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The Center of the Lattice of Factorization Structures

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Abstract. The center of the lattice of factorization structures of the category of locally convex topological vector spaces is studied. The center consists of those structures for which the projections class is contained in the class of universal epimorphisms. The injective class is contained in the class of universal monomorphisms. It is proved that every element of the center defines two commuting functors: a coreflector and a reflector functor. The relationships of these pair of functors with left and right products of two subcategories and with the theories of relative torsion are examined. Some concrete examples are constructed.

Mathematics subject classification: 18A40, 46A03.
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1 Introduction

The conditions for a reflector functor and a coreflector functor to commute were examined by many authors. Various examples of this kind were constructed in [8, 9, 10], in the category of uniform spaces and in [2] - in the category $\mathcal{C}_2\mathcal{V}$ of locally convex vector Hausdorff spaces. We use the terminology from [7, 11, 12]. We denote by $\mathbb{B}$ the class of factorization structures of the category $\mathcal{C}_2\mathcal{V}$.

The epimorphism p is called up orthogonal on a morphism m, and m is called down orthogonal on p (see [1, 3]), if for any commutative square

$$m \cdot g = g \cdot p$$

it results that there a morphism t (diagonal of the square) so that

$$f = t \cdot p$$

and

$$g = m \cdot t$$

This is denoted by $p \perp m$. If $\mathcal{P}$ (respectively: $\mathcal{I}$) is a class of epimorphisms (respectively: of monomorphisms), then $\mathcal{P}^{\lfloor}$ (respectively: $\mathcal{I}^{\rceil}$) is the class of all down orthogonal monomorphisms (respectively: up orthogonal epimorphisms) for all element of $\mathcal{P}$ (respectively: of $\mathcal{I}$).

The following statement is well known.

Let $\mathcal{E}$ (respectively: $\mathcal{M}$) be a class of epimorphisms (respectively: of monomorphisms), in the category $\mathcal{C}_2\mathcal{V}$. Then $(\mathcal{E}^{\lfloor\rceil}, \mathcal{E}^{\lfloor})$ (respectively: $(\mathcal{M}^{\rceil}, \mathcal{M}^{\rceil\lfloor})$ is a factorization structure.

Definition 1 [3]. 1. *The monomorphism m is called a universal monomorphism if for every pushout*

$$f^{'} \cdot m = m^{'} \cdot f,$$

the morphism $m^{'}$ is a monomorphism.

2. *An epimorphism e is called exact if it is up orthogonal to every universal monomorphism.*

Definition 2 [3]. *Let $\mathcal{A}$ and $\mathcal{B}$ be two classes of morphisms of the category $\mathcal{C}$. The class of all morphisms of the category $\mathcal{C}$, of the form $a \cdot b$, with $a \in \mathcal{A}$ and $b \in \mathcal{B}$, for which this composition exists, is named the composition of the classes $\mathcal{A}$ and $\mathcal{B}$ and is denoted by $\mathcal{A} \circ \mathcal{B}$.*

Definition 3 [3]. *Let $\mathcal{A}$ and $\mathcal{B}$ be two classes of morphisms of the category $\mathcal{C}$. The class $\mathcal{A}$ is called $\mathcal{B}$-hereditary ($\mathcal{B}$-cohereditary) if from the fact that $f \cdot g \in \mathcal{A}$ and $f \in \mathcal{B}$ (respectively: $g \in \mathcal{B}$), it follows that $g \in \mathcal{A}$ (respectively: $f \in \mathcal{A}$).*

Theorem 1 [3]. 1. *A monomorphism $m : X \longrightarrow Y$ is an universal monomorphism in the category $\mathcal{C}_2\mathcal{V}$ if every continuous functional $f : X \longrightarrow K$ extends through m: $f = g \cdot m$ for a morphism g.*

2. *A monomorphism $f : (E, u) \longrightarrow (F, v)$ belongs to the class $\mathcal{E}_p$ iff:*

a) *f is a surjective mapping*;

b) *$v = min(u'', m(v))$, where u'' is the factor topology defined by the mapping $f : (E, u) \longrightarrow (F, v)$, and $m(v)$ is the Mackey topology defined on the space F and compatible with duality of topology v: $(F, v)' = (F, m(v))'$.*

In the category $\mathcal{C}_2\mathcal{V}$ we denote by $\mathcal{M}_u$ the class of universal monomorphisms and by $\mathcal{E}_p$ - the class of exact epimorphisms.

Dual notions and notations: a universal epimorphism, an exact monomorphism, $\mathcal{E}_u$ - class of universal epimorphisms, $\mathcal{M}_p$ - class of exact monomorphisms.

In the category $\mathcal{C}_2\mathcal{V}$ the following structures are well known (see [3]):

$(\mathcal{E}_u, \mathcal{M}_p)$=(the class of universal epimorphisms, the class of exact monomorphisms)=(the class of surjective morphisms, the class of topological embeddings);

$(\mathcal{E}_p, \mathcal{M}_u)$=(the class of exact epimorphisms, the class of universal monomorphisms);

$(\mathcal{E}_p^{'}, \mathcal{M}_u^{'})$=($\mathcal{E}_p$, the class of universal monomorphisms with closed image);

$(\mathcal{E}_f, \mathcal{M}ono)$=(the class of cokernels, the class of monomorphisms)=(the class of factorial morphisms, the class of injective morphisms);

$(\mathcal{E}pi, \mathcal{M}_f)$ - (the class of epimorphisms, the class of kernels) = (the class of morphisms with dense image, the class of topological inclusions with closed images).

The properties of factorization structures $(\mathcal{E}_f, \mathcal{M}ono)$ and $(\mathcal{E}pi, \mathcal{M}_f)$ characterize the category $\mathcal{C}_2\mathcal{V}$ as a semiabelian category. The factorization structures $(\mathcal{E}_u, \mathcal{M}_p)$ and $(\mathcal{E}_p, \mathcal{M}_u)$ play an important role in the study of the reflective and coreflective subcategories. We need some notions and results from [3], [4] and [6].

We use the following notations for some subcategories of the category $\mathcal{C}_2\mathcal{V}$.

Π - the subcategory of complete spaces with a weak topology and with respective functor $\pi : \mathcal{C}_2\mathcal{V} \to \Pi$;

$\mathcal{S}$ - the subcategory of spaces endowed with a weak topology, $s : \mathcal{C}_2\mathcal{V} \to \mathcal{S}$;

Γ_0 - the subcategory of complete spaces, $g_o : \mathcal{C}_2\mathcal{V} \to \Gamma_o$;

$\sum$ - the coreflective subcategory of spaces with the strongest locally convex topology, $\sigma : \mathcal{C}_2\mathcal{V} \to \sum$;

$\widetilde{\mathcal{M}}$ - the subcategory of spaces endowed with the Mackey topology, $m : \mathcal{C}_2\mathcal{V} \to \widetilde{\mathcal{M}}$.

Any factorization structure $(\mathcal{P}, \mathcal{I})$ in the category $\mathcal{C}_2\mathcal{V}$ divides the lattice $\mathbb{R}$ (respectively: the lattice $\mathbb{K}$) of reflective (respectively: coreflective) subcategories into three classes

$\mathbb{R}(\mathcal{P}) = \{\mathcal{R} \in \mathbb{R} \mid \mathcal{R}$ is a $\mathcal{P}$-reflective subcategory$\}$,

$\mathbb{R}(\mathcal{I}) = \{\mathcal{R} \in \mathbb{R} \mid \mathcal{R}$ is an $\mathcal{I}$-reflective subcategory$\}$,

$\mathbb{R}(\mathcal{P}, \mathcal{I}) = (\mathbb{R} \backslash \{\mathbb{R}(\mathcal{P}) \cup \mathbb{R}(\mathcal{I})\}) \cup \{\mathcal{C}_2\mathcal{V}\}$.

Respectively - the division class $\mathbb{K}$:

$\mathbb{K}(\mathcal{P}) = \{\mathcal{L} \in \mathbb{K} \mid \mathcal{L}$ is a $\mathcal{P}$-coreflective subcategory$\}$,

$\mathbb{K}(\mathcal{I}) = \{\mathcal{V} \in \mathbb{K} \mid \mathcal{V}$ is a $\mathcal{I}$-coreflective subcategory$\}$,

$\mathbb{K}(\mathcal{P}, \mathcal{I}) = (\mathbb{K} \backslash \{\mathbb{K}(\mathcal{P}) \cup \mathbb{K}(\mathcal{I})\}) \cup \{\mathcal{C}_2\mathcal{V}\}$.

