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Semi-integral filters and semi-integral BL-algebras

Somayeh Motamed

Abstract. In this paper, we introduced the concepts of semi-integral filters and semi-integral BL-algebras. With respect to these concepts, we give some related results. In particular, we give some relations among semi-integral BL-algebras, integral BL-algebras and local BL-algebra. Also, we give some relations among semi-integral filters and other types of filters in BL-algebras, such as prime, maximal, primary, perfect, normal, positive implicative and obstinate filters.

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1 Introduction

BL-algebras are the algebraic structure for Hájek basic logic introduced in order to investigate many valued logic by algebraic means. His motivations for introducing BL-algebras were of two kinds. The first one was providing an algebraic counterpart of a propositional logic, called Basic Logic, which embodies a fragment common to some of the important many-valued logics, namely Lukasiewicz Logic, Godel Logic and Product Logic. This Basic Logic (BL for short) is proposed as the most general many-valued logic with truth values in $[0,1]$ and BL-algebras are the corresponding Lindenbaum-Tarski algebras. The second one was to provide algebraic means for the study of continuous t-norms (or triangular norms) on $[0,1]$, [6]. Turunen introduced the notion of an implicative filter and a Boolean filter and proved that these notions are equivalent in BL-algebras, [11]. Boolean filters are an important class of filters, because the quotient BL-algebras induced by these filters are Boolean algebras.

2 Preliminaries

Definition 1 (see [6])**.** A BL-algebra is an algebra $(A, \wedge, \vee, *, \rightarrow, 0, 1)$ with four binary operations $\wedge, \vee, *, \rightarrow$ and two constants $0, 1$ such that:

(BL_1) $(A, \wedge, \vee, 0, 1)$ is a bounded lattice $L(A)$,

(BL_2) $(A, *, 1)$ is a commutative monoid,

(BL_3) $*$ and $\rightarrow$ form an adjoint pair, i.e. $c \leq a \rightarrow b$ if and only if $a * c \leq b$, for all $a, b, c \in A$,

(BL_4) $a \wedge b = a * (a \rightarrow b)$,

(BL_5) $(a \rightarrow b) \vee (b \rightarrow a) = 1$.

A BL-algebra A is called

– an MV-algebra if for all $x \in A$, $x^{--} = x$, where $x^- = x \to 0$.

– a Godel algebra if for all $x \in A$, $x^2 = x$.

– an Integral BL-algebra if for all $x, y \in A$, $x * y = 0$ imlpies $x = 0$ or $y = 0$, [3].

It is easy to prove that if A is a BL-algebra and $x, y, z \in A$, we have the following rules of calculus (for more details see [4], [5], [6], [12]):

(BL_6) $x \leq y$ if and only if $x \to y = 1$,

(BL_7) $1 \to x = x$ and $x \leq y \to x$,

(BL_8) $x \to (y \to z) = (x * y) \to z = y \to (x \to z)$,

(BL_9) If $x \leq y$, then $y \to z \leq x \to z$, $z \to x \leq z \to y$ and $y^- \leq x^-$,

(BL_{10}) $x \leq x^{--}$, $x^{---} = x^-$, $x * 0 = 0$ and $x * x^- = 0$,

(BL_{12}) $x \to y^- = y \to x^- = x^{--} \to y^- = (x * y)^-$.

Hájek defined a filter of a BL-algebra A to be a nonempty subset F of A such that (i) $a, b \in F$ implies $a * b \in F$, and (ii) if $a \in F$, $a \leq b$, then $b \in F$,[6]. Turunen defined a deductive system of a BL-algebra A to be a nonempty subset D of A such that (i) $1 \in D$ and (ii) $x \in D$ and $x \to y \in D$ imply $y \in D$. Note that a subset F of a BL-algebra A is a deductive system of A if and only if F is a filter of A, [11]. Let F be a filter of a BL-algebra A. F is proper if $F \neq A$. A proper filter F of A is called a prime filter of A if for all $x, y \in A$, $x \vee y \in F$ implies $x \in F$ or $y \in F$. Equivalently, F is a prime filter of A if and only if for all $x, y \in A$, either $x \to y \in F$ or $y \to x \in F$. A filter of A is maximal if it is proper and it is not contained in any other proper filter of A.

