a_5a_6^2b_4b_5 - 10a_3a_6a_9b_4b_5 - 2a_5^2a_6b_5^2 + 6a_4a_6^2b_5^2 - 42a_3a_6a_8b_5^2 + 13a_3a_5a_9b_5^2 - 3a_2a_6a_9b_5^2 + 25a_3a_6b_{12}b_5^2 + 3a_6a_9b_4b_5^2 + 10a_6a_8b_5^3 - 3a_5a_9b_5^3 - 6a_6b_{12}b_5^3 - 9a_3^2a_6^2b_7 + 13a_3a_6^2b_5b_7 - 4a_6^2b_5^2b_7)y^2 - 2a_6^2(-a_3a_4a_6^2 + 7a_3^2a_6a_8 - a_3^2a_5a_9 + a_2a_3a_6a_9 - 6a_3^2a_6b_{12} - a_3a_6a_9b_4 - 9a_3a_6a_8b_5 + a_3a_5a_9b_5 + 8a_3a_6b_{12}b_5 + 2a_6a_8b_5^2 - 2a_6b_{12}b_5^2 + a_3a_6^2b7)y^3 + a_6^3(a_4a_6^2 + a_3a_5a_9 - a_2a_6a_9 - a_3a_6b_{12} + a_6a_9b_4 + 3a_6a_8b_5 - a_5a_9b_5 - 2a_6b_{12}b_5 - a_6^2b_7)y^4 + 2a_6^5(a_8 - b_{12})y^5)/a_6^4 \equiv 0 \Rightarrow$

$$b_4 = (a_3a_5 + 2a_2a_6)/(2a_6),\ b_5 = 3a_3/2,\ b_7 = (a_4a_6^2 - a_3a_6a_8 + a_3a_5a_9 - a_2a_6a_9 + a_6a_9b_4 + a_6a_8b_5 - a_5a_9b_5)/a_6^2,\ b_{12} = a_8; \tag{38}$$

$$b_4 = (8a_2a_6^2 + a_3^2a_9)/(4a_6^2),\ b_5 = 3a_3/2,\ b_7 = (a_4a_6^2 - a_3a_6a_8 + a_3a_5a_9 - a_2a_6a_9 + a_6a_9b_4 + a_6a_8b_5 - a_5a_9b_5)/a_6^2,\ b_{12} = a_8; \tag{39}$$

$$b_3 = -a_4(a_3 - b_5)/a_6,\ b_4 = (-a_3a_5 + a_2a_6 + a_5b_5)/a_6,\ b_7 = (a_4a_6^2 - a_3a_6a_8 + a_3a_5a_9 - a_2a_6a_9 + a_6a_9b_4 + a_6a_8b_5 - a_5a_9b_5)/a_6^2,\ b_{12} = a_8; \tag{40}$$

10) (10), (25) $\Rightarrow A_3(y) = -(a_{10}a_3 + 2a_6^2 - 3a_6b_9 + b_9^2 + 2a_{10}a_6y - a_{10}b_9y + a_{10}^2y^2)(B_0 + B_1y + B_2y^2 + B_3y^3 + B_4y^4 + B_5y^5 + B_6y^6)/a_{10}^7$, where

$B_0 = a_{10}^4a_3^2a_5^2 + 3a_{10}^3a_3a_5^2a_6^2 - 2a_{10}^3a_3a_4a_6^3 + 2a_{10}^3a_2a_5a_6^3 + 2a_{10}^2a_5^2a_6^4 - 6a_{10}^2a_4a_6^5 + a_{10}^2a_3a_6^4a_8 + 2a_{10}a_6^6a_8 - 2a_{10}^3a_3^2a_5a_6a_9 - 7a_{10}^2a_3a_5a_6^3a_9 - 2a_{10}^2a_2a_6^4a_9 - 6a_{10}a_5a_6^5a_9 + a_{10}^2a_3^2a_6^2a_9^2 + 4a_{10}a_3a_6^4a_9^2 + 4a_6^6a_9^2 - a_{10}^4a_3a_6^2b_3 - 4a_{10}^3a_6^4b_3 + a_{10}^3a_3a_6^3b_7 + 2a_{10}^2a_6^5b_7 - 2a_{10}^4a_3^2a_5b_8 - 7a_{10}^3a_3a_5a_6^2b_8 - 2a_{10}^3a_2a_6^3b_8 - 6a_{10}^2a_5a_6^4b_8 + 2a_{10}^3a_3^2a_6a_9b_8 + 8a_{10}^2a_3a_6^3a_9b_8 + 8a_{10}a_6^5a_9b_8 + a_{10}^4a_3^2b_8^2 + 4a_{10}^3a_3a_6^2b_8^2 + 4a_{10}^2a_6^4b_8^2 - 4a_{10}^3a_3a_5^2a_6b_9 + 6a_{10}^3a_3a_4a_6^2b_9 - 5a_{10}^3a_2a_5a_6^2b_9 - 5a_{10}^2a_5^2a_6^3b_9 + 27a_{10}^2a_4a_6^4b_9 - 4a_{10}^2a_3a_6^3a_8b_9 - 11a_{10}a_6^5a_8b_9 + 2a_{10}^3a_3^2a_5a_9b_9 + 17a_{10}^2a_3a_5a_6^2a_9b_9 + 7a_{10}^2a_2a_6^3a_9b_9 + 23a_{10}a_5a_6^4a_9b_9 - 2a_{10}^2a_3^2a_6a_9^2b_9 - 14a_{10}a_3a_6^3a_9^2b_9 - 20a_6^5a_9^2b_9 + 2a_{10}^4a_3a_6b_3b_9 + 14a_{10}^3a_6^3b_3b_9 - 3a_{10}^3a_3a_6^2b_7b_9 - 9a_{10}^2a_6^4b_7b_9 + 10a_{10}^3a_3a_5a_6b_8b_9 + 5a_{10}^3a_2a_6^2b_8b_9 + 17a_{10}^2a_5a_6^3b_8b_9 - 2a_{10}^3a_3^2a_9b_8b_9 - 20a_{10}^2a_3a_6^2a_9b_8b_9 - 32a_{10}a_6^4a_9b_8b_9 - 6a_{10}^3a_3a_6b_8^2b_9 - 12a_{10}^2a_6^3b_8^2b_9 + a_{10}^3a_3a_5^2b_9^2 - 6a_{10}^3a_3a_4a_6b_9^2 + 4a_{10}^3a_2a_5a_6b_9^2 + 4a_{10}^2a_5^2a_6^2b_9^2 - 48a_{10}^2a_4a_6^3b_9^2 + 6a_{10}^2a_3a_6^2a_8b_9^2 + 25a_{10}a_6^4a_8b_9^2 - 13a_{10}^2a_3a_5a_6a_9b_9^2 - 9a_{10}^2a_2a_6^2a_9b_9^2 - 34a_{10}a_5a_6^3a_9b_9^2 + a_{10}^2a_3^2a_9^2b_9^2 + 18a_{10}a_3a_6^2a_9^2b_9^2 + 41a_6^4a_9^2b_9^2 - a_{10}^4a_3b_3b_9^2 - 18a_{10}^3a_6^2b_3b_9^2 + 3a_{10}^3a_3a_6b_7b_9^2 + 16a_{10}^2a_6^3b_7b_9^2 - 3a_{10}^3a_3a_5b_8b_9^2 - 4a_{10}^3a_2a_6b_8b_9^2 - 17a_{10}^2a_5a_6^2b_8b_9^2 + 16a_{10}^2a_3a_6a_9b_8b_9^2 + 50a_{10}a_6^3a_9b_8b_9^2 + 2a_{10}^3a_3b_8^2b_9^2 + 13a_{10}^2a_6^2b_8^2b_9^2 + 2a_{10}^3a_3a_4b_9^3 - a_{10}^3a_2a_5b_9^3 - a_{10}^2a_5^2a_6b_9^3 + 42a_{10}^2a_4a_6^2b_9^3 - 4a_{10}^2a_3a_6a_8b_9^3 - 30a_{10}a_6^3a_8b_9^3 + 3a_{10}^2a_3a_5a_9b_9^3 + 5a_{10}^2a_2a_6a_9b_9^3 + 24a_{10}a_5a_6^2a_9b_9^3 - 10a_{10}a_3a_6a_9^2b_9^3 - 44a_6^3a_9^2b_9^3 + 10a_{10}^3a_6b_3b_9^3 - a_{10}^3a_3b_7b_9^3 - 14a_{10}^2a_6^2b_7b_9^3 + a_{10}^3a_2b_8b_9^3 + 7a_{10}^2a_5a_6b_8b_9^3 - 4a_{10}^2a_3a_9b_8b_9^3 - 38a_{10}a_6^2a_9b_8b_9^3 - 6a_{10}^2a_6b_8^2b_9^3 - 18a_{10}^2a_4a_6b_9^4 + a_{10}^2a_3a_8b_9^4 + 20a_{10}a_6^2a_8b_9^4 - a_{10}^2a_2a_9b_9^4 - 8a_{10}a_5a_6a_9b_9^4 + 2a_{10}a_3a_9^2b_9^4 + 26a_6^2a_9^2b_9^4 -$