Let $\mathcal{R}$ be a reflective non-zero subcategory of category $\mathcal{C}_2\mathcal{V}$. We examine $\mathcal{R}$ and the Π-replica of an arbitrary object $X \in |\mathcal{C}_2\mathcal{V}| \quad r^X : X \longrightarrow rX$ and $\pi^X : X \longrightarrow \pi X$. Because $\Pi \subset \mathcal{R}$ we have $\pi^X = v^X \cdot r^X$ for a morphism v^X. Let

$$\mathcal{U} = \mathcal{U}(\mathcal{R}) = \{r^X \mid X \in |\mathcal{C}_2\mathcal{V}|\}, \quad \mathcal{V} = \mathcal{V}(\mathcal{R}) = \{v^X \mid X \in |\mathcal{C}_2\mathcal{V}|\}.$$

We build factorization structures:

$$(\mathcal{P}'', \mathcal{I}'') = (\mathcal{P}''(\mathcal{R}), \mathcal{I}''(\mathcal{R})) = ((\mathcal{V}(\mathcal{R}))^{\urcorner}, (\mathcal{V}(\mathcal{R}))^{\urcorner\llcorner}),$$

$$(\mathcal{P}', \mathcal{I}') = (\mathcal{P}'(\mathcal{R}), \mathcal{I}'(\mathcal{R})) = ((\mathcal{U}(\mathcal{R}))^{\llcorner\urcorner}, (\mathcal{U}(\mathcal{R}))^{\llcorner}).$$

We establish notations for the dual case. Let $k : \mathcal{C}_2\mathcal{V} \to \mathcal{K}$ be a non-zero coreflector functor. We examine $\mathcal{K}$ and the $\sum$-coreplica of an arbitrary object $X \in |\mathcal{C}_2\mathcal{V}| : k^X : kX \to X$ and $\sigma^X : \sigma X \to X$. Because $\sum \subset \mathcal{K}$, we have

$$\sigma^X = k^X \cdot v_c^X$$

for a morphism $v_c^X : \sigma X \to kX$. Let

$\mathcal{U}_c = \mathcal{U}_c(\mathcal{K}) = \{k^X | X \in |\mathcal{C}_2\mathcal{V}|\}$, $\mathcal{V}_c = \mathcal{V}_c(\mathcal{K}) = \{v_c^X | X \in |\mathcal{C}_2\mathcal{V}|\}$.

We build factorization structures:

$$(\mathcal{E}', \mathcal{M}') = (\mathcal{E}'(\mathcal{K}), \mathcal{M}'(\mathcal{K})) = ((\mathcal{V}_c(\mathcal{K}))^{\llcorner\urcorner}, (\mathcal{V}_c(\mathcal{K}))^{\llcorner}),$$

$$(\mathcal{E}'', \mathcal{M}'') = (\mathcal{E}''(\mathcal{K}), \mathcal{M}''(\mathcal{K})) = ((\mathcal{U}_c(\mathcal{K}))^{\urcorner}, (\mathcal{V}_c(\mathcal{K}))^{\urcorner\llcorner}).$$

For $\mathcal{R} \in \mathbb{R}$ and $\mathcal{K} \in \mathbb{K}$ set $\mathbb{L}_\rho(\mathcal{R}) = \{(\mathcal{P}, \mathcal{I}) \in \mathbb{B} \mid \mathcal{P}'(\mathcal{R}) \subset \mathcal{P} \subset \mathcal{P}''(\mathcal{R})\}$, $\mathbb{L}_\kappa(\mathcal{K}) = \{(\mathcal{P}, \mathcal{I}) \in \mathbb{B} \mid \mathcal{E}'(\mathcal{K}) \subset \mathcal{P} \subset \mathcal{E}''(\mathcal{K})\}$.

The class $\mathbb{B}$ is divided into disjoint subclasses of the form $\mathbb{L}_\kappa(\mathcal{R})$, where $\mathcal{K} \in \mathbb{K}$.

For a reflective non-zero subcategory $\mathcal{R}$ of category $\mathcal{C}_2\mathcal{V}$ with reflector functor $r : \mathcal{C}_2\mathcal{V} \to \mathcal{R}$ denote by: $\varepsilon\mathcal{R} = \{e \in \mathcal{E}pi \mid r(e) \in \mathcal{I}so$.

Dually, for a coreflective non-zero subcategory $\mathcal{K}$ of category $\mathcal{C}_2\mathcal{V}$ with coreflector functor $k : \mathcal{C}_2\mathcal{V} \to \mathcal{K}$ set $\mu\mathcal{K} = \{m \in \mathcal{M}ono \mid r(m) \in \mathcal{I}so\}$.

Theorem 2 [3]. 1. *For any reflective non-zero subcategory $\mathcal{R}$ of category $\mathcal{C}_2\mathcal{V}$, the factorization structure $(\mathcal{P}''(\mathcal{R}), \mathcal{I}''(\mathcal{R}))$ has $\mathcal{M}_u$-hereditary class $\mathcal{P}''(\mathcal{R})$, of projections.*

2. *The application $\mathcal{R} \longmapsto (\mathcal{P}''(\mathcal{R}), \mathcal{I}''(\mathcal{R}))$ establishes a bijective correspondence between the lattice $\mathbb{R}$ of reflective non-zero subcategories of category $\mathcal{C}_2\mathcal{V}$ and the lattice $\mathbb{B}_{\varepsilon p}$ of factorization structure $(\mathcal{P}, \mathcal{I})$ with properties:*

a) $\mathcal{I} \subset \mathcal{M}_u$;

b) *the class $\mathcal{P}$ is $\mathcal{M}_u$-hereditary.*

3. *Consider $f : X \longrightarrow Y \in \mathcal{E}pi$, and $f = m \cdot e$ the $(\mathcal{E}_p, \mathcal{M}_u)$-factorization of this morphism. The epimorphism $f \in \mathcal{P}''(\mathcal{R})$ iff the reflector functor $r : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{R}$ takes m to an isomorphism : $r(m) \in \mathcal{I}so$. In other words $\mathcal{P}''(\mathcal{R}) = (\varepsilon\mathcal{R}) \circ \mathcal{E}_p$ where $\varepsilon\mathcal{R} = \{e \in \mathcal{E}pi \mid r(e) \in \mathcal{I}so\}$.*

4. *Consider $f : X \longrightarrow Y \in \mathcal{M}_u$. $f \in \mathcal{I}''(\mathcal{R})$ iff the square $r(f) \cdot r^X = r^Y \cdot f$ is a pullback.*

Let $\mathcal{K}$ be a coreflective subcategory, and $\mathcal{R}$ - a reflective subcategory of the category $\mathcal{C}_2\mathcal{V}$ with corresponding functors $k : \mathcal{C}_2\mathcal{V} \to \mathcal{K}$ and $r : \mathcal{C}_2\mathcal{V} \to \mathcal{R}$. For an arbitrary object X of the category $\mathcal{C}_2\mathcal{V}$ let $k^X : kX \to X$ be the $\mathcal{K}$-coreplica of the object X, and $r^X : X \to rX$ and $r^{kX} : kX \to rkX$ $\mathcal{R}$-replicas of respective objects. Further, let $r(k^X) : rkX \to rX$ be the unique morphism for which $r(k^X) \cdot r^{kX} = r^X \cdot k^X$.

For the morphisms r^X and r^{kX} we build the pullback

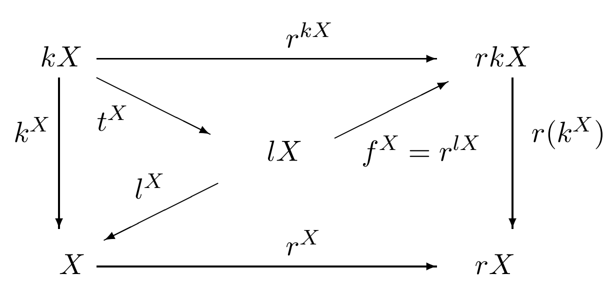
$$r^X \cdot l^X = r(k^X) \cdot f^X.$$

Then there exists a morphism t^X so that:

$$l^X \cdot t^X = k^X,$$

$$f^X \cdot t^X = r^{kX},$$

and



k and r commutes.