Let F be a proper filter of A. The intersection of all maximal filters of A containing F is called the radical of F and it is denoted by $Rad(F)$. We proved that $Rad(F) = \{a \in A : (a^n)^- \to a \in F, \text{ for all } n \in N\}$, for any filter F of A (for details, see [9]). It is clear that $F \subseteq Rad(F)$, for any filter F of A.

Definition 2 (see [1–3])**.** Let A be a BL-algebra and F be a nonempty subset of A. Then

★ F is called a normal filter of A if F is a filter of A and $z \to ((y \to x) \to x) \in F$ and $z \in F$ imply that $(x \to y) \to y \in F$,

★ a proper filter F is called obstinate filter of A if F is a filter of A and $x, y \notin F$ imply $x \to y \in F$ and $y \to x \in F$,

★ a proper filter F is called primary filter of A if F is a proper filter of A and $(x * y)^- \in F$ implies $(x^n)^- \in F$ or $(y^n)^- \in F$, for some $n \in N$,

★ a proper filter F is called integral filter of A if $(x * y)^- \in P$ implies $x^- \in P$ or $y^- \in P$,

for all $x, y, z \in A$.

If F is a proper filter of A, then the relation $\sim_F$ defined on A by $(x, y) \in \sim_F$ if and only if $x \to y \in F$ and $y \to x \in F$ is a congruence relation on A. The quotient algebra $A/\sim_F$ denoted by A/F becomes a BL-algebra in a natural way, with the operations induced from those of A. So, the order relation on A/F is given by $x/F \leq y/F$ if and only if $x \to y \in F$. Hence $x/F = 1/F$ if and only if $x \in F$ and $x/F = 0/F$ if and only if $x^- \in F$.

Theorem 1 (see [11]). *Let P be a proper filter of BL-algebra A. The following are equivalent:*
(1) A/P is a linearly BL-algebra,
(2) P is a prime filter of A.

3 Main Results in BL-algebras

In this section, first we obtain some new properties of X^- in BL-algebras, that it defined in [10]. Further we define a new radical of a non-empty subset of BL-algebras. Also, we introduce the notions of semi-integral BL-algebras and semi-integral filters and characterized them.

Let X be a non-empty subset of BL-algebra A. We have $X^- = \{a \in A : a^- \in X\}$, for details see [10].

Proposition 1. *Let A and B be BL-algebras, X, Y and X_i $(i \in I)$, be non-empty subsets of A, F be a proper filter of A, Z be a non-empty subset of B and $f : A \longrightarrow B$ be a BL-homomorphism. Then*
(i) If $X \subseteq Y$, then $X^- \subseteq Y^-$. If A is an MV-algebra, the converse is true.
(ii) $F \cap F^- = \emptyset$.
(iii) $F^{--} = \{a \in A : a^{--} \in F\}$, $F \subseteq F^{--}$ and if A is an MV-algebra, then $F = F^{--}$.
(iv) $X^{---} = X^-$.
(v) $(\cap_{i\in I} X_i)^- = \cap_{i\in I} X_i^-$.
(vi) F is an obstinate filter if and only if $F \cup F^- = A$.
(vii) $(\cup_{i\in I} X_i)^- = \cup_{i\in I} X_i^-$.
(viii) $X^- = \cup_{a\in X}\{a\}^-$, $\{0\}^- = D(A)$ and $\{1\}^- = \{0\}$, where $\{a\}^- = \{x \in A : x^- = a\}$.
(ix) $f(X^-) \subseteq f(X)^-$ and if A is an MV-algebra, then $f(X^-) = f(X)^-$.
(x) $f^{-1}(Z^-) = (f^{-1}(Z))^-$.