$2a_{10}^3b_3b_9^4+6a_{10}^2a_6b_7b_9^4-a_{10}^2a_5b_8b_9^4+14a_{10}a_6a_9b_8b_9^4+a_{10}^2b_8^2b_9^4+3a_{10}^2a_4b_9^5-7a_{10}a_6a_8b_9^5+a_{10}a_5a_9b_9^5-8a_6a_9^2b_9^5-a_{10}^2b_7b_9^5-2a_{10}a_9b_8b_9^5+a_{10}a_8b_9^6+a_9^2b_9^6$,

$B_1 = 2a_{10}(3a_{10}^3a_3a_5^2a_6+3a_{10}^3a_3a_4a_6^2-a_{10}^3a_2a_5a_6^2+6a_{10}^2a_5^2a_6^3+7a_{10}^2a_4a_6^4-3a_{10}^2a_3a_6^3a_8-6a_{10}a_6^5a_8-6a_{10}^2a_3a_5a_6^2a_9+a_{10}^2a_2a_6^3a_9-12a_{10}a_5a_6^4a_9+3a_{10}a_3a_6^3a_9^2+6a_6^5a_9^2+a_{10}^2a_3a_6^3b_{12}+2a_{10}a_6^5b_{12}+a_{10}^4a_3a_6b_3+3a_{10}^3a_6^3b_3-2a_{10}^3a_3a_6^2b_7-4a_{10}^2a_6^4b_7-5a_{10}^3a_3a_5a_6b_8+a_{10}^3a_2a_6^2b_8-10a_{10}^2a_5a_6^3b_8+5a_{10}^2a_3a_6^2a_9b_8+10a_{10}a_6^4a_9b_8+2a_{10}^3a_3a_6b_8^2+4a_{10}^2a_6^3b_8^2-2a_{10}^3a_3a_5^2b_9-6a_{10}^3a_3a_4a_6b_9+a_{10}^3a_2a_5a_6b_9-13a_{10}^2a_5^2a_6^2b_9-24a_{10}^2a_4a_6^3b_9+9a_{10}^2a_3a_6^2a_8b_9+27a_{10}a_6^4a_8b_9+10a_{10}^2a_3a_5a_6a_9b_9-2a_{10}^2a_2a_6^2a_9b_9+38a_{10}a_5a_6^3a_9b_9-8a_{10}a_3a_6^2a_9^2b_9-25a_6^4a_9^2b_9-3a_{10}^2a_3a_6^2b_{12}b_9-9a_{10}a_6^4b_{12}b_9-a_{10}^4a_3b_3b_9-7a_{10}^3a_6^2b_3b_9+4a_{10}^3a_3a_6b_7b_9+14a_{10}^2a_6^3b_7b_9+3a_{10}^3a_3a_5b_8b_9-a_{10}^3a_2a_6b_8b_9+21a_{10}^2a_5a_6^2b_8b_9-8a_{10}^2a_3a_6a_9b_8b_9-31a_{10}a_6^3a_9b_8b_9-a_{10}^3a_3b_8^2b_9-8a_{10}^2a_6^2b_8^2b_9+3a_{10}^3a_3a_4b_9^2+9a_{10}^2a_5^2a_6b_9^2+30a_{10}^2a_4a_6^2b_9^2-9a_{10}^2a_3a_6a_8b_9^2-48a_{10}a_6^3a_8b_9^2-4a_{10}^2a_3a_5a_9b_9^2+a_{10}^2a_2a_6a_9b_9^2-44a_{10}a_5a_6^2a_9b_9^2+7a_{10}a_3a_6a_9^2b_9^2+41a_6^3a_9^2b_9^2+3a_{10}^2a_3a_6b_{12}b_9^2+16a_{10}a_6^3b_{12}b_9^2+5a_{10}^3a_6b_3b_9^2-2a_{10}^3a_3b_7b_9^2-18a_{10}^2a_6^2b_7b_9^2-14a_{10}^2a_5a_6b_8b_9^2+3a_{10}^2a_3a_9b_8b_9^2+35a_{10}a_6^2a_9b_8b_9^2+5a_{10}^2a_6b_8^2b_9^2-2a_{10}^2a_5^2b_9^3-16a_{10}^2a_4a_6b_9^3+3a_{10}^2a_3a_8b_9^3+42a_{10}a_6^2a_8b_9^3+22a_{10}a_5a_6a_9b_9^3-2a_{10}a_3a_9^2b_9^3-33a_6^2a_9^2b_9^3-a_{10}^2a_3b_{12}b_9^3-14a_{10}a_6^2b_{12}b_9^3-a_{10}^3b_3b_9^3+10a_{10}^2a_6b_7b_9^3+3a_{10}^2a_5b_8b_9^3-17a_{10}a_6a_9b_8b_9^3-a_{10}^2b_8^2b_9^3+3a_{10}^2a_4b_9^4-18a_{10}a_6a_8b_9^4-4a_{10}a_5a_9b_9^4+13a_6a_9^2b_9^4+6a_{10}a_6b_{12}b_9^4-2a_{10}^2b_7b_9^4+3a_{10}a_9b_8b_9^4+3a_{10}a_8b_9^5-2a_9^2b_9^5-a_{10}b_{12}b_9^5)$,

$B_2 = -a_{10}^2(-a_{10}^3a_3a_5^2+6a_{10}^3a_3a_4a_6+2a_{10}^3a_2a_5a_6-7a_{10}^2a_5^2a_6^2+6a_{10}^2a_4a_6^3-12a_{10}^2a_3a_6^2a_8-19a_{10}a_6^4a_8-a_{10}^2a_3a_5a_6a_9-2a_{10}^2a_2a_6^2a_9+9a_{10}a_5a_6^3a_9+2a_{10}a_3a_6^2a_9^2-2a_6^4a_9^2+7a_{10}^2a_3a_6^2b_{12}+12a_{10}a_6^4b_{12}+a_{10}^4a_3b_3-a_{10}^3a_6^2b_3-5a_{10}^3a_3a_6b_7-7a_{10}^2a_6^3b_7+3a_{10}^3a_3a_5b_8-2a_{10}^3a_2a_6b_8+15a_{10}^2a_5a_6^2b_8-10a_{10}a_6^3a_9b_8-2a_{10}^3a_3b_8^2-8a_{10}^2a_6^2b_8^2-6a_{10}^3a_3a_4b_9-3a_{10}^3a_2a_5b_9+9a_{10}^2a_5^2a_6b_9-12a_{10}^2a_4a_6^2b_9+24a_{10}^2a_3a_6a_8b_9+66a_{10}a_6^3a_8b_9+a_{10}^2a_3a_5a_9b_9+5a_{10}^2a_2a_6a_9b_9-19a_{10}a_5a_6^2a_9b_9-4a_{10}a_3a_6a_9^2b_9+5a_6^3a_9^2b_9-14a_{10}^2a_3a_6b_{12}b_9-42a_{10}a_6^3b_{12}b_9+4a_{10}^3a_6b_3b_9+5a_{10}^3a_3b_7b_9+17a_{10}^2a_6^2b_7b_9+3a_{10}^3a_2b_8b_9-19a_{10}^2a_5a_6b_8b_9+21a_{10}a_6^2a_9b_8b_9+10a_{10}^2a_6b_8^2b_9-3a_{10}^2a_5^2b_9^2+6a_{10}^2a_4a_6b_9^2-12a_{10}^2a_3a_8b_9^2-84a_{10}a_6^2a_8b_9^2-3a_{10}^2a_2a_9b_9^2+13a_{10}a_5a_6a_9b_9^2+2a_{10}a_3a_9^2b_9^2-4a_6^2a_9^2b_9^2+7a_{10}^2a_3b_{12}b_9^2+54a_{10}a_6^2b_{12}b_9^2-3a_{10}^3b_3b_9^2-13a_{10}^2a_6b_7b_9^2+6a_{10}^2a_5b_8b_9^2-14a_{10}a_6a_9b_8b_9^2-3a_{10}^2b_8^2b_9^2+46a_{10}a_6a_8b_9^3-3a_{10}a_5a_9b_9^3+a_6a_9^2b_9^3-30a_{10}a_6b_{12}b_9^3+3a_{10}^2b_7b_9^3+3a_{10}a_9b_8b_9^3-9a_{10}a_8b_9^4+6a_{10}b_{12}b_9^4)$,