Lemma 1 [4]. *For any object X of the category $\mathcal{C}_2\mathcal{V}$ the morphism f^X is the $\mathcal{R}$-replica of object lX.*

We denote by $\mathcal{L} = \mathcal{K} *_s \mathcal{R}$ the full subcategory of the category $\mathcal{C}_2\mathcal{V}$ consisting of all objects isomorphic with the objects of the form lX when $X \in |\mathcal{C}_2\mathcal{V}|$.

Definition 4 [4]. 1. *The subcategory $\mathcal{L} = \mathcal{K} *_s \mathcal{R}$ is called the $\mathcal{S}$-product or left product of the subcategories $\mathcal{K}$ and $\mathcal{R}$.*

2. *The diagram is called the left product diagram for X.*

Dual notions: the right product $\mathcal{V} = \mathcal{K} *_d \mathcal{R}$ of the subcategories $\mathcal{K}$ and $\mathcal{R}$, the right product diagram are presented to fix notations.

For arbitrary object X of the category $\mathcal{C}_2\mathcal{V}$ let $r^X : X \to rX$ be the $\mathcal{R}$-replica, and $k^X : kX \to X$ and $k^{rX} : krX \to rX$ the $\mathcal{K}$-coreplicas of the respective objects. Then

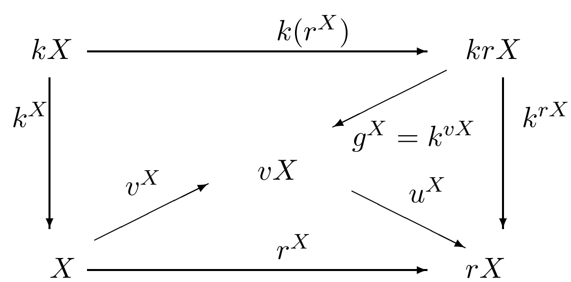
$$r^X \cdot k^X = k^{rX} \cdot k(r^X)$$

for a morphism $k(r^X) : kX \to krX$. On the morphisms k^X and $k(r^X)$ we build the pushout $v^X \cdot k^X = g^X = k(r^X)$. Then there exists a morphism $u^X : v^X \to rX$ so that

$$r^X = u^X \cdot v^X,$$

$$k^{rX} = u^X \cdot g^X,$$

and



k and r commutes.

Theorem 3 [4]. *For the pair of subcategories $\mathcal{K}$ and $\mathcal{R}$, the following affirmations are equivalent:*

1. *$r(k^X)$ is a monomorphism for any object $X \in |\mathcal{C}_2\mathcal{V}|$.*
2. *$\mathcal{L}$ is a coreflective subcategory of the category $\mathcal{C}_2\mathcal{V}$.*
3. *l^X is a monomorphism for any object $X \in |\mathcal{C}_2\mathcal{V}|$.*
4. *l^X is the $\mathcal{L}$-coreplica of X for any object $X \in |\mathcal{C}_2\mathcal{V}|$.*
5. *$l^X \in \mathcal{E}_u \cap \mathcal{M}ono$ for any object $X \in |\mathcal{C}_2\mathcal{V}|$.*
6. *$t^X \in \mathcal{E}_u$ for any object $X \in |\mathcal{C}_2\mathcal{V}|$.*
7. *t^X is the $\mathcal{K}$-coreplica of lX for any object $X \in |\mathcal{C}_2\mathcal{V}|$.*
8. *$t^X \in \mathcal{E}_u \cap \mathcal{M}ono$ for any object $X \in |\mathcal{C}_2\mathcal{V}|$.*
9. *$l^X, t^X \in \mathcal{E}_u \cap \mathcal{M}ono$ for any object $X \in |\mathcal{C}_2\mathcal{V}|$.*
10. *$X \in |\mathcal{L}| \Leftrightarrow r^X \cdot k^X$ is the $\mathcal{R}$-replica of kX.*
11. *The equality $k^X = l^X \cdot t^X$ is the $(\varepsilon\mathcal{R}, (\varepsilon\mathcal{R})^\perp)$-factorization of the morphism k^X by the right factorization structure $(\varepsilon\mathcal{R}, (\varepsilon\mathcal{R})^\perp)$ for any object $X \in |\mathcal{C}_2\mathcal{V}|$.*

The previous theorem indicates a string of necessary and sufficient conditions for the left product of two subcategories to be a coreflective subcategory. This affirmation is not always true.

Proposition 1 [6]. *In the category $\mathcal{C}_2\mathcal{V}$ the following assertions are true:*

1. $\sum *_s \prod$ *is not a coreflective subcategory.*
2. $\sum *_d \prod$ *is not a reflective subcategory.*

At the same time the following theorem indicates the cases when the left product is a coreflective subcategory, and the right product is a reflective subcategory.

Theorem 4 (see [6], Theorem 6). 1. *Let $\mathcal{K}$ be a coreflective subcategory of the category $\mathcal{C}_2\mathcal{V}$ and $\widetilde{\mathcal{M}} \subset \mathcal{K}$. Then, for any reflective subcategory $\mathcal{R}$ of category $\mathcal{C}_2\mathcal{V}$:*

a) *the left product $\mathcal{K} *_s \mathcal{R}$ is a coreflective subcategory of category $\mathcal{C}_2\mathcal{V}$;*

b) *the right product $\mathcal{K} *_d \mathcal{R}$ is a reflective subcategory of category $\mathcal{C}_2\mathcal{V}$.*

2. *Let $\mathcal{R}$ a reflective subcategory of category $\mathcal{C}_2\mathcal{V}$ and $\mathcal{S} \subset \mathcal{R}$. Then, for any coreflective subcategory $\mathcal{K}$ of category $\mathcal{C}_2\mathcal{V}$:*

a) *the left product $\mathcal{K} *_s \mathcal{R}$ is a coreflective subcategory of category $\mathcal{C}_2\mathcal{V}$;*

b) *the right product $\mathcal{K} *_d \mathcal{R}$ is a reflective subcategory of category $\mathcal{C}_2\mathcal{V}$.*

2 The center of the lattice of factorization structures

Let $(\mathcal{P},\mathcal{I})$ be a factorization structure in the category $\mathcal{C}_2\mathcal{V}$, and $\mathcal{A}$ be an arbitrary subcategory. We denote by:

$\mathbf{Q}_{\mathcal{P}}(\mathcal{A})$ - the full subcategory of all $\mathcal{P}$-factorobjects of the objects belonging to the subcategory $\mathcal{A}$,

$\mathbf{S}_{\mathcal{I}}(\mathcal{A})$ - the full subcategory of all $\mathcal{I}$-subobjects of the objects belonging to the subcategory $\mathcal{A}$.

Proposition 2. *Let $(\mathcal{P},\mathcal{I})$ be a factorization structure in the category $\mathcal{C}_2\mathcal{V}$, $\mathcal{K}$ a coreflective subcategory, and $\mathcal{R}$ a reflective subcategory in the category $\mathcal{C}_2\mathcal{V}$.*

1. *$\mathcal{L} = \mathbf{Q}_{\mathcal{P}}(\mathcal{K})$ is an $\mathcal{I}$-coreflective subcategory of the category $\mathcal{C}_2\mathcal{V}$,*
2. *$\mathcal{V} = \mathbf{S}_{\mathcal{I}}(\mathcal{R})$ is an $\mathcal{P}$- reflective subcategory of the category $\mathcal{C}_2\mathcal{V}$.*