Proof. (i) If $X \subseteq Y$, then it is clear that $X^- \subseteq Y^-$. Now let $X^- \subseteq Y^-$, A be an MV-algebra and $x \in X$. Then $(x^-)^- = x \in X$ and therefore $x^- \in X^-$. Thus $x^- \in Y^-$. So $x = (x^-)^- \in Y$. Hence $X \subseteq Y$.
(ii) Let $x \in F \cap F^-$. So $x \in F$ and $x^- \in F$. Hence $0 \in F$, which is a contradiction. Thus $F \cap F^- = \emptyset$.
$(iii), (iv), (v), (vi), (vii)$ The proofs are clear.
$(viii)$ Since $\{a\} \subseteq X$, for all $a \in X$, by item (i), $\{a\}^- \subseteq X^-$ and so $\cup_{a\in X}\{a\}^- \subseteq X^-$. Now let $x \in X^-$. Then $x^- \in X$ and so there exists $a \in X$ such that $x^- = a \in \{a\}$. Thus $x \in \{a\}^-$ and therefore $x \in \cup_{a\in X}\{a\}^-$.
(ix) Let $a \in X^-$. Then $a^- \in X$ and so $f(a)^- = f(a^-) \in f(X)$. Therefore $f(X^-) \subseteq f(X)^-$. Now assume that A is an MV-algebra and $y \in f(X)^-$. Then $y^- \in f(X)$ and so $y^- = f(a)$, for some $a \in X$. As A is an MV-algebra and $a^{--} = a \in X$, we have $a^- \in X^-$ and $y = y^{--} = f(a)^- = f(a^-) \in f(X^-)$. Thus $f(X^-) = f(X)^-$.

(x) Let $a \in f^{-1}(Z^-)$. Then $f(a) \in Z^-$ and $f(a^-) = f(a)^- \in Z$. So $a^- \in f^{-1}(Z)$ and $a \in (f^{-1}(Z))^-$. So $f^{-1}(Z^-) \subseteq (f^{-1}(Z))^-$. Now let $a \in (f^{-1}(Z))^-$. Then $a^- \in f^{-1}(Z)$ and $f(a)^- = f(a^-) \in Z$. Thus $f(a) \in Z^-$ and so $a \in f^{-1}(Z^-)$, i.e. $(f^{-1}(Z))^- \subseteq f^{-1}(Z^-)$. $\square$

Definition 3. Let X be a non-empty subset of BL-algebra A. Define

$$\sqrt{X} = \{a \in A : a^t \in X, \text{ for some } t \in \mathbb{N}\}.$$

Example 1. Let $A = \{0, a, b, c, d, 1\}$, where $0 < d < c < a, b < 1$. Define $*$ and $\to$ as follows:

$*$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	c	c	d	a
b	0	c	b	c	d	b
c	0	c	c	c	d	c
d	0	d	d	d	0	d
1	0	a	b	c	d	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	0	1	b	b	d	1
b	0	a	1	a	d	1
c	0	1	1	1	d	1
d	d	1	1	1	1	1
1	0	a	b	c	d	1

Then $(A, \wedge, \vee, *, \to, 0, 1)$ is a BL-algebra. Now for $X = \{a, b\}$ and $Y = \{0\}$, we have $\sqrt{X} = X$ and $Y = \{0\} \subset \sqrt{Y} = \{0, d\}$.

An element $a \in A$ is called nilpotent if $a^n = 0$, for some $n \in \mathbb{N}$. We have $Nil(A) = \{a \in A : a \text{ is a nilpotent element of } A\}$.

Proposition 2. *Let A and B be BL-algebras, X, Y and X_i $(i \in I)$ be non-empty subsets of A, F and G be filters of A, Z be a non-empty subset of B and $f : A \longrightarrow B$ be a BL-homomorphism. Then*
(i) $X \subseteq \sqrt{X}$, $F = \sqrt{F}$ *and if A is a Godel algebra, then* $\sqrt{X} = X$.
(ii) If $X \subseteq Y$, *then* $\sqrt{X} \subseteq \sqrt{Y}$.
(iii) $\sqrt{\sqrt{X}} = \sqrt{X}$.
(iv) $\sqrt{(\cap_{i \in I} X_i)} \subseteq \cap_{i \in I} \sqrt{X_i}$.
(v) $\sqrt{(\cup_{i \in I} X_i)} = \cup_{i \in I} \sqrt{X_i}$.
(vi) $f(\sqrt{X}) \subseteq \sqrt{f(X)}$ *and if f is an isomorphism, then* $f(\sqrt{X}) = \sqrt{f(X)}$.
(vii) $f^{-1}(\sqrt{Z}) = \sqrt{(f^{-1}(Z))}$.
(viii) $\sqrt{0} = Nil(A)$.
(ix) $\sqrt{X} = \cup_{x \in X} \sqrt{x}$, *where* $\sqrt{x} = \{a \in A : a^t = x, \text{ for some } t \in \mathbb{N}\}$.