$B_3 = 2a_{10}^3(a_{10}^3a_3a_4+a_{10}^3a_2a_5+a_{10}^2a_5^2a_6-3a_{10}^2a_4a_6^2-5a_{10}^2a_3a_6a_8-3a_{10}a_6^3a_8-a_{10}^2a_3a_5a_9-a_{10}^2a_2a_6a_9-3a_{10}a_5a_6^2a_9+a_{10}a_3a_6a_9^2+2a_6^3a_9^2+4a_{10}^2a_3a_6b_{12}+4a_{10}a_6^3b_{12}-2a_{10}^3a_6b_3-a_{10}^3a_3b_7+a_{10}^2a_6^2b_7-a_{10}^3a_2b_8-3a_{10}^2a_5a_6b_8+a_{10}^2a_3a_9b_8+4a_{10}a_6^2a_9b_8+2a_{10}^2a_6b_8^2+6a_{10}^2a_4a_6b_9+5a_{10}^2a_3a_8b_9+7a_{10}a_6^2a_8b_9+a_{10}^2a_2a_9b_9+4a_{10}a_5a_6a_9b_9-a_{10}a_3a_9^2b_9-5a_6^2a_9^2b_9-4a_{10}^2a_3b_{12}b_9-10a_{10}a_6^2b_{12}b_9+2a_{10}^3b_3b_9-2a_{10}^2a_6b_7b_9+a_{10}^2a_5b_8b_9-6a_{10}a_6a_9b_8b_9-a_{10}^2b_8^2b_9-3a_{10}^2a_4b_9^2-5a_{10}a_6a_8b_9^2-a_{10}a_5a_9b_9^2+4a_6a_9^2b_9^2+8a_{10}a_6b_{12}b_9^2+a_{10}^2b_7b_9^2+2a_{10}a_9b_8b_9^2+a_{10}a_8b_9^3-a_9^2b_9^3-2a_{10}b_{12}b_9^3)$,

$B_4 = a_{10}^4(2a_{10}^2a_5^2+4a_{10}^2a_4a_6+3a_{10}^2a_3a_8-6a_{10}a_6^2a_8-5a_{10}a_5a_6a_9+3a_6^2a_9^2-3a_{10}^2a_3b_{12}+3a_{10}a_6^2b_{12}+a_{10}^3b_3-3a_{10}^2a_6b_7-3a_{10}^2a_5b_8+4a_{10}a_6a_9b_8+a_{10}^2b_8^2-3a_{10}^2a_4b_9+9a_{10}a_6a_8b_9+4a_{10}a_5a_9b_9-5a_6a_9^2b_9-4a_{10}a_6b_{12}b_9+2a_{10}^2b_7b_9-3a_{10}a_9b_8b_9-3a_{10}a_8b_9^2+2a_9^2b_9^2+a_{10}b_{12}b_9^2)$,

$B_5 = 2a_{10}^6a_6(a_8-b_{12})$, $B_6 = a_{10}^7(a_8-b_{12})$.

In this case the identity $A_3(y) \equiv 0$ holds if at least one of the following three series of conditions is satisfied:

$$\begin{array}{l} a_2=(-5a_{10}a_5a_6+3a_6^2a_9+4a_{10}a_6b_8+4a_{10}a_5b_9-5a_6a_9b_9-3a_{10}b_8\cdot \\ b_9+2a_9b_9^2)/a_{10}^2, \quad a_3=(5a_6-3b_9)(b_9-a_6)/a_{10}, \quad b_3=(-2a_{10}^2a_5^2- \\ -4a_{10}^2a_4a_6+3a_{10}a_6^2a_8+5a_{10}a_5a_6a_9-3a_6^2a_9^2+3a_{10}^2a_6b_7+3a_{10}^2a_5b_8- \\ -4a_{10}a_6a_9b_8-a_{10}^2b_8^2+3a_{10}^2a_4b_9-5a_{10}a_6a_8b_9-4a_{10}a_5a_9b_9+5a_6a_9^2\cdot \\ b_9-2a_{10}^2b_7b_9+3a_{10}a_9b_8b_9+2a_{10}a_8b_9^2-2a_9^2b_9^2)/a_{10}^3,\ b_{12}=a_8; \end{array} \tag{41}$$

$$\begin{array}{l} b_3=a_4(b_9-a_6)/a_{10},\ b_7=(a_{10}a_4-a_6a_8+a_8b_9)/a_{10}, \\ b_8=(a_{10}a_5-a_6a_9+a_9b_9)/a_{10},\ b_{12}=a_8; \end{array} \tag{42}$$

$$\begin{array}{l} a_3=(5a_6-3b_9)(b_9-a_6)/a_{10},\ b_3=(-2a_{10}^2a_5^2-4a_{10}^2a_4a_6+3a_{10}a_6^2a_8 \\ +5a_{10}a_5a_6a_9-3a_6^2a_9^2+3a_{10}^2a_6b_7+3a_{10}^2a_5b_8-4a_{10}a_6a_9b_8-a_{10}^2b_8^2+ \\ +3a_{10}^2a_4b_9-5a_{10}a_6a_8b_9-4a_{10}a_5a_9b_9+5a_6a_9^2b_9-2a_{10}^2b_7b_9+3a_{10}a_9\cdot \\ b_8b_9+2a_{10}a_8b_9^2-2a_9^2b_9^2)/a_{10}^3,\ b_8=(a_{10}a_5-a_6a_9+a_9b_9)/a_{10}, \\ b_{12}=a_8; \end{array} \tag{43}$$

11) (10), (26) $\Rightarrow A_3(y)=(a_1+a_3y+a_6y^2+a_{10}y^3)(3a_1a_{10}a_4a_6+a_3a_4a_6^2-a_1a_6^2a_8+2a_1a_{10}^2b_3+a_{10}a_3a_6b_3-a_1a_{10}a_6b_7-3a_1a_{10}a_4b_9-2a_3a_4a_6b_9+2a_1a_6a_8b_9-a_{10}a_3b_3b_9+a_1a_{10}b_7b_9+a_3a_4b_9^2-a_1a_8b_9^2+(a_{10}a_3a_4a_6-3a_1a_{10}^2a_4+2a_4a_6^3+5a_1a_{10}a_6a_8-2a_1a_{10}a_6b_{12}+a_{10}^2a_3b_3+2a_{10}a_6^2b_3+3a_1a_{10}^2b_7-a_{10}a_3a_4b_9-4a_4a_6^2b_9-5a_1a_{10}a_8b_9+2a_1a_{10}b_{12}b_9-2a_{10}a_6b_3b_9+2a_4a_6b_9^2)y+(2a_{10}a_4a_6^2-2a_{10}^2a_3a_4-4a_1a_{10}^2a_8+3a_{10}a_3a_6a_8+a_6^3a_8+4a_1a_{10}^2b_{12}-a_{10}a_3a_6b_{12}+3a_{10}^2a_6b_3+2a_{10}^2a_3b_7+a_{10}a_6^2b_7-5a_{10}a_4a_6b_9-3a_{10}a_3a_8b_9-2a_6^2a_8b_9+a_{10}a_3b_{12}b_9-3a_{10}^2b_3b_9-a_{10}a_6b_7b_9+3a_{10}a_4b_9^2+a_6a_8b_9^2)y^2-a_{10}(4a_{10}a_4a_6+3a_{10}a_3a_8-3a_6^2a_8-3a_{10}a_3b_{12}+a_{10}^2b_3-3a_{10}a_6b_7-3a_{10}a_4b_9+5a_6a_8b_9+2a_{10}b_7b_9-2a_8b_9^2)y^3-a_{10}^2(a_8-b_{12})(3a_6-b_9)y^4-a_{10}^3(a_8-b_{12})y^5)/a_{10}^2\equiv 0\Rightarrow$

$$b_3=-a_4(a_6-b_9)/a_{10},\ b_7=(a_{10}a_4-a_6a_8+a_8b_9)/a_{10},\ b_{12}=a_8; \tag{44}$$

$$\begin{array}{l} b_3=(-4a_{10}a_4a_6+3a_6^2a_8+3a_{10}a_6b_7+3a_{10}a_4b_9-5a_6a_8b_9- \\ \quad -2a_{10}b_7b_9+2a_8b_9^2)/a_{10}^2,\ a_3=(-5a_6^2+8a_6b_9-3b_9^2)/a_{10}, \\ a_1=(3a_6-2b_9)(a_6-b_9)^2/a_{10}^2,\quad b_{12}=a_8; \end{array} \tag{45}$$

It is easy to see that the set of conditions {(8), (21), (34)} is a particular case for the set of conditions {(8), (22), (35)}, the set {(9), (24), (40)} for {(9), (23), (36)} and {(10), (25), (42)} for {(10), (26), (44)}. The sets of conditions: {(9), (24), (38)} and {(9), (23), (37)}; {(10), (26), (45)} and {(10), (25), (43)} are the same.