Proof. 1. $k^X : kX \to X$ is $\mathcal{K}$-coreplica of object X, and

$$k^X = l^X \cdot t^X \tag{1}$$

is $(\mathcal{P},\mathcal{I})$-factorization of coreplica. We proved that l^X is $\mathcal{L}$-coreplica of object X. Let $A \in\mid \mathcal{K} \mid$, and $p : A \to B \in \mathcal{P}$, i.e. $B \in\mid \mathcal{L} \mid$. Further, let $f : B \to X \in \mathcal{C}_2\mathcal{V}$. Then

$$f \cdot p = k^X \cdot g \tag{2}$$

for a morphism g. Equality (2) can be written as

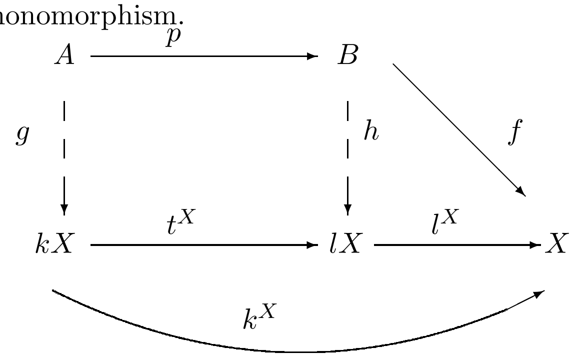
$$f \cdot p = l^X \cdot (t^X \cdot g) \tag{3}$$

where $p \in \mathcal{P}$, and $l^X \in \mathcal{I}$. Since $p \perp l^X$ it follows that there exists a morphism $h : B \to lX$ so that

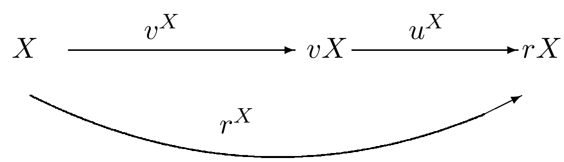
$$f = l^X \cdot h, \tag{4}$$

$$t^X \cdot g = h \cdot p. \tag{5}$$

The uniqueness of the morphism h which verifies the equality (4) follows from the fact that l^X is an monomorphism.



2. It is proved in dual mode. We indicate $(\mathcal{P},\mathcal{I})$-factorization of $\mathcal{R}$-replica of an arbitrary object X.



Let $(\mathcal{P},\mathcal{I})$ be a factorization structure in the category $\mathcal{C}_2\mathcal{V}$. Then there exists a unique coreflective subcategory $\mathcal{K}$ and a unique reflective subcategory $\mathcal{R}$ such that

$$(\mathcal{P},\mathcal{I}) \in \mathbb{L}_\kappa(\mathcal{K}) \text{ and } (\mathcal{P},\mathcal{I}) \in \mathbb{L}_\rho(\mathcal{R}) \tag{6}$$

Since ε $(\mathcal{R} \mapsto \varepsilon\mathcal{R})$ and μ $(\mathcal{K} \mapsto \mu\mathcal{K})$ are contravariant operations, we have that

Lemma 2. *Let $\mathcal{K}$ be a coreflective subcategory in category $\mathcal{C}_2\mathcal{V}$. $\widetilde{\mathcal{M}} \subset \mathcal{K}$ iff $\mathcal{E}_p \subset \mathcal{E}'(\mathcal{K})$. In particular, $(\mathcal{E}_p,\mathcal{M}_u) = (\mathcal{E}'(\widetilde{\mathcal{M}}),\mathcal{M}'(\widetilde{\mathcal{M}}))$.*

2. *Let $\mathcal{R}$ be a reflective subcategory in the category $\mathcal{C}_2\mathcal{V}$. $\mathcal{S} \subset \mathcal{R}$ iff $\mathcal{P}''(\mathcal{R}) \subset \mathcal{E}_u$. In particular, $(\mathcal{E}_u,\mathcal{M}_p) = (\mathcal{P}''(\mathcal{S}),\mathcal{I}''(\mathcal{S}))$.*

Proof. 1. Because $\widetilde{\mathcal{M}} \subset \mathcal{K}$, we deduce that $\mu\mathcal{K} \subset \mu\widetilde{\mathcal{M}} = \mathcal{E}_u \cap \mathcal{M}_u$. Hence $\mathcal{M}'(\mathcal{K}) = \mathcal{M}_p \cdot (\mu\mathcal{K}) \subset \mathcal{M}_p \cdot (\mathcal{E}_u \cap \mathcal{M}_u) = \mathcal{M}_u$. It follows that $\mathcal{M}'(\mathcal{K}) \subset \mathcal{M}'(\widetilde{\mathcal{M}})$, and $\mathcal{E}'(\widetilde{\mathcal{M}}) \subset \mathcal{E}'(\mathcal{K})$. That is $\mathcal{E}_p \subset \mathcal{E}'(\mathcal{K})$.

The inverse statement. Consider $\mathcal{E}_p \subset \mathcal{E}'(\mathcal{K})$. Then $\mathcal{M}'(\mathcal{K}) \subset \mathcal{M}_u$. For any object X of the category $\mathcal{C}_2\mathcal{V}$ we have the following equation

$$m^X \cdot t^X = k^X \cdot v_c^X, \tag{7}$$

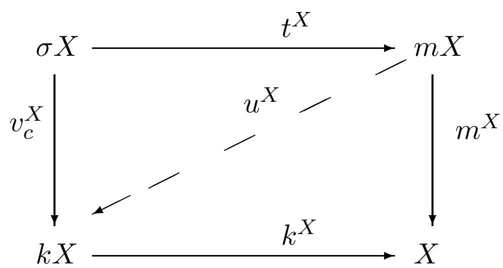
where $k^X : kX \to X$ is the $\mathcal{K}$-coreplica, $m^X : mX \to X$ is the $\widetilde{\mathcal{M}}$-coreplica of the object X, and $t^X : \sigma X \to mX$ and $v_c^X : \sigma X \to X$ are the $\sum$-coreplicas of the respective objects. In equality (7) $t^X \in \mathcal{E}_p$, and $v_c^X \in \mathcal{M}'(\mathcal{K}) \subset \mathcal{M}_u$.

Hence $t^X \perp k^X$. There is a morphism $u^X : mX \to kX$ such that

$$v_c^X = u^X \cdot t^X, \tag{8}$$

$$m^X = k^X \cdot u^X. \tag{9}$$

Equality (9) proves that $\widetilde{\mathcal{M}} \subset \mathcal{K}$.

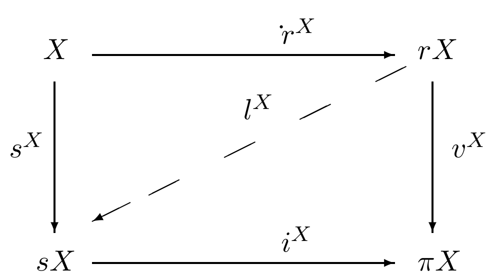


2. Consider $\mathcal{S} \subset \mathcal{R}$. Then $\varepsilon\mathcal{R} \subset \varepsilon\mathcal{S} = \mathcal{E}_u \cap \mathcal{M}_u$, and $\mathcal{P}''(\mathcal{R}) = (\varepsilon\mathcal{R}) \cdot \mathcal{E}_p \subset (\mathcal{E}_u \cap \mathcal{M}_u) \cdot \mathcal{E}_p = \mathcal{E}_u$. So $\mathcal{P}''(\mathcal{R}) \subset \mathcal{E}_u$.