In the following, the concepts of semi-integral BL-algebras and semi-integral filters are introduced and also characterized.

Definition 4. A BL-algebra A is called semi-integral if $x * y = 0$, for $x, y \in A$, implies $x = 0$ or $y^n = 0$, for some $n \in \mathbb{N}$.

Lemma 1. *Any integral BL-algebra is a semi-integral BL-algebra.*

Example 2. In Example 1, A is a semi-integral BL-algebra; but it is not integral. Since $d^2 = 0$, but $d \neq 0$.

Definition 5. A proper filter P of BL-algebra A is called a semi-integral filter if for $x, y \in A$, $(x * y)^- \in P$ implies $x^- \in P$ or $(y^n)^- \in P$, for some $n \in \mathbb{N}$.

Example 3. (i) In Example 1, $\{1\}$, $\{a, 1\}$ and $\{a, b, c, 1\}$ are semi-integral filters.
(ii) Let $A = \{0, a, b, c, d, 1\}$, where $0 < a < b, d < 1$ and $0 < c < d < 1$. Define $*$ and $\rightarrow$ as follows:

$*$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	a	0	0	a
b	0	a	b	0	a	b
c	0	0	0	c	c	c
d	0	0	a	c	c	d
1	0	a	b	c	d	1

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	1	d	1	1
b	c	d	1	c	d	1
c	b	b	b	1	1	1
d	a	b	b	d	1	1
1	0	a	b	c	d	1

Then $(A, \wedge, \vee, *, \rightarrow, 0, 1)$ is a BL-algebra and $\{1\}$ is not a semi-integral filter. Since $(b * c)^- = \{1\}$ and $b^n = b,\ c^n = c,\ b^- = c,\ c^- = b \notin \{1\}$, for all $n \geq 1$.

Corollary 1. *In any BL-algebra, every integral filter is a semi-integral filter.*

Example 4. In Example 1, $\{1\}$ is a semi-integral filter; but it is not an integral filter. Since $(d * d)^- \in \{1\}$ while $d^- = d \notin \{1\}$.

Theorem 2. *Let P be a proper filter of BL-algebra A. Then A/P is a semi-integral BL-algebra if and only if P is a semi-integral filter of A.*

Proof. Let A/P be a semi-integral BL-algebra and $(x * y)^- \in P$, for $x, y \in A$. Then $[x] * [y] = [0]$. Hence $[x] = [0]$ or $[y^n] = [y]^n = [0]$, for some $n \in \mathbb{N}$, since A/P is semi-integral BL-algebra. Therefore $x^- \in P$ or $(y^n)^- \in P$, for some $n \in \mathbb{N}$ and so P is a semi-integral filter of A. Now let P be a semi-integral filter and $[x] * [y] = [0]$, for $x, y \in A$. Then $[x * y] = [0]$ and therefore $(x * y)^- \in P$. So $x^- \in P$ or $(y^n)^- \in P$, for some $n \in \mathbb{N}$. Hence $[x] = [0]$ or $[y]^n = [y^n] = [0]$, for some $n \in \mathbb{N}$, i.e. A/P is a semi-integral BL-algebra. $\square$

Theorem 3. *Every linearly ordered BL-algebra is a semi-integral BL-algebra.*

Proof. Let $x * y = 0$, for $x, y \in A$. As A is a linearly ordered BL-algebra, we have $x \leq y$ or $y \leq x$. Therefore $x^2 \leq x * y$ or $y^2 \leq x * y$, so $x^2 = 0$ or $y^2 = 0$. Thus, A is a semi-integral BL-algebra. $\square$

Example 5. In Example 1, A is a semi-integral BL-algebra; but it is not a linearly ordered BL-algebra, since $a \not\leq b$ and $b \not\leq a$.