Lemma 3. *For quartic differential system* {(4), (3)} *the algebraic multiplicity of the invariant straight line* $x=0$ *is greater than or equal to four if and only if at least one of the following fourteen sets of conditions holds:*

1) (6), (16), (27), *2)* (6), (16), (28), *3)* (6), (16), (29),
4) (6), (17), (30), *5)* (6), (18), (31), *6)* (6), (19), (32),
7) (7), (20), (33), *8)* (8), (22), (35), *9)* (9), (23), (36),
10) (9),(23), (37), *11)* (9), (24), (39), *12)* (10), (25), (41),
13) (10), (25), (43), *14)* (10), (26), (44).

The algebraic multiplicity $m_a \geq 5$ if in each of the cases 1)–14) of Lemma 3 the identity $A_4(y) \equiv 0$ holds. Proceeding as in the previous case ($m_a \geq 4$) and taking into account (3), we will examine each case separately.

1) (6), (16), (27) $\Rightarrow A_4(y) = a_7(b_2 + 2b_5y + 3b_9y^2 + 4b_{14}y^3)(b_0 + b_2y + b_5y^2 + b_9y^3 + b_{14}y^4) \equiv 0 \Rightarrow$

$$b_2 = b_5 = b_9 = b_{14} = 0; \tag{46}$$

2) (6), (16), (28) $\Rightarrow A_4(y) = a_4b_0(b_4 + 2b_8y + 3b_{13}y^2) \equiv 0 \Rightarrow$

$$b_4 = b_8 = b_{13} = 0,\ b_0 \neq 0; \tag{47}$$

3) (6), (16), (29) $\Rightarrow A_4(y) = b_2(a_4+a_8y)(-a_8b_1+a_7b_2+a_4b_4+2a_4b_8y+3a_4b_{13}y^2+a_8b_8y^2 + 2a_8b_{13}y^3)/a_8 \equiv 0 \Rightarrow$

$$b_1 = (a_7b_2 + a_4b_4)/a_8,\ b_8 = 0,\ b_{13} = 0; \tag{48}$$

4) (6), (17), (30) $\Rightarrow A_4(y) = -5a_2a_4b_0 + 3a_2^2b_1 + a_4b_0b_4 - a_2b_1b_4 + a_2b_0b_7 + 10a_2^3y + 2a_2b_0b_{12}y - 4a_2^2b_4y \equiv 0 \Rightarrow$

$$\begin{aligned} &b_7 = (5a_2a_4b_0 - 3a_2^2b_1 - a_4b_0b_4 + a_2b_1b_4)/(a_2b_0),\\ &b_{12} = a_2(2b_4 - 5a_2)/b_0; \end{aligned} \tag{49}$$

5) (6), (18), (31) $\Rightarrow A_4(y) = -(a_2+a_5y)(2a_2^3a_5^2+4a_2a_4a_5^2b_2-a_2^2a_5a_8b_2-a_5^2a_7b_2^2+a_4a_5a_8b_2^2 - a_2a_8^2b_2^2 + a_5^3b_2b_3 - a_2^2a_5^2b_4 - a_4a_5^2b_2b_4 + a_2a_5a_8b_2b_4 - a_2a_5^2b_2b_7 + 2a_2^2a_5^3y + 6a_2a_5^2a_8b_2y - 2a_2a_5^2b_{12}b_2y - 2a_2a_5^3b_4y - 2a_2a_5^4y^2 + 3a_5^3a_8b_2y^2 - a_5^3b_{12}b_2y^2 - a_5^4b_4y^2 - 2a_5^5y^3)/a_5^3 \not\equiv 0;$

6) (6), (19), (32) $\Rightarrow A_4(y) = (a_2+a_5y+a_9y^2)(-2a_2^2a_5a_9-2a_4a_5^2b_5+2a_2a_5a_8b_5-2a_2a_4a_9b_5-a_5a_9b_3b_5+a_5a_7b_5^2+a_2a_9b_5b_7+a_2^2a_9b_8+a_4a_5b_5b_8-a_2a_8b_5b_8-4a_2a_5^2a_9y+2a_2^2a_9^2y-4a_4a_5a_9b_5y-4a_2a_8a_9b_5y+2a_2a_9b_{12}b_5y-2a_9^2b_3b_5y+2a_7a_9b_5^2y+2a_2a_5a_9b_8y+2a_4a_9b_5b_8y-2a_5^3a_9y^2-4a_5a_8a_9b_5y^2+2a_4a_9^2b_5y^2+a_5a_9b_{12}b_5y^2-a_9^2b_5b_7y^2+a_5^2a_9b_8y^2+2a_2a_9^2b_8y^2 + a_8a_9b_5b_8y^2 - 2a_5^2a_9^2y^3 + 4a_2a_9^3y^3 + 2a_5a_9^2b_8y^3 + 2a_5a_9^3y^4 + a_9^3b_8y^4 + 2a_9^4y^5)/a_9^2 \not\equiv 0$

7) (7), (20), (33) $\Rightarrow A_4(y) = -a_1(4a_7b_0 - b_0b_{11} - 3a_1b_6 + 4a_1a_7y - 4a_1b_{11}y) \equiv 0 \Rightarrow$

$$b_6 = a_7b_0/a_1,\ b_{11} = a_7; \tag{50}$$

8) (8), (22), (35) $\Rightarrow A_4(y) = -(a_1 + a_3y)(-5a_1^2a_7 + a_1^2b_{11} + 6a_1a_7b_2 - a_1b_{11}b_2 - a_7b_2^2 - 4a_1a_3b_6 + a_3b_2b_6 + 2a_1a_3a_7y - 4a_1a_3b_{11}y + 2a_3a_7b_2y - 2a_3^2b_6y + 3a_3^2a_7y^2 - 3a_3^2b_{11}y^2)/a_3 \equiv 0 \Rightarrow$

$$b_6 = a_7(b_2 - a_1)/a_3,\ b_{11} = a_7; \tag{51}$$

9) (9), (23), (36) $\Rightarrow A_4(y) = -(a_1+a_3y+a_6y^2)(-a_3^3a_7-4a_1a_3a_6a_7+a_1a_3a_6b_{11}+2a_3^2a_7b_5+4a_1a_6a_7b_5-a_1a_6b_{11}b_5-a_3a_7b_5^2-a_3^2a_6b_6-3a_1a_6^2b_6+a_3a_6b_5b_6-4a_3^2a_6a_7y+4a_1a_6^2a_7y - 4a_1a_6^2b_{11}y + 6a_3a_6a_7b_5y - 2a_6a_7b_5^2y - 4a_3a_6^2b_6y + 2a_6^2b_5b_6y + 3a_3a_6^2a_7y^2 - 4a_3a_6^2b_{11}y^2 + a_6^2b_{11}b_5y^2 - a_6^3b_6y^2 + 2a_6^3a_7y^3 - 2a_6^3b_{11}y^3)/a_6^2 \equiv 0 \Rightarrow$

$$b_6 = a_7(b_5 - a_3)/a_6,\ b_{11} = a_7; \tag{52}$$

10) (9),(23), (37)⇒ $A_4(y) = -(a_3 + 2a_6y)^2(-2a_3^2a_4a_5 + 4a_2a_3a_4a_6 + 2a_3^3a_7 - a_3^3b_{11} + 4a_3a_5a_6b_3 - 8a_2a_6^2b_3 - 2a_3^2a_6b_6 + 12a_3^2a_6a_7y - 4a_3^2a_4a_9y - 8a_3^2a_6b_{11}y + 8a_3a_6a_9b_3y - 8a_3a_6^2b_6y + 24a_3a_6^2a_7y^2 - 4a_3a_4a_6a_9y^2 - 20a_3a_6^2b_{11}y^2 + 8a_6^2a_9b_3y^2 - 8a_6^3b_6y^2 + 16a_6^3a_7y^3 - 16a_6^3b_{11}y^3)/32a_6^3 \equiv 0 \Rightarrow$

$$\begin{array}{l} a_2 = a_3(2a_5a_6 - a_3a_9)/(4a_6^2), \\ b_6 = (a_3a_6a_7 - a_3a_4a_9 + 2a_6a_9b_3)/(2a_6^2),\ b_{11} = a_7; \end{array} \tag{53}$$