The inverse statement. Let $\mathcal{P}''(\mathcal{R}) \subset \mathcal{E}_u$. Then $\mathcal{M}_p \subset \mathcal{I}''(\mathcal{R})$. For any object X of the category $\mathcal{C}_2\mathcal{V}$ we have the following equation

$$v^X \cdot r^X = i^X \cdot s^X, \tag{10}$$

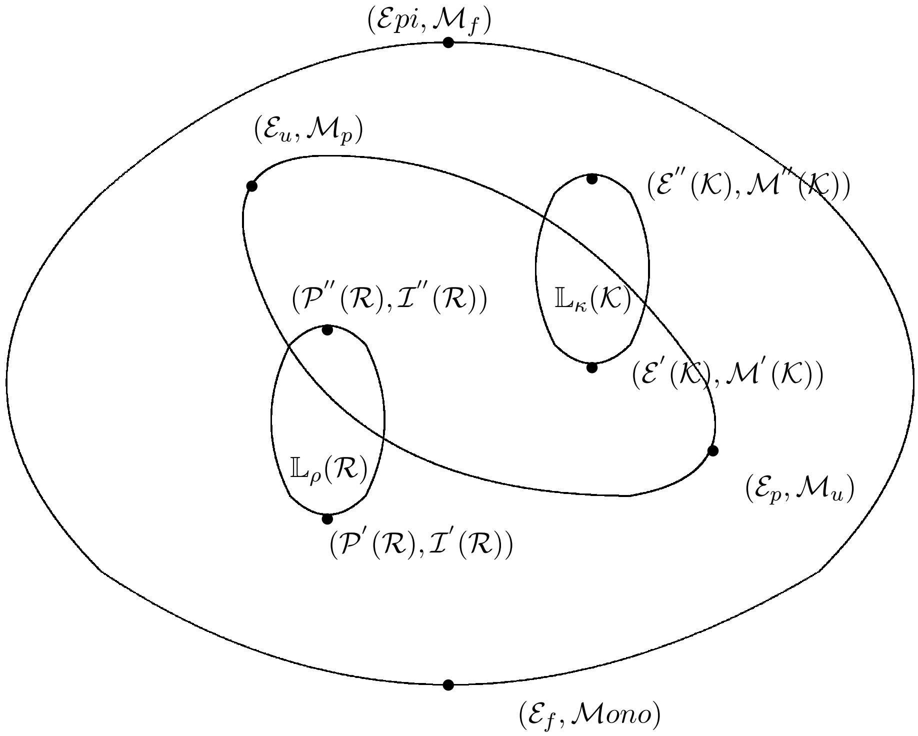
where $s^X : X \to sX$ is the $\mathcal{S}$-replica, $r^X : X \to rX$ is the $\mathcal{R}$-replica of object X, and $i^X : sX \to \pi X$ and $v^X : rX \to \pi X$ are the $\prod$-replicas of the respective objects. In equality (10) $r^X \in \mathcal{P}''(\mathcal{R})$, and $i^X \in \mathcal{M}_p \subset \mathcal{I}''(\mathcal{R})$. Hence $r^X \perp i^X$. There is a morphism l^X with properties



$$s^X = l^X \cdot r^X, \tag{11}$$

$$v^X = i^X \cdot l^X. \tag{12}$$

Equality (11) proves that $\mathcal{S} \subset \mathcal{R}$.



A schematic interpretation of the lattices $\mathbb{L}_\rho(\mathcal{R})$ and $\mathbb{L}_\kappa(\mathcal{K})$ in the case when $\mathcal{S} \subset \mathcal{R}$ and $\widetilde{\mathcal{M}} \subset \mathcal{K}$ is shown above.

We denote by $\mathbb{C}(\mathbb{B})$ the class of all factorization structures $(\mathcal{P}, \mathcal{I})$ with the property

$$\mathcal{E}_p \subset \mathcal{P} \subset \mathcal{E}_u.$$

$\mathbb{C}(\mathbb{B})$ is called the lattice center of the lattice $\mathbb{B}$ of factorization structures of the category $\mathcal{C}_2\mathcal{V}$.

Lemma 3. *Let $(\mathcal{P}, \mathcal{I})$ be a factorization structure in the category $\mathcal{C}_2\mathcal{V}$, $\mathcal{K}$ and $\mathcal{R}$ subcategories for which*

$$(\mathcal{P}, \mathcal{I}) \in \mathbb{L}_\kappa(\mathcal{K}) \text{ and } (\mathcal{P}, \mathcal{I}) \in \mathbb{L}_\rho(\mathcal{R}).$$

The following affirmations are true:

1. $\widetilde{\mathcal{M}} \subset \mathcal{K}$ *iff* $\mathcal{E}_p \subset \mathcal{P}$.
2. $\mathcal{S} \subset \mathcal{R}$ *iff* $\mathcal{P} \subset \mathcal{E}_u$.
3. $\widetilde{\mathcal{M}} \subset \mathcal{K}$ and $\mathcal{S} \subset \mathcal{R}$ *iff* $(\mathcal{P}, \mathcal{I}) \in \mathbb{C}(\mathbb{B})$.

Proof. 1. In the previous lemma $\mathcal{E}_p \subset \mathcal{E}'(\mathcal{K})$, and $\mathcal{E}'(\mathcal{K}) \subset \mathcal{P}$.

2. Dually.

3. Follows from 1. and 2.

Corollary 1. *For any coreflective subcategory $\mathcal{K}$ of the category $\mathcal{C}_2\mathcal{V}$ with the property $\widetilde{\mathcal{M}} \subset \mathcal{K}$, it follows that $(\mathcal{E}'(\mathcal{K}), \mathcal{M}'(\mathcal{K})) \in \mathbb{C}(\mathbb{B})$.*

2. *For any reflective subcategory $\mathcal{R}$ with the property $\mathcal{S} \subset \mathcal{R}$, it follows that $(\mathcal{P}''(\mathcal{R}), \mathcal{I}''(\mathcal{R})) \in \mathbb{C}(\mathbb{B})$.*

3. $\mathbb{C}(\mathbb{B})$ *is a complete lattice with first element and last element:* $(\mathcal{E}_p, \mathcal{M}_u)$ *and* $(\mathcal{E}_u, \mathcal{M}_p)$.

4. $\mathbb{C}(\mathbb{B})$ *contains two proper classes of elements:*

$$\{(\mathcal{E}'(\mathcal{K}), \mathcal{M}'(\mathcal{K})) \mid \widetilde{\mathcal{M}} \subset \mathcal{K} \in \mathbb{K}\},$$

$$\{(\mathcal{P}''(\mathcal{R}), \mathcal{I}''(\mathcal{R})) \mid \mathcal{S} \subset \mathcal{R} \in \mathbb{R}\}.$$

3 Commutative functors

We shall examine some cases when the coreflector and reflector functor commute in the category $\mathcal{C}_2\mathcal{V}$.

Theorem 5. *Consider* $(\mathcal{P}, \mathcal{I}) \in \mathbb{C}(\mathbb{B})$ *and let* $\mathcal{K}$ *and* $\mathcal{R}$ *be subcategories of the category* $\mathcal{C}_2\mathcal{V}$ *for which* $(\mathcal{P}, \mathcal{I}) \in \mathbb{L}_\kappa(\mathcal{K}) \cap \mathbb{L}_\rho(\mathcal{R})$. *Then the coreflector functor* $k : \mathcal{C}_2\mathcal{V} \to \mathcal{K}$ *and the reflector functor* $r : \mathcal{C}_2\mathcal{V} \to \mathcal{R}$ *commute*: $k \cdot r = r \cdot k$.

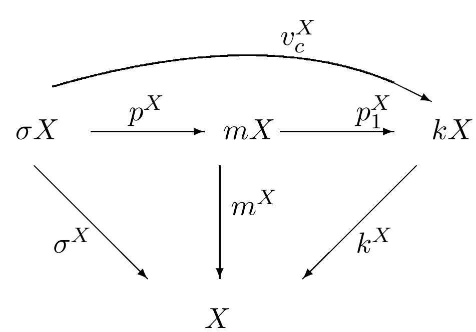
Proof. We construct the $\mathcal{K}$-coreplica an arbitrary object X of the category $\mathcal{C}_2\mathcal{V}$. Let $\sigma^X : \sigma X \to X$, $m^X : mX \to X$ and $k^X : kX \to X$ respectively the Σ-coreplica, $\widetilde{\mathcal{M}}$-coreplica and $\mathcal{K}$-coreplica of X. Because $\Sigma \subset \widetilde{\mathcal{M}} \subset \mathcal{K}$ it follows that:

$$\sigma^X = k^X \cdot v_c^X, \tag{13}$$

$$\sigma^X = m^X \cdot p^X, \tag{14}$$

$$m^X = k^X \cdot p_1^X \tag{15}$$

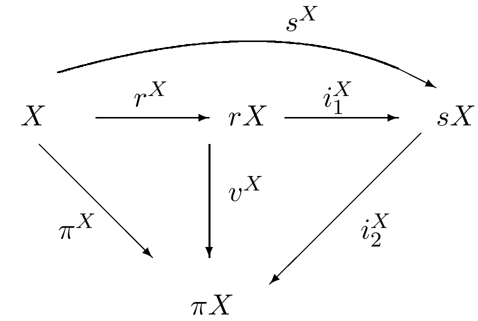
for three morphisms v_c^X, p^X, p_1^X. We have $p^X \subset \mathcal{P}$. So both equalities (13) and (15) are $(\mathcal{P}, \mathcal{I})$-factorizations of respective morphisms.