Corollary 2. *In any BL-algebra, every prime filter is semi-integral.*

Proof. Let F be a prime filter of BL-algebra A. Then by Theorem 1, A/F is a linearly ordered BL-algebra. So by Theorem 3, A/F is a semi-integral BL-algebra. Therefore by Theorem 2, F is a semi-integral filter. $\square$

Corollary 3. *In any BL-algebra, every maximal filter is semi-integral.*

Example 6. In Example 1, $F = \{1\}$ is a semi-integral filter; but it is not a prime and so maximal filter. Since $a \to b \notin F$ and $b \to a \notin F$.

Lemma 2. *In any BL-algebra, every semi-integral filter is primary.*

Proof. Let $(x * y)^- \in P$, for $x, y \in A$. Then $x^- \in P$ or $(y^n)^- \in P$, for some $n \in \mathbb{N}$. As $x^n \leq x$, by (BL_9), we have $(x^n)^- \in P$ or $(y^n)^- \in P$, for some $n \in \mathbb{N}$ and so P is a primary filter of A. $\square$

Open Problem. Is every primary filter semi-integral?

Proposition 3. *Let P be an obstinate filter of BL-algebra A. Then A/P is semi-integral BL-algebra.*

Proof. Let $[x] * [y] = [0]$, for $x, y \in A$. So $(x * y) \to 0 \in P$. Thus by (BL_8), $x \to (y \to 0) \in P$, (I). Similarly $y \to (x \to 0) \in P$, (II). If $x \in P$, then by (I), $y^- \in P$ and so $[y] = [0]$. If $y \in P$, then by (II), $x^- \in P$ and so $[x] = [0]$. If $x, y \notin P$, then by hypothesis $x^- \in P$ and $y^- \in P$. Therefore $[x] = [y] = [0]$, i.e A/P is semi-integral BL-algebra.

$\square$

Corollary 4. *In any BL-algebra, every obstinate filter is semi-integral.*

In the following example, we show that the converse of the above corollary is not true in general.

Example 7. In Example 1, $\{1\}$ is a semi-integral filter; but it is not an obstinate filter. Since $a, b \notin \{1\}$ and $a \to b,\ b \to a \notin \{1\}$.

Proposition 4. *In any BL-algebra, every positive implicative and semi-integral filter is obstinate.*

Proof. Let F be a positive implicative and semi-integral filter of BL-algebra A and $x, y \notin F$, for $x, y \in A$. As $(x^- * x)^- \in F$ and F is semi-integral, we have $(x^-)^- \in F$ or $(x^n)^- \in F$, for some $n \in \mathbb{N}$. If $(x^-)^- \in F$, then since F is normal, $(x^-)^- \to x \in F$ and so $x \in F$, which is a contradiction. Thus $(x^n)^- \in F$, for some $n \in \mathbb{N}$. As $(x^n)^- \leq x^n \to y^n$, we have $x^n \to y^n \in F$. Similarly $y^n \to x^n \in F$. Therefore $[x^n] = [y^n]$. So $[x]^n = [y]^n$ and since F is implicative, we have $[x] = [y]$. Hence $x \to y \in F$ and $y \to x \in F$, i.e. F is obstinate. $\square$

By Propositions 4.6 [2], 4 and Corollary 4 we have:

Theorem 4. *Let F be a proper filter of BL-algebra A. Then the following conditions are equivalent.*
(i) F is a maximal and positive implicative filter;
(ii) F is a maximal and implicative filter;
(iii) F is an obstinate filter;
(iv) F is a semi-integral and positive implicative filter.

By Theorems 4.14 [3] and Corollary 1, we have the following theorem.