$$b_3 = a_3a_4/(2a_6),\ b_6 = a_3a_7/(2a_6),\ b_{11} = a_7; \tag{54}$$

11) (9), (24), (39) ⇒ $A_4(y) = (a_3+2a_6y)(-8a_3^3a_5^3a_6^3 - 16a_3^3a_4a_5a_6^4 + 64a_2a_3^2a_5^2a_6^4 + 32a_2a_3^2a_4a_6^5 - 160a_2^2a_3a_5a_6^5 + 128a_2^3a_6^6 - 8a_3^4a_6^4a_7 + 4a_3^4a_5a_6^3a_8 - 8a_2a_3^3a_6^4a_8 + 12a_3^4a_5^2a_6^2a_9 + 12a_3^4a_4a_6^3a_9 - 72a_2a_3^3a_5a_6^3a_9 + 96a_2^2a_3^2a_6^4a_9 - 2a_3^5a_6^2a_8a_9 - 6a_3^5a_5a_6a_9^2 + 20a_2a_3^4a_6^2a_9^2 + a_3^6a_9^3 + 4a_3^4a_6^4b_{11} + 32a_3^2a_5a_6^5b_3 - 64a_2a_3a_6^6b_3 - 24a_3^3a_6^4a_9b_3 + 8a_3^3a_6^5b_6 - 2a_6(-8a_3^2a_5^3a_6^3 + 16a_3^2a_4a_5a_6^4 + 32a_2a_3a_5^2a_6^4 - 32a_2a_3a_4a_6^5 - 32a_2^2a_5a_6^5 + 32a_3^3a_6^4a_7 - 12a_3^3a_5a_6^3a_8 + 24a_2a_3^2a_6^4a_8 + 20a_3^3a_5^2a_6^2a_9 - 20a_3^3a_4a_6^3a_9 - 40a_2a_3^2a_5a_6^3a_9 + 6a_3^4a_6^2a_8a_9 - 14a_3^4a_5a_6a_9^2 + 12a_2a_3^3a_6^2a_9^2 + 3a_3^5a_9^3 - 20a_3^3a_6^4b_{11} - 32a_3a_5a_6^5b_3 + 64a_2a_6^6b_3 + 40a_3^2a_6^4a_9b_3 - 24a_3^2a_6^5b_6)y - 12a_3a_6^2(16a_3a_6^4a_7 - 4a_3a_5a_6^3a_8 + 8a_2a_6^4a_8 + 4a_3a_5^2a_6^2a_9 - 4a_3a_4a_6^3a_9 - 8a_2a_5a_6^3a_9 + 2a_3^2a_6^2a_8a_9 - 4a_3^2a_5a_6a_9^2 + 4a_2a_3a_6^2a_9^2 + a_3^3a_9^3 - 12a_3a_6^4b_{11} + 8a_6^4a_9b_3 - 8a_6^5b_6)y^2 - 8a_6^3(32a_3a_6^4a_7 - 4a_3a_5a_6^3a_8 + 8a_2a_6^4a_8 + 4a_3a_5^2a_6^2a_9 - 4a_3a_4a_6^3a_9 - 8a_2a_5a_6^3a_9 + 2a_3^2a_6^2a_8a_9 - 4a_3^2a_5a_6a_9^2 + 4a_2a_3a_6^2a_9^2 + a_3^3a_9^3 - 28a_3a_6^4b_{11} + 8a_6^4a_9b_3 - 8a_6^5b_6)y^3 - 128a_6^8(a_7 - b_{11})y^4)/(128a_6^7) \equiv 0 \Rightarrow$

$$\begin{array}{l} a_2 = a_3(2a_5a_6 - a_3a_9)/(4a_6^2), \\ b_6 = (a_3a_6a_7 - a_3a_4a_9 + 2a_6a_9b_3)/(2a_6^2),\ b_{11} = a_7; \end{array} \tag{55}$$

$$\begin{array}{l} a_2 = a_3(2a_5a_6 - a_3a_9)/(4a_6^2), \\ b_3 = a_3a_4/(2a_6),\ b_6 = a_3a_7/(2a_6),\ b_{11} = a_7; \end{array} \tag{56}$$

12) (10), (25), (41) ⇒ $A_4(y) = (3a_6 - 2b_9 + a_{10}y)(a_6 - b_9 - a_{10}y)(B_0 + B_1y + B_2y^2 + B_3y^3 + B_4y^4 + B_5y^5)/a_{10}^6$, where

$B_0 = 14a_{10}^2a_5^3a_6^2 - 21a_{10}^2a_4a_5a_6^3 + 7a_{10}a_6^5a_7 + 23a_{10}a_5a_6^4a_8 - 29a_{10}a_5^2a_6^3a_9 + 22a_{10}a_4a_6^4a_9 - 24a_6^5a_8a_9 + 15a_5a_6^4a_9^2 - 3a_{10}a_6^5b_{11} + 4a_{10}^2a_6^4b_6 + 20a_{10}^2a_5a_6^3b_7 - 21a_{10}a_6^4a_9b_7 - 44a_{10}^2a_5^2a_6^2b_8 + 20a_{10}^2a_4a_6^3b_8 - 22a_{10}a_6^4a_8b_8 + 63a_{10}a_5a_6^3a_9b_8 - 18a_6^4a_9^2b_8 - 19a_{10}^2a_6^3b_7b_8 + 46a_{10}^2a_5a_6^2b_8^2 - 34a_{10}a_6^3a_9b_8^2 - 16a_{10}^2a_6^2b_8^3 - 18a_{10}^2a_5^3a_6b_9 + 51a_{10}^2a_4a_5a_6^2b_9 - 33a_{10}a_6^4a_7b_9 - 79a_{10}a_5a_6^3a_8b_9 + 66a_{10}a_5^2a_6^2a_9b_9 - 75a_{10}a_4a_6^3a_9b_9 + 106a_6^4a_8a_9b_9 - 49a_5a_6^3a_9^2b_9 + 14a_{10}a_6^4b_{11}b_9 - 15a_{10}^2a_6^3b_6b_9 - 48a_{10}^2a_5a_6^2b_7b_9 + 71a_{10}a_6^3a_9b_7b_9 + 57a_{10}^2a_5^2a_6b_8b_9 - 48a_{10}^2a_4a_6^2b_8b_9 + 75a_{10}a_6^3a_8b_8b_9 - 145a_{10}a_5a_6^2a_9b_8b_9 + 60a_6^3a_9^2b_8b_9 + 45a_{10}^2a_6^2b_7b_8b_9 - 60a_{10}^2a_5a_6b_8^2b_9 + 79a_{10}a_6^2a_9b_8^2b_9 + 21a_{10}^2a_6b_8^3b_9 + 6a_{10}^2a_5^3b_9^2 - 41a_{10}^2a_4a_5a_6b_9^2 + 62a_{10}a_6^3a_7b_9^2 + 101a_{10}a_5a_6^2a_8b_9^2 - 49a_{10}a_5^2a_6a_9b_9^2 + 95a_{10}a_4a_6^2a_9b_9^2 - 186a_6^3a_8a_9b_9^2 + 59a_5a_6^2a_9^2b_9^2 - 26a_{10}a_6^3b_{11}b_9^2 + 21a_{10}^2a_6^2b_6b_9^2 + 38a_{10}^2a_5a_6b_7b_9^2 - 89a_{10}a_6^2a_9b_7b_9^2 - 19a_{10}^2a_5^2b_8b_9^2 + 38a_{10}^2a_4a_6b_8b_9^2 - 95a_{10}a_6^2a_8b_8b_9^2 + 109a_{10}a_5a_6a_9b_8b_9^2 - 74a_6^2a_9^2b_8b_9^2 - 35a_{10}^2a_6b_7b_8b_9^2 + 20a_{10}^2a_5b_8^2b_9^2 - 60a_{10}a_6a_9b_8^2b_9^2 - 7a_{10}^2b_8^3b_9^2 + 11a_{10}^2a_4a_5b_9^3 - 58a_{10}a_6^2a_7b_9^3 - 57a_{10}a_5a_6a_8b_9^3 + 12a_{10}a_5^2a_9b_9^3 - 53a_{10}a_4a_6a_9b_9^3 + 162a_6^2a_8a_9b_9^3 - 31a_5a_6a_9^2b_9^3 + 24a_{10}a_6^2b_{11}b_9^3 - 13a_{10}^2a_6b_6b_9^3 - 10a_{10}^2a_5b_7b_9^3 + 49a_{10}a_6a_9b_7b_9^3 - 10a_{10}^2a_4b_8b_9^3 + 53a_{10}a_6a_8b_8b_9^3 - 27a_{10}a_5a_9b_8b_9^3 + 40a_6a_9^2b_8b_9^3 + 9a_{10}^2b_7b_8b_9^3 + 15a_{10}a_9b_8^2b_9^3 + 27a_{10}a_6a_7b_9^4 + 12a_{10}a_5a_8b_9^4 + 11a_{10}a_4a_9b_9^4 -$