Dual. Let $r^X : X \to rX$ and $s^X : X \to sX$ be respectively the $\mathcal{R}$-replica and the $\mathcal{S}$-replica of the object X. Then

$$s^X = i_1^X \cdot r^X \tag{16}$$

for any morphism i_1^X and this equality is $(\mathcal{P}, \mathcal{I})$-factorization of $\mathcal{S}$-replica s^X of object X.



So we obtained the following rule:

1. The $\mathcal{K}$-coreplica of an arbitrary object X of the category $\mathcal{C}_2\mathcal{V}$ is obtained by the $(\mathcal{P},\mathcal{I})$-factorization of the $\widetilde{\mathcal{M}}$-coreplica of this object

$$m^X = k^X \cdot p_1^X.$$

2. The $\mathcal{R}$-replica of the object X is obtained by the $(\mathcal{P},\mathcal{I})$-factorization of the $\mathcal{S}$-replica of this object

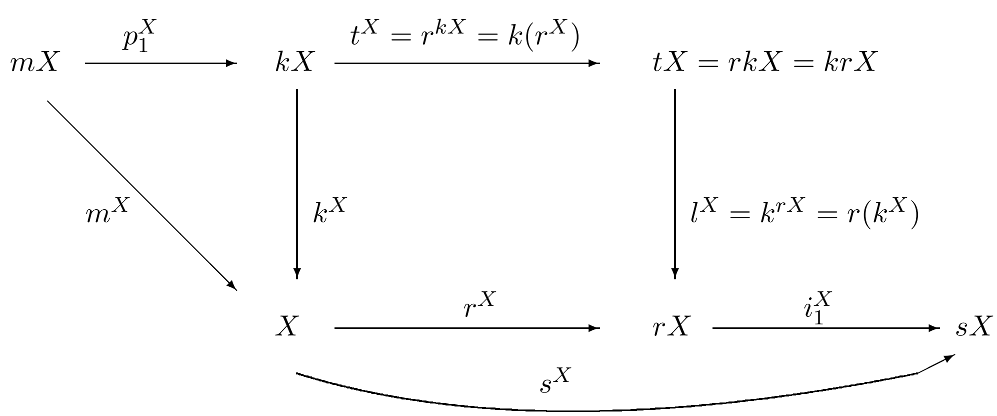
$$s^X = i_1^X \cdot r^X.$$

We apply this rule to obtain the $\mathcal{R}$-replica of the object kX and the $\mathcal{K}$-coreplica of the object rX.

Because $k^X \in \mathcal{E}_u \bigcap \mathcal{M}_u$, it follows that the morphism $s^X \cdot k^X$ is the $\mathcal{S}$-replica of the object kX. So to obtain the $\mathcal{R}$-replica of the object kX it is necessary to execute the $(\mathcal{P},\mathcal{I})$-factorization of the morphism $s^X \cdot k^X$. Considering that $i_1^X \in \mathcal{I}$, remains to be performed the $(\mathcal{P},\mathcal{I})$-factorization of morphism $r^X \cdot k^X$. Let

$$r^X \cdot k^X = l^X \cdot t^X \tag{17}$$

be this factorization.



Hence $t^X : kX \to tX$ is the $\mathcal{R}$-replica of the object kX. Since $r^X \cdot m^X \in \mathcal{E}_u \bigcap \mathcal{M}_u$, it follows that this morphism is the $\widetilde{\mathcal{M}}$-coreplica of the object rX. To obtain the $\mathcal{K}$-coreplica of the object rX should to perform the $(\mathcal{P},\mathcal{I})$-factorization of this morphism, or the morphism $r^X \cdot k^X$, because $p_1^X \in \mathcal{P}$, and

$$r^X \cdot m^X = r^X \cdot k^X \cdot p_1^X. \tag{18}$$

But equality (11) is the $(\mathcal{P},\mathcal{I})$-factorization of the morphism $r^X \cdot k^X$. So we proved that the functors k and r commute: $k \cdot r = r \cdot k$.

Assume that the conditions of the previous theorem are met and return to the last diagram. Equalities

$$m^X = k^X \cdot p_1^X, \tag{19}$$

$$s^X = i_1^X \cdot r^X, \tag{20}$$

$$r^X \cdot k^X = k^{rX} \cdot r^{kX} \tag{21}$$

are $(\mathcal{P},\mathcal{I})$-factorizations of respective morphisms (the left part of equality). Hence

$$r^X,\ p_1^X,\ r^{kX} \in \mathcal{P},\quad i_1^X,\ k^{rX},\ k^X \in \mathcal{I}. \tag{22}$$

The equalities (19) and (21) are $(\mathcal{E},\mathcal{M})$-factorizations of respective morphisms for any element $(\mathcal{E},\mathcal{M}) \in \mathbb{L}_\kappa(\mathcal{K})$, in particular, for the element $(\mathcal{E}'(\mathcal{K}),\mathcal{M}'(\mathcal{K}))$. So

$$r^{kX},\ p_1^X \in \mathcal{E}'(\mathcal{K});\quad k^{rX},\ k^X \in \mathcal{M}'(\mathcal{K}). \tag{23}$$

Dual

$$r^X,\ r^{kX} \in \mathcal{P}''(\mathcal{R});\quad i_1^X, k^{rX} \in \mathcal{I}''(\mathcal{R}). \tag{24}$$

We have the following equalities:

$$k^{rX} = r(k^X), \tag{25}$$

$$r^{kX} = k(r^X). \tag{26}.$$

Theorem 6. *Consider* $(\mathcal{P},\mathcal{I}) \in \mathbb{C}(\mathbb{B})$, $(\mathcal{P},\mathcal{I}) \in \mathbb{L}_\kappa(\mathcal{K}) \bigcap \mathbb{L}_\rho(\mathcal{R})$ *and let the class* $\mathcal{P}$ *of projections be* $\mathcal{M}_u$*-hereditary. Then*:

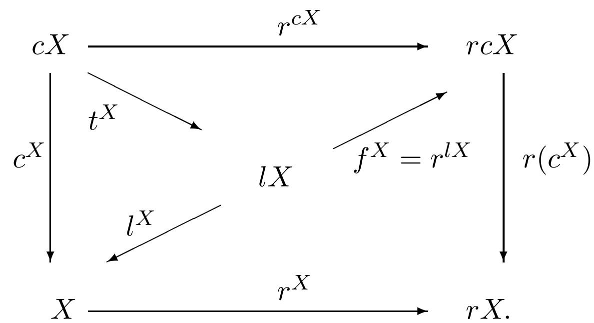
1. $(\mathcal{P},\mathcal{I}) = (\mathcal{P}''(\mathcal{R}),\mathcal{I}''(\mathcal{R}))$.
2. *For any object* $X \in |\mathcal{C}_2\mathcal{V}|$ *the square* $r^X \cdot k^X = k^{rX} \cdot r^{kX}$ *is a pullback.*
3. $\mathcal{K} = \mathcal{C} *_s \mathcal{R}$ *for any coreflective subcategory* $\mathcal{C}$ *with the property* $\mathcal{K} \subset \mathcal{C} \subset \widetilde{\mathcal{M}}$.
4. *The subcategory* $\mathcal{K}$ *is closed in relation to* $(\varepsilon\mathcal{R})$*-subobjects.*

Proof. 1. In the lattice $\mathbb{L}_\rho(\mathcal{R})$ there is an unique factorization structure whose class of projections is $\mathcal{M}_u$-hereditary, i.e. $(\mathcal{P}''(\mathcal{R}),\mathcal{I}''(\mathcal{R}))$.