Theorem 5. *Let A be a finite BL-algebra. Then any perfect filter of A is a semi-integral filter.*

Proposition 5. *Let F and G be proper filters of BL-algebra A such that $F \subseteq G$ and F be a semi-integral filter. Then G is a semi-integral filter of A.*

Proof. Let $(x * y)^- \in G$, for $x, y \in A$. As $((x * y) * (x * y)^-)^- = 1 \in F$ and F is a semi-integral filter, we have $(x * y)^- \in F$ or $(((x * y)^-)^n)^- \in F$, for some $n \in \mathbb{N}$. If $(((x*y)^-)^n)^- \in F$, for some $n \in \mathbb{N}$, then $(((x*y)^-)^n)^- \in G$, which is contradictions with $((x * y)^-)^n \in G$. So $(x * y)^- \in F$ and as F is a semi-integral filter, we have $x^- \in F$ or $(y^n)^- \in F$, for some $n \in \mathbb{N}$. Therefore $x^- \in G$ or $(y^n)^- \in G$, for some $n \in \mathbb{N}$. Thus G is a semi-integral filter of A. □

By Lemma 2 and Theorem 3.3 [8], we have:

Proposition 6. *Let F be a semi-integral filter of BL-algebra A. Then $Rad(F)$ is a maximal filter.*

Corollary 5. *Let F be a prime or semi-integral filter of BL-algebra A. Then $Rad(F)$ is a prime, semi-integral and primary filter.*

Theorem 6. *Let A be a BL-algebra. Then the following conditions are equivalent.*
(i) $\{1\}$ is a semi-integral filter;
(ii) Any filter of A is a semi-integral filter;
(iii) A is a semi-integral BL-algebra.

Proof. $(i) \Rightarrow (ii)$ By Proposition 5, the proof is easy.
$(ii) \Rightarrow (iii)$ As $\{1\}$ is a semi-integral filter, by Theorem 2, $A/\{1\}$ is semi-integral. So A is a semi-integral BL-algebra.
$(iii) \Rightarrow (i)$ $A/\{1\}$ is semi-integral, so by Theorem 2, $\{1\}$ is a semi-integral filter of A. □

Theorem 7. *Let F be a proper filter of BL-algebra A. Then F is a semi-integral filter if and only if every filter of the quotient algebra A/F is a semi-integral filter.*

Proof. Let F be a semi-integral filter and $([x]*[y])^- \in \{[1]\}$. Then $(x*y)^- \in F$ and so $x^- \in F$ or $(y^n)^- \in F$, for some $n \in \mathbb{N}$. Thus $[x]^- \in \{[1]\}$ or $([y]^n)^- \in \{[1]\}$, for some $n \in \mathbb{N}$. Therefore $\{[1]\}$ is a semi-integral filter and by Theorem 6 $(i) \Rightarrow (ii)$, every filter of the quotient algebra A/F is a semi-integral filter. Conversely, assume that every filter of the quotient algebra A/F is a semi-integral filter. So $\{[1]\}$ is a semi-integral filter. Let $(x*y)^- \in F$, for $x, y \in A$. Then $([x]*[y])^- \in \{[1]\}$. Therefore $[x]^- \in \{[1]\}$ or $([y]^n)^- \in \{[1]\}$, for some $n \in \mathbb{N}$. Hence $x^- \in F$ or $(y^n)^- \in F$, for some $n \in \mathbb{N}$. Thus F is a semi-integral filter of A. □

Proposition 7. *Let A be a Godel algebra. Then any proper filter F of A is semi-integral if and only if F is primary.*

Lemma 3. *A Godel algebra A is a semi-integral BL-algebra if and only if A is an Integral BL-algebra.*

By Theorem 4.15 [3] and Lemma 3, we have

Corollary 6. *Let A be a Godel algebra and F be a proper filter of A. The following conditions are equivalent:*
(i) F is an integral filter of A;
(ii) F is a primary filter of A;
(iii) F is a semi-integral filter of A;
(iv) A/F is a local BL-algebra.

By Theorem 3.6 [8] and Proposition 5, we have:

Proposition 8. *Let F be a filter of BL-algebra A. Then we have $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v)$.*
(i) F is a semi-integral filter.
(ii) F is a primary filter.
(iii) Rad(F) is a prime filter.
(iv) Rad(F) is a primary filter.
(v) There exists a unique maximal filter M of A containing F.