$70a_6a_8a_9b_9^4+6a_5a_9^2b_9^4-11a_{10}a_6b_{11}b_9^4+3a_{10}^2b_6b_9^4-10a_{10}a_9b_7b_9^4-11a_{10}a_8b_8b_9^4-8a_9^2b_8b_9^4-5a_{10}a_7b_9^5+12a_8a_9b_9^5+2a_{10}b_{11}b_9^5$,

$B_1 = 12a_{10}^3a_5^3a_6+15a_{10}^3a_4a_5a_6^2-27a_{10}^2a_6^4a_7-21a_{10}^2a_5a_6^3a_8-45a_{10}^2a_5^2a_6^2a_9-18a_{10}^2a_4a_6^3a_9+24a_{10}a_6^4a_8a_9+57a_{10}a_5a_6^3a_9^2-24a_6^4a_9^3+15a_{10}^2a_6^4b_{11}-12a_{10}^3a_6^3b_6-12a_{10}^3a_5a_6^2b_7+15a_{10}^2a_6^3a_9b_7-36a_{10}^3a_5^2a_6b_8-12a_{10}^3a_4a_6^2b_8+18a_{10}^2a_6^3a_8b_8+87a_{10}^2a_5a_6^2a_9b_8-54a_{10}a_6^3a_9^2b_8+9a_{10}^3a_6^2b_7b_8+36a_{10}^3a_5a_6b_8^2-42a_{10}^2a_6^2a_9b_8^2-12a_{10}^3a_6b_8^3-6a_{10}^3a_5^3b_9-22a_{10}^3a_4a_5a_6b_9+100a_{10}^2a_6^3a_7b_9+52a_{10}^2a_5a_6^2a_8b_9+67a_{10}^2a_5^2a_6a_9b_9+43a_{10}^2a_4a_6^2a_9b_9-82a_{10}a_6^3a_8a_9b_9-140a_{10}a_5a_6^2a_9^2b_9+82a_6^3a_9^3b_9-55a_{10}^2a_6^3b_{11}b_9+33a_{10}^3a_6^2b_6b_9+16a_{10}^3a_5a_6b_7b_9-34a_{10}^2a_6^2a_9b_7b_9+19a_{10}^3a_5^2b_8b_9+16a_{10}^3a_4a_6b_8b_9-43a_{10}^2a_6^2a_8b_8b_9-130a_{10}^2a_5a_6a_9b_8b_9+132a_{10}a_6^2a_9^2b_8b_9-10a_{10}^3a_6b_7b_8b_9-20a_{10}^3a_5b_8^2b_9+63a_{10}^2a_6a_9b_8^2b_9+7a_{10}^3b_8^3b_9+9a_{10}^3a_4a_5b_9^2-138a_{10}^2a_6^2a_7b_9^2-43a_{10}^2a_5a_6a_8b_9^2-24a_{10}^2a_5^2a_9b_9^2-34a_{10}^2a_4a_6a_9b_9^2+104a_{10}a_6^2a_8a_9b_9^2+113a_{10}a_5a_6a_9^2b_9^2-104a_6^2a_9^3b_9^2+75a_{10}^2a_6^2b_{11}b_9^2-30a_{10}^3a_6b_6b_9^2-6a_{10}^3a_5b_7b_9^2+25a_{10}^2a_6a_9b_7b_9^2-6a_{10}^3a_4b_8b_9^2+34a_{10}^2a_6a_8b_8b_9^2+47a_{10}^2a_5a_9b_8b_9^2-106a_{10}a_6a_9^2b_8b_9^2+3a_{10}^3b_7b_8b_9^2-23a_{10}^2a_9b_8^2b_9^2+84a_{10}^2a_6a_7b_9^3+12a_{10}^2a_5a_8b_9^3+9a_{10}^2a_4a_9b_9^3-58a_{10}a_6a_8a_9b_9^3-30a_{10}a_5a_9^2b_9^3+58a_6a_9^3b_9^3-45a_{10}^2a_6b_{11}b_9^3+9a_{10}^3b_6b_9^3-6a_{10}^2a_9b_7b_9^3-9a_{10}^2a_8b_8b_9^3+28a_{10}a_9^2b_8b_9^3-19a_{10}^2a_7b_9^4+12a_{10}a_8a_9b_9^4-12a_9^3b_9^4+10a_{10}^2b_{11}b_9^4$,

$B_2 = 12a_{10}^3a_5^3a_6+15a_{10}^3a_4a_5a_6^2-27a_{10}^2a_6^4a_7-21a_{10}^2a_5a_6^3a_8-45a_{10}^2a_5^2a_6^2a_9-18a_{10}^2a_4a_6^3a_9+24a_{10}a_6^4a_8a_9+57a_{10}a_5a_6^3a_9^2-24a_6^4a_9^3+15a_{10}^2a_6^4b_{11}-12a_{10}^3a_6^3b_6-12a_{10}^3a_5a_6^2b_7+15a_{10}^2a_6^3a_9b_7-36a_{10}^3a_5^2a_6b_8-12a_{10}^3a_4a_6^2b_8+18a_{10}^2a_6^3a_8b_8+87a_{10}^2a_5a_6^2a_9b_8-54a_{10}a_6^3a_9^2b_8+9a_{10}^3a_6^2b_7b_8+36a_{10}^3a_5a_6b_8^2-42a_{10}^2a_6^2a_9b_8^2-12a_{10}^3a_6b_8^3-6a_{10}^3a_5^3b_9-22a_{10}^3a_4a_5a_6b_9+100a_{10}^2a_6^3a_7b_9+52a_{10}^2a_5a_6^2a_8b_9+67a_{10}^2a_5^2a_6a_9b_9+43a_{10}^2a_4a_6^2a_9b_9-82a_{10}a_6^3a_8a_9b_9-140a_{10}a_5a_6^2a_9^2b_9+82a_6^3a_9^3b_9-55a_{10}^2a_6^3b_{11}b_9+33a_{10}^3a_6^2b_6b_9+16a_{10}^3a_5a_6b_7b_9-34a_{10}^2a_6^2a_9b_7b_9+19a_{10}^3a_5^2b_8b_9+16a_{10}^3a_4a_6b_8b_9-43a_{10}^2a_6^2a_8b_8b_9-130a_{10}^2a_5a_6a_9b_8b_9+132a_{10}a_6^2a_9^2b_8b_9-10a_{10}^3a_6b_7b_8b_9-20a_{10}^3a_5b_8^2b_9+63a_{10}^2a_6a_9b_8^2b_9+7a_{10}^3b_8^3b_9+9a_{10}^3a_4a_5b_9^2-138a_{10}^2a_6^2a_7b_9^2-43a_{10}^2a_5a_6a_8b_9^2-24a_{10}^2a_5^2a_9b_9^2-34a_{10}^2a_4a_6a_9b_9^2+104a_{10}a_6^2a_8a_9b_9^2+113a_{10}a_5a_6a_9^2b_9^2-104a_6^2a_9^3b_9^2+75a_{10}^2a_6^2b_{11}b_9^2-30a_{10}^3a_6b_6b_9^2-6a_{10}^3a_5b_7b_9^2+25a_{10}^2a_6a_9b_7b_9^2-6a_{10}^3a_4b_8b_9^2+34a_{10}^2a_6a_8b_8b_9^2+47a_{10}^2a_5a_9b_8b_9^2-106a_{10}a_6a_9^2b_8b_9^2+3a_{10}^3b_7b_8b_9^2-23a_{10}^2a_9b_8^2b_9^2+84a_{10}^2a_6a_7b_9^3+12a_{10}^2a_5a_8b_9^3+9a_{10}^2a_4a_9b_9^3-58a_{10}a_6a_8a_9b_9^3-30a_{10}a_5a_9^2b_9^3+58a_6a_9^3b_9^3-45a_{10}^2a_6b_{11}b_9^3+9a_{10}^3b_6b_9^3-6a_{10}^2a_9b_7b_9^3-9a_{10}^2a_8b_8b_9^3+28a_{10}a_9^2b_8b_9^3-19a_{10}^2a_7b_9^4+12a_{10}a_8a_9b_9^4-12a_9^3b_9^4+10a_{10}^2b_{11}b_9^4$,

$B_3 = a_{10}^2(5a_{10}^3a_4a_5-22a_{10}^2a_6^2a_7-7a_{10}^2a_5a_6a_8-3a_{10}^2a_5^2a_9-6a_{10}^2a_4a_6a_9+8a_{10}a_6^2a_8a_9+7a_{10}a_5a_6a_9^2-4a_6^2a_9^3+18a_{10}^2a_6^2b_{11}-4a_{10}^3a_6b_6-4a_{10}^3a_5b_7+5a_{10}^2a_6a_9b_7-4a_{10}^3a_4b_8+6a_{10}^2a_6a_8b_8+5a_{10}^2a_5a_9b_8-6a_{10}a_6a_9^2b_8+3a_{10}^3b_7b_8-2a_{10}^2a_9b_8^2+36a_{10}^2a_6a_7b_9+6a_{10}^2a_5a_8b_9+5a_{10}^2a_4a_9b_9-14a_{10}a_6a_8a_9b_9-6a_{10}a_5a_9^2b_9+7a_6a_9^3b_9-29a_{10}^2a_6b_{11}b_9+3a_{10}^3b_6b_9-4a_{10}^2a_9b_7b_9-5a_{10}^2a_8b_8b_9+5a_{10}a_9^2b_8b_9-14a_{10}^2a_7b_9^2+6a_{10}a_8a_9b_9^2-3a_9^3b_9^2+11a_{10}^2b_{11}b_9^2)$,
$B_4 = a_{10}^5(a_7-b_{11})(3a_6-b_9)$, $B_5 = a_{10}^6(a_7-b_{11})$.