2. Because $k^X \in \mathcal{I} = \mathcal{I}''(\mathcal{R})$ by equality

$$r^X \cdot k^X = r(k^X) \cdot r^{kX}. \tag{27}$$

3. Let $\mathcal{C}$ be a coreflective subcategory of the category $\mathcal{C}_2\mathcal{V}$ and $\mathcal{K} \subset \mathcal{C} \subset \widetilde{\mathcal{M}}$. For an arbitrary object $X \in |\mathcal{C}_2\mathcal{V}|$ construct the diagram



From the equality

$$r^{cX} = f^X \cdot t^X \tag{28}$$

and from the fact that t^X is an epimorphism it follows that $t^X \in \varepsilon\mathcal{R} \subset \mathcal{P}''(\mathcal{R})$. Further, the morphisms c^X and l^X are universal monomorphisms. Because

$$r^X \cdot l^X = r(c^X) \cdot f^X \tag{29}$$

is a pullback, we deduce that $l^X \in \mathcal{I}''(\mathcal{R})$. So the equality

$$c^X = l^X \cdot t^X \tag{30}$$

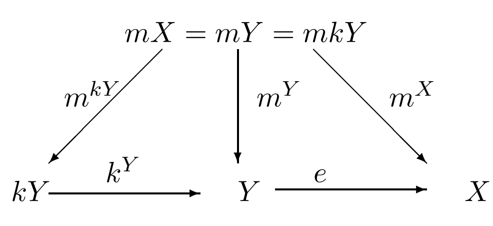
is the $(\mathcal{P}''(\mathcal{R}), \mathcal{I}''(\mathcal{R}))$-factorization of the morphism c^X.

In particular we consider $\mathcal{C} = \widetilde{\mathcal{M}}$. Then $\mathcal{L} = \mathcal{K}$ and we have proved the equality

$$\mathcal{K} = \widetilde{\mathcal{M}} *_s \mathcal{R}.$$

In Theorem 2.4 [3] we deduce that $\mathcal{K} = \mathcal{K} *_s \mathcal{R}$. The relations $\mathcal{K} \subset \mathcal{C} \subset \widetilde{\mathcal{M}}$ and $\mathcal{K} *_s \mathcal{R} = \widetilde{\mathcal{M}} *_s \mathcal{R} = \mathcal{K}$ imply that $\mathcal{C} *_s \mathcal{R} = \mathcal{K}$.

4. Let $X \in |\mathcal{K}|$, and $e : Y \to X \in \varepsilon\mathcal{R}$. To prove that $Y \in |\mathcal{K}|$ we examine the $\mathcal{K}$-coreplica of this object. By the hypothesis $k^Y : k^Y \to Y \in \mathcal{I}''(\mathcal{R})$, and kY is a the $\mathcal{P}''(\mathcal{R})$-factorobject of the object mY. Because e and kY belong to the class $\mathcal{E}_u \cap \mathcal{M}_u$, it follows that the $\widetilde{\mathcal{M}}$-coreplicas of kY, Y and X coincide: $mkY = mY = mX$. On the other hand, because $X \in |\mathcal{K}|$, we deduce that $m^X \in \mathcal{P}''(\mathcal{R})$.



We have

$$m^X = e \cdot k^Y \cdot m^{kY}. \tag{31}$$

So $m^X \in \mathcal{P}''(\mathcal{R})$, $e \in \mathcal{M}_u$ and the class $\mathcal{P}''(\mathcal{R})$ is $\mathcal{M}_u$-hereditary. Hence $k^Y \cdot m^{kY} \in \mathcal{P}''(\mathcal{R})$ and $k^Y \in \mathcal{P}''(\mathcal{R})$. Finally, $k^Y \in \mathcal{P}''(\mathcal{R}) \cap \mathcal{I}''(\mathcal{R}) = \mathcal{I}so$. So we have proved that $Y \in |\mathcal{K}|$.

Corollary 2. *Take $\mathcal{R} \in \mathbb{R}$ and $S \subset \mathcal{R}$. Then:*

1. $\mathbf{Q}_{\mathcal{P}''(\mathcal{R})}(\widetilde{\mathcal{M}}) = \widetilde{\mathcal{M}} *_s \mathcal{R}$.

2. *The coreflector functor $l : \mathcal{C}_2\mathcal{V} \to \widetilde{\mathcal{M}} *_s \mathcal{R}$ and the reflector functor $r : \mathcal{C}_2\mathcal{V} \to \mathcal{R}$ commute: $l \cdot r = r \cdot l$.*

We formulate the dual result.

Theorem 7. *Let $(\mathcal{P},\mathcal{I}) \in \mathbb{C}(\mathbb{B})$, $(\mathcal{P},\mathcal{I}) \in \mathbb{L}_\kappa(\mathcal{K}) \cap \mathbb{L}_\rho(\mathcal{R})$, and let the class $\mathcal{I}$ of injections be $\mathcal{E}_u$-cohereditary. Then*:

1. $((\mathcal{P},\mathcal{I}) = (\mathcal{E}'(\mathcal{K}), \mathcal{M}'(\mathcal{K}))$.

2. *For any object $X \in \mid \mathcal{C}_2\mathcal{V} \mid$, the square $r^X \cdot k^X = k^{rX} \cdot r^{kX}$ is a pushout.*

3. *$\mathcal{R} = \mathcal{K} *_d \mathcal{C}_1$ for any reflective subcategory $\mathcal{C}_1$ of the category $\mathcal{C}_2\mathcal{V}$ with the condition $\mathcal{S} \subset \mathcal{C}_1 \subset \mathcal{R}$.*

4. *The subcategory $\mathcal{R}$ is closed in relation to $(\mu\mathcal{K})$-factorobjects.*

Corollary 3. *Let $\mathcal{K} \in \mathbb{K}$ and $\widetilde{\mathcal{M}} \subset \mathcal{K}$. Then:*

1.$\mathbf{S}_{\mathcal{M}'(\mathcal{K})}(\mathcal{S}) = \mathcal{K} *_d \mathcal{S}$.

2. *The coreflector functor $k : \mathcal{C}_2\mathcal{V} \to \mathcal{K}$ and the reflector functor $v : \mathcal{C}_2\mathcal{V} \to \mathcal{K} *_d \mathcal{S}$ commute:*

$$k \cdot v = v \cdot k.$$

Definition 5 [2]. *Let $\mathcal{K}$ be a coreflective non-zero subcategory, and $\mathcal{R}$ - a reflective non-zero subcategory of the category $\mathcal{C}_2\mathcal{V}$. The pair $(\mathcal{K},\mathcal{R})$ is called a relative torsion theory (RTT) if the functors k and r commute and for any object X of the category $\mathcal{C}_2\mathcal{V}$ the square $r^X \cdot k^X = k^{rX} \cdot r^{kX}$ is a pullback and a pushout.*

Considering the last two theorems we obtain:

Theorem 8. *Consider $(\mathcal{P},\mathcal{I}) \in \mathbb{C}(\mathbb{B})$, $(\mathcal{P},\mathcal{I}) \in \mathbb{L}_\kappa(\mathcal{K}) \cap \mathbb{L}_\rho(\mathcal{R})$ where the class $\mathcal{P}$ (respectively: the class $\mathcal{I}$) is $\mathcal{M}_u$-hereditary (respectively: $\mathcal{E}_u$-cohereditary). Then*:

1.$(\mathcal{P},\mathcal{I}) = (\mathcal{E}'(\mathcal{K}), \mathcal{M}'(\mathcal{K})) = (\mathcal{P}''(\mathcal{R}), \mathcal{I}''(\mathcal{R}))$.

2. *For any object $X \in \mid \mathcal{C}_2\mathcal{V} \mid$ the square $r^X \cdot k^X = k^{rX} \cdot r^{kX}$ is a pushout and a pullback.*

3. *$\mathcal{K} = \mathcal{C} *_s \mathcal{R}$ and $\mathcal{R} = \mathcal{K} *_d \mathcal{C}_1$ for any coreflective subcategory $\mathcal{C}$, with the condition $\mathcal{K} \subset \mathcal{C} \subset \widetilde{\mathcal{M}}$ and any reflective subcategory $\mathcal{C}_1$ with the condition $\mathcal{S} \subset \mathcal{C}_1 \subset \mathcal{R}$,*

4. *The pair $(\mathcal{K},\mathcal{R})$ is a relative torsion theory* (RTT).