In this case the identity $A_4(y) \equiv 0$ holds if at least one of the following four series of conditions is satisfied:

$$a_5 = (a_6a_9+a_{10}b_8-a_9b_9)/a_{10},\ b_6 = (-4a_{10}a_6^2a_7-a_{10}a_4a_6a_9+ +a_6^2a_8a_9+a_{10}a_6a_9b_7+a_{10}^2a_4b_8-a_{10}a_6a_8b_8-a_{10}^2b_7b_8+7a_{10}a_6a_7b_9- -a_6a_8a_9b_9+a_{10}a_8b_8b_9-3a_{10}a_7b_9^2)/(a_{10}^2(4a_6-3b_9)),\quad b_{11}=a_7; \tag{57}$$

$$a_4 = b_7/2,\ a_6 = 3b_9/4,\ b_8 = 3a_5/2,\ b_{11} = a_7; \tag{58}$$

$$a_4=(4a_{10}b_7-a_8b_9)/(4a_{10}),\ a_6=3b_9/4,\\ b_8=(4a_{10}a_5+a_9b_9)/(4a_{10}),\ b_{11}=a_7; \tag{59}$$

$$a_5=a_9b_9/(2a_{10}),\ a_6=3b_9/4,\ b_8=3a_9b_9/(4a_{10}),\ b_{11}=a_7; \tag{60}$$

13) (10), (25), (43) $\Rightarrow A_4(y)=(3a_6-2b_9+a_{10}y)(a_6-b_9-a_{10}y)^2(B_0'+B_1'y+B_2'y^2+B_3'y^3+B_4'y^4)/a_{10}^5$, where

$B_0'=2a_{10}^2a_2a_4a_6+3a_{10}a_4a_5a_6^2-7a_6^4a_7-2a_{10}a_2a_6^2a_8-3a_5a_6^3a_8+3a_6^4b_{11}-4a_{10}a_6^3b_6-2a_{10}^2a_2a_6b_7-3a_{10}a_5a_6^2b_7-a_{10}^2a_2a_4b_9-5a_{10}a_4a_5a_6b_9+26a_6^3a_7b_9+3a_{10}a_2a_6a_8b_9+8a_5a_6^2a_8b_9-11a_6^3b_{11}b_9+11a_{10}a_6^2b_6b_9+a_{10}^2a_2b_7b_9+5a_{10}a_5a_6b_7b_9+2a_{10}a_4a_5b_9^2-36a_6^2a_7b_9^2-a_{10}a_2a_8b_9^2-7a_5a_6a_8b_9^2+15a_6^2b_{11}b_9^2-10a_{10}a_6b_6b_9^2-2a_{10}a_5b_7b_9^2+22a_6a_7b_9^3+2a_5a_8b_9^3-9a_6b_{11}b_9^3+3a_{10}b_6b_9^3-5a_7b_9^4+2b_{11}b_9^4$,

$B_1'=2(a_{10}^3a_2a_4+10a_{10}a_6^3a_7-a_{10}^2a_2a_6a_8+3a_{10}a_4a_6^2a_9-3a_6^3a_8a_9-6a_{10}a_6^3b_{11}+4a_{10}^2a_6^2b_6-a_{10}^3a_2b_7-3a_{10}a_6^2a_9b_7-27a_{10}a_6^2a_7b_9+a_{10}^2a_2a_8b_9-5a_{10}a_4a_6a_9b_9+8a_6^2a_8a_9b_9+16a_{10}a_6^2b_{11}b_9-7a_{10}^2a_6b_6b_9+5a_{10}a_6a_9b_7b_9+24a_{10}a_6a_7b_9^2+2a_{10}a_4a_9b_9^2-7a_6a_8a_9b_9^2-14a_{10}a_6b_{11}b_9^2+3a_{10}^2b_6b_9^2-2a_{10}a_9b_7b_9^2-7a_{10}a_7b_9^3+2a_8a_9b_9^3+4a_{10}b_{11}b_9^3)$,

$B_2'=a_{10}(a_{10}^2a_4a_5-18a_{10}a_6^2a_7-a_{10}a_5a_6a_8-2a_{10}a_4a_6a_9+2a_6^2a_8a_9+14a_{10}a_6^2b_{11}-4a_{10}^2a_6b_6-a_{10}^2a_5b_7+2a_{10}a_6a_9b_7+30a_{10}a_6a_7b_9+a_{10}a_5a_8b_9+a_{10}a_4a_9b_9-3a_6a_8a_9b_9-23a_{10}a_6b_{11}b_9+3a_{10}^2b_6b_9-a_{10}a_9b_7b_9-12a_{10}a_7b_9^2+a_8a_9b_9^2+9a_{10}b_{11}b_9^2)$,

$B_3'=2a_{10}^3(a_7-b_{11})(2a_6-b_9),\ B_4'=a_{10}^4(a_7-b_{11})$.

In this case the identity $A_4(y)\equiv 0$ holds if at least one of the following four series of conditions is satisfied:

$$a_4=(4a_{10}b_7-a_8b_9)/(4a_{10}),\ a_6=3b_9/4,\ b_{11}=a_7; \tag{61}$$

$$a_4=(a_6a_8+a_{10}b_7-a_8b_9)/a_{10},\ b_6=a_7(b_9-a_6)/a_{10},\ b_{11}=a_7; \tag{62}$$

$$a_2=(b_9-a_6)(a_{10}a_5+a_6a_9-a_9b_9)/a_{10}^2,\qquad b_6=(a_{10}^2a_4a_5-\\ -4a_{10}a_6^2a_7-a_{10}a_5a_6a_8-2a_{10}a_4a_6a_9+2a_6^2a_8a_9-a_{10}^2a_5b_7+\\ +2a_{10}a_6a_9b_7+7a_{10}a_6a_7b_9+a_{10}a_5a_8b_9+a_{10}a_4a_9b_9-3a_6a_8a_9b_9-\\ -a_{10}a_9b_7b_9-3a_{10}a_7b_9^2+a_8a_9b_9^2)/(a_{10}^2(4a_6-3b_9)),\ b_{11}=a_7; \tag{63}$$

$$a_2=a_9b_9^2/(16a_{10}^2),\ a_5=a_9b_9/(2a_{10}),\ a_6=3b_9/4,\ b_{11}=a_7; \tag{64}$$

14) (10), (26), (44) $\Rightarrow A_4(y)=(a_1+a_3y+a_6y^2+a_{10}y^3)(4a_1a_{10}a_6a_7+a_3a_6^2a_7-a_1a_{10}a_6b_{11}+3a_1a_{10}^2b_6+a_{10}a_3a_6b_6-4a_1a_{10}a_7b_9-2a_3a_6a_7b_9+a_1a_{10}b_{11}b_9-a_{10}a_3b_6b_9+a_3a_7b_9^2-2(2a_1a_{10}^2a_7-a_{10}a_3a_6a_7-a_6^3a_7-2a_1a_{10}^2b_{11}-a_{10}^2a_3b_6-a_{10}a_6^2b_6+a_{10}a_3a_7b_9+2a_6^2a_7b_9+a_{10}a_6b_6b_9-a_6a_7b_9^2)y-a_{10}(3a_{10}a_3a_7-3a_6^2a_7-3a_{10}a_3b_{11}-a_6^2b_{11}-4a_{10}a_6b_6+6a_6a_7b_9+a_6b_{11}b_9+3a_{10}b_6b_9-3a_7b_9^2)y^2-2a_{10}^2(a_7-b_{11})(2a_6-b_9)y^3-a_{10}^3(a_7-b_{11})y^4)/a_{10}^2\equiv 0\Rightarrow$

$$b_6=a_7(b_9-a_6)/a_{10},\ b_{11}=a_7; \tag{65}$$

$$a_1=b_9^3/(64a_{10}^2),\ a_3=3b_9^2/(16a_{10}),\ a_6=3b_9/4,\ b_{11}=a_7; \tag{66}$$

It is easy to see that:

– the sets of conditions {(10), (25), (43), (64)} and {(10), (25), (41), (60)}; {(9), (24), (39), (55)} and {(9), (23), (37), (53)} are the same;

– the set of conditions {(10), (26), (44), (66)} and {(10), (25), (43), (61)} are equivalent.

– the set of conditions {(9), (24), (39), (56)} is a particular case for the set of conditions {(9), (23), (37), (54)} and {(10), (25), (41), (59)} is a particular case for {(10), (25), (43), (61)}.

Lemma 4. *For quartic differential system {(4), (3)} the algebraic multiplicity of the invariant straight line $x = 0$ is greater than or equal to five if and only if at least one of the following sixteen sets of conditions holds:*

1) (6), (16), (27), (46), *2)* (6), (16), (28), (47),
3) (6), (16), (29), (48), *4)* (6), (17), (30), (49),
5) (7), (20), (33), (50), *6)* (8), (22), (35), (51),
7) (9), (23), (36), (52), *8)* (9), (23), (37), (53),
9) (9), (23), (37), (54), *10)* (10), (25), (41), (57),
11) (10), (25), (41), (58), *12)* (10), (25), (41), (60),
13) (10), (25), (43), (61), *14)* (10), (25), (43), (62),
15) (10), (25), (43), (63), *16)* (10), (26), (44), (65).

The algebraic multiplicity $m_a \geq 6$ if in each of the cases 1)–16) of Lemma 4 the identity $A_5(y) \equiv 0$ holds. Taking into account (3), we will examine each case separately.

1) (6), (16), (27), (46) $\Rightarrow A_5(y) = a_7 b_0 (b_4 + 2b_8 y + 3b_{13} y^2) \equiv 0 \Rightarrow$

$$b_4 = b_8 = b_{13} = 0, \ a_7 \neq 0, \ b_0 \neq 0; \qquad (67)$$

2) (6), (16), (28), (47) $\Rightarrow A_5(y) = -a_4 b_0 (3a_4 - b_7 - 2b_{12} y) \equiv 0 \Rightarrow$

$$b_{12} = 0, \ b_7 = 3a_4; \qquad (68)$$

3) (6), (16), (29), (48) $\Rightarrow A_5(y) = -b_2(a_4 + a_8 y)(3a_4^2 + a_8 b_3 - a_7 b_4 - a_4 b_7 + 6a_4 a_8 y - 2a_4 b_{12} y + 3a_8^2 y^2 - a_8 b_{12} y^2)/a_8 \equiv 0 \Rightarrow$

$$b_3 = (a_7 b_4 + a_4(b_7 - 3a_4))/a_8, \ b_{12} = 3a_8, \ b_2 \neq 0; \qquad (69)$$

4) (6), (17), (30), (49) $\Rightarrow A_5(y) = (2a_2 a_4^2 b_0^2 - 6a_2^2 a_7 b_0^2 - a_2^2 a_4 b_0 b_1 - a_2^3 b_1^2 + a_2^2 b_0^2 b_{11} + 4a_2^3 b_0 b_3 - a_4^2 b_0^2 b_4 + a_2 a_7 b_0^2 b_4 + a_2 a_4 b_0 b_1 b_4 - a_2^2 b_0 b_3 b_4 + 2a_2^2(6a_2 a_4 b_0 - 5a_2^2 b_1 - a_4 b_0 b_4 + a_2 b_1 b_4) y - 3a_2^4(4a_2 - b_4) y^2)/(a_2 b_0) \equiv 0 \Rightarrow$

$$b_1 = 2a_4 b_0/a_2, \ b_4 = 4a_2, \ b_{11} = 2a_7. \qquad (70)$$

In each of cases 5)–16) of Lemma 4, the identity $A_5(y) \equiv 0$ and the conditions (3) are not compatible.

Lemma 5. *For quartic differential system {(4), (3)} the algebraic multiplicity of the invariant straight line $x = 0$ is greater than or equal to six if and only if at least one of the following four sets of conditions holds:*

1) (6), (16), (27), (46), (67), *2)* (6), (16), (28), (47), (68),
3) (6), (16), (29), (48), (69), *4)* (6), (17), (30), (49), (70).

Taking into account (3) in each of the cases 1)–4) of Lemma 5 we have respectively the implications:

1) (6), (16), (27), (46), (67) $\Rightarrow A_6(y) = a_7 b_0(b_7 + 2b_{12} y) \equiv 0 \Rightarrow$

$$b_7 = b_{12} = 0; \tag{71}$$

2) (6), (16), (28), (47), (68) $\Rightarrow A_6(y) = a_4(-4a_7 b_0 + a_4 b_1 + b_0 b_{11}) \equiv 0 \Rightarrow$

$$b_1 = b_0(4a_7 - b_{11})/a_4; \tag{72}$$

3) (6), (16), (29), (48), (69) $\Rightarrow A_6(y) = (a_4 + a_8 y)(-6a_4 a_7 b_2 + a_4 b_{11} b_2 + a_4^2 b_4 - a_8 b_2 b_6 + a_7 b_2 b_7 + 2a_4 a_8 b_4 y + a_8^2 b_4 y^2)/a_8 \equiv 0 \Rightarrow$

$$b_4 = 0,\ b_6 = (a_4 b_{11} - 6a_4 a_7 + a_7 b_7)/a_8,\ b_2 \neq 0; \tag{73}$$

4) (6), (17), (30), (49), (70) $\Rightarrow A_6(y) = (2a_4^3 b_0^3 + 2a_2 a_4 a_7 b_0^3 - 2a_2^2 a_4 b_0^2 b_3 + a_2^3 b_0^2 b_6 + 6a_2^2 a_4^2 b_0^2 y + 6a_2^3 a_7 b_0^2 y - 4a_2^4 b_0 b_3 y + 6a_2^4 a_4 b_0 y^2 + 4a_2^6 y^3)/(a_2 b_0^2) \not\equiv 0.$

Lemma 6. *For quartic differential system* $\{(4), (3)\}$ *the algebraic multiplicity of the invariant straight line* $x = 0$ *is greater than or equal to seven if and only if at least one of the following three sets of conditions holds:*

1) (6), (16), (27), (46), (67), (71), *2)* (6), (16), (28), (47), (68), (72),
3) (6), (16), (29), (48), (69), (73).

In the conditions 1) (6), (16), (27), (46), (67), (71) we have $A_7 = a_7 b_0(b_{11} - 4a_7) \equiv 0 \Rightarrow$

$$b_{11} = 4a_7, \tag{74}$$

but in the cases 2) and 3) of Lemma 6 $A_7 \not\equiv 0$.

Lemma 7. *For quartic differential system* $\{(4), (3)\}$ *the algebraic multiplicity of the invariant straight line* $x = 0$ *is greater than or equal to eight if the following set of conditions:* (6), (16), (27), (46), (67), (71), (74) *holds.*

In the conditions of Lemma 7 we have $A_8 = a_7^2 b_1 \equiv 0 \Rightarrow$

$$b_1 = 0. \tag{75}$$

Lemma 8. *For quartic differential system* $\{(4), (3)\}$ *the algebraic multiplicity of the invariant straight line* $x = 0$ *is greater than or equal to nine if the following set of conditions:* (6), (16), (27), (46), (67), (71), (74), (75) *holds.*

The conditions of Lemma 8 give us $A_9 = 2a_7^2 b_3 \equiv 0 \Rightarrow$

$$b_3 = 0 \tag{76}$$

$\Rightarrow A_{10} = 3a_7^2(b_6 + 4a_7 y) \not\equiv 0$, $m_a = 10$ and the quartic differential system (4) takes the form

$$\dot{x} = a_7 x^4, \quad \dot{y} = b_0 + b_6 x^3 + b_{10} x^4 + 4a_7 x^3 y. \tag{77}$$

For system (77) we have $E_1(X) = a_7^2 x^{10}(3b_6 + 4b_{10}x + 12a_7 y)$.

Via the affine transformation of coordinates: $x \to x$, $y \to (-3b_6 - 4b_{10}x + 12b_0 y)/(12a_7)$ and the rescaling of time $t = \tau/a_7$ the system (77) can be written into the form:

$$\dot{x} = x^4, \quad \dot{y} = 1 + 4x^3 y. \tag{78}$$

In this way we have proved the following theorem.

Theorem. *In the class of quartic differential system* $\{(4), (3)\}$ *the maximal algebraic multiplicity of an affine real invariant straight line is equal to 10. Via an affine transformation of coordinates and time rescaling each quartic system which has an invariant straight line of algebraic multiplicity 10 can be written in the form* (78).

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