5. *The subcategory $\mathcal{K}$ is closed in relation to $(\varepsilon\mathcal{R})$-subobjects, and the subcategory $\mathcal{R}$ is closed in relation to $(\mu\mathcal{K})$-factorobjects.*

4 Examples and problems

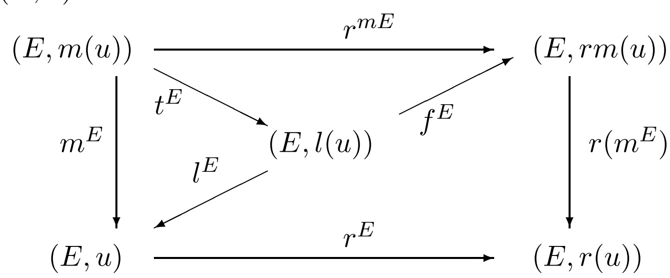
Example 1. For the reflective subcategory $\mathcal{S}$ of spaces with weak topology, we have

$$\mathcal{P}''(\mathcal{S}) = (\varepsilon\mathcal{S}) \cdot \mathcal{E}_p = (\mathcal{E}_u \cap \mathcal{M}_u) \cdot \mathcal{E}_p = \mathcal{E}_u.$$

Hence $\mathcal{I}''(\mathcal{S}) = \mathcal{M}_p$. Then $Q_{\mathcal{E}_u}(\widetilde{\mathcal{M}}) = \mathcal{C}_2\mathcal{V}$. For the pair $(\mathcal{C}_2\mathcal{V}, \mathcal{S})$, the coreflector functor $i : \mathcal{C}_2\mathcal{V} \to \mathcal{C}_2\mathcal{V}$ and the reflector functor $s : \mathcal{C}_2\mathcal{V} \to \mathcal{S}$ commute.

Example 2. Let $\mathcal{R}$ be a reflective subcategory of the category $\mathcal{C}_2\mathcal{V}$, $\mathcal{S} \subset \mathcal{R}$ and $\mathcal{S} \neq \mathcal{R}$.

Let (E, u) be an arbitrary object of the category $\mathcal{C}_2\mathcal{V}$, $r(E, u) = (E, r(u))$ - the $\mathcal{R}$-replica of this object, and $m(E, u) = (E, m(u))$ and $r(E, m(u)) = (E, rm(u))$ - the $\widetilde{\mathcal{M}}$-coreplica and the $\mathcal{R}$-replica respectively of (E, u). We denote by $\mathcal{L}$ the subcategory $\mathbf{Q}_{\mathcal{P}''(\mathcal{R})}(\widetilde{\mathcal{M}})$. Hence $\mathcal{L} = \widetilde{\mathcal{M}} *_s \mathcal{R}$. Let $l(E, u) = (E, l(u))$ be the $\mathcal{L}$-coreplica of (E, u).



Because
$$r^E \cdot m^E = r(m^E) \cdot r^{mE}$$
is an pullback we deduce that
$$l(u) = sup(u, rm(u)),$$
where the supremum is taken in the class of locally convex topologies.

Let l^E be an isomorphism. We consider $l^E = 1$. For the morphism l^E there exists a morphism $g : (E, r(u)) \to (E, rm(u))$ so that
$$f^E = g \cdot r^E.$$
We have
$$g \cdot r(m^E) \cdot r^{mE} = g \cdot r^E \cdot m^E = f^E \cdot m^E = r^{mE}$$
i.e.
$$g \cdot r(m^E) \cdot r^{mE} = r^{mE}.$$
Since r^{mE} is an epimorphism, we deduce that
$$g \cdot r(m^E) = 1.$$
But $r(m^E)$ is an epimorphism. Hence $r(m^E)$ is an isomorphism. Finally, if l^E is an isomorphism, then $r(E, u) = r(E, m(u))$. So, if for any object (E, u) of the category $\mathcal{C}_2\mathcal{V}$, the morphism l^E is an isomorphism, we deduce that this is the case also when $(E, u) \in |\mathcal{S}|$:
$$(E, u) = r(E, u) = r(E, m(u)).$$
The $\mathcal{R}$-replica of objects of subcategory $\widetilde{\mathcal{M}}$ belongs to the subcategory $\mathcal{S}$. From here it results that the $\mathcal{R}$-replica of any object belongs to the subcategory $\mathcal{S}$. So $\mathcal{R} = \mathcal{S}$.

Theorem 9. *Let $\mathcal{R}$ be a reflective subcategory of the category $\mathcal{C}_2\mathcal{V}$, $\mathcal{S} \subset \mathcal{R}$ and $\mathcal{S} \neq \mathcal{R}$. Then:*

1. $\widetilde{\mathcal{M}} *_s \mathcal{R} \neq \mathcal{C}_2\mathcal{V}$.

2. *The coreflector functor $l : \mathcal{C}_2\mathcal{V} \to \widetilde{\mathcal{M}} *_s \mathcal{R}$ and the reflector functor $r : \mathcal{C}_2\mathcal{V} \to \mathcal{R}$ commute: $l \cdot r = r \cdot l$.*

3. *For any object $(E, u) \in |\mathcal{C}_2\mathcal{V}|$, its the $\mathcal{L}$-coreplica $(E, l(u))$ has the property $l(u) = max(u, rm(u))$, where $(E, rm(u))$ is the $\mathcal{R}$-replica of the object $(E, m(u))$, and the maximum is taken in the class of locally convex topologies.*

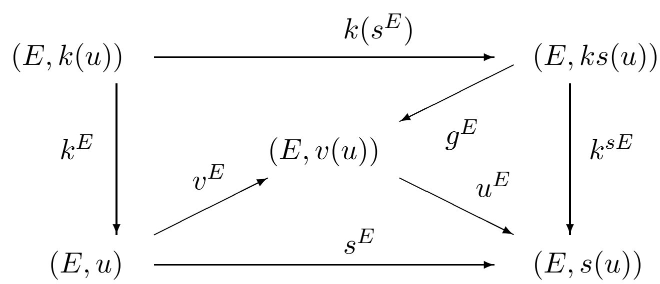
We formulate the dual result.

Theorem 10. *Let $\mathcal{K}$ a coreflective subcategory of the category $\mathcal{C}_2\mathcal{V}$, $\widetilde{\mathcal{M}} \subset \mathcal{K}$ and $\widetilde{\mathcal{M}} \neq \mathcal{K}$. Then:*

1. $\mathcal{K} *_d \mathcal{S} \neq \mathcal{C}_2\mathcal{V}$.

2. *The reflector functor $v : \mathcal{C}_2\mathcal{V} \to \mathcal{K} *_d \mathcal{S}$ and the coreflector functor $k : \mathcal{C}_2\mathcal{V} \to \mathcal{K}$ commute: $k \cdot v = v \cdot k$.*

3. *For any object $(E, u) \in |\mathcal{C}_2\mathcal{V}|$, its the $\mathcal{V}$-replica $(E, v(u))$ has the property $v(u) = min(u, ks(u))$, where the minimum is taken in the class of locally convex topologies, and $(E, ks(u))$ is the $\mathcal{K}$-coreplica of $(E, s(u))$.*



Problem 1. Let $(\mathcal{P}_1, \mathcal{I}_1)$ and $(\mathcal{P}_2, \mathcal{I}_2)$ be two factorization structures in the category $\mathcal{C}_2\mathcal{V}$, ($\mathcal{P}_1 \subset \mathcal{P}_2$ and $\mathcal{P}_1 \neq \mathcal{P}_2$). Is there always a factorization structure $(\mathcal{P}, \mathcal{I})$ with property $\mathcal{P}_1 \subset \mathcal{P} \subset \mathcal{P}_2$ and $\mathcal{P}_1 \neq \mathcal{P} \neq \mathcal{P}_2$?

Problem 2. Let $\mathcal{K}$ and $\mathcal{R}$ be two subcategories of the category $\mathcal{C}_2\mathcal{V}$, the first coreflective, and the second reflective.

If the right product $\mathcal{K} *_d \mathcal{R}$ is a coreflective subcategory, then is $\widetilde{\mathcal{M}} \subset \mathcal{K}$ or $\mathcal{S} \subset \mathcal{R}$?

Problem 3. Is it true that for any factorization structure $(\mathcal{P}, \mathcal{I})$ which belongs to the center $\mathbb{C}(\mathbb{B})$, the class $\mathcal{P}$ must be $\mathcal{M}_u$-hereditary and the class $\mathcal{I}$ must be $\mathcal{E}_u$-cohereditary?

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