By Theorem 4.8 [3] and Lemma 1, we have

Proposition 9. *If BL-algebra A has the Godel negation, then A is a semi-integral BL-algebra.*

Lemma 4. *Let P be a semi-integral filter of BL-algebra A and $[x]*[y] = [0]$, for $[x]$ and $[y] \in A/P$. Then $[x]$ or $[y]$ is nilpotent.*

Proof. As $[x]*[y] = [0]$, we have $(x*y)^- \in P$. By Theorem 2, A/P is a semi-integral BL-algebra and so $[x] = [0]$ or $[y^n] = [0]$, for some $n \in \mathbb{N}$. Thus $[x]$ or $[y]$ is nilpotent. □

An element $a \in A$ is called zero divisor element of A if $a*b = 0$, for some $0 \neq b \in A$.

Proposition 10. *Let F be a proper filter of BL-algebra A. If any zero divisor element of A/F is nilpotent, then F is a semi-integral filter of A.*

Proof. Let $(x * y)^- \in F$, for $x, y \in A$. So $[x] * [y] = 0$. Let $x^- \notin F$. If $[y] = 0$, then $y^- \in F$. Otherwise $[x]$ and $[y]$ are zero divisors. Thus there exists $n \in \mathbb{N}$ such that $[y^n] = 0$. Therefore $(y^n)^- \in F$. Hence F is a semi-integral filter. □

Theorem 8. *Let A be a semi-integral BL-algebra. Then $(Nil(A))^- = A - \{0\}$. The converse is true, when $(Nil(A))^-$ is a $(*)$-closed subset.*

Proof. It is clear that $(Nil(A))^- \subseteq A - \{0\}$. Let $x \in A - \{0\}$. Then $x * x^- = 0$ and so $(x * x^-)^- = 1 \in \{1\}$. By Theorem 6 $(iii) \Rightarrow (i)$, $\{1\}$ is a semi-integral filter of A. Thus $x^- = 1$ or $((x^-)^n)^- = 1$, for some $n \in \mathbb{N}$. If $x^- = 1$, then as $x \le x^{--} = 1^- = 0$, we have $x = 0$, which is a contradiction. So $((x^-)^n)^- = 1$, for some $n \in \mathbb{N}$. Since $(x^-)^n \le ((x^-)^n)^{--} = 1^- = 0$, we have $(x^-)^n = 0$. Therefore $x \in (Nil(A))^-$, i.e. $(Nil(A))^- = A - \{0\}$. Now assume that $(Nil(A))^-$ is a $*$-closed subset, $(Nil(A))^- = A - \{0\}$ and $x * y = 0$, for $x, y \in A$. If $x \notin (Nil(A))^-$, then $x = 0$. Similarly if $y \notin (Nil(A))^-$, then $y = 0$. If $x,\ y \in (Nil(A))^-$, then by hypothesis $x * y \in (Nil(A))^-$, which is a contradiction. Therefore A is semi-integral BL-algebra. □

Definition 6. Let F be a semi-integral filter of BL-algebra A. If $m = Rad(F)$ (which by Proposition 6, is an maximal filter), then F is called a m-semi-integral filter of A

Example 8. In Example 1, $\{1\}$, $\{a, 1\}$ and $\{a, b, c, 1\}$ are $\{a, b, c, 1\}$-semi-integral filters.

By Theorem 3.1 (1), (2) [8] and Lemma 2, we have:

Proposition 11. *Let F be a semi-integral filter of BL-algebra A. Then the following conditions hold:*
(i) $(x^n)^- \in F$ or $((x^-)^m)^- \in F$, for some $n, m \in \mathbb{N}$, for all $x \in A$.
(ii) $(a * b)^- \in F$ implies that $(a^m)^- \in F$ or $(b^n)^- \in F$, for some $n, m \in \mathbb{N}$, for all $a, b \in A$.

Theorem 9. *Let F be a proper filter of BL-algebra A. Then the following conditions are equivalent.*
(i) A/F is a semi-integral BL-algebra;
(ii) For all $x \in A - \sqrt{F^-}$, $F^- = \{a \in A : (a * x)^- \in F\}$;
(iii) For any subset X of A such that $X \nsubseteq \sqrt{F^-}$, $F^- = \{a \in A : (a * x)^- \in F,\ \text{for all } x \in X\}$.

Proof. $(i) \Rightarrow (ii)$ Assume that $x \in A - \sqrt{F^-}$. Let $a \in F^-$. So $a^- \in F$ and as $a^- \le (a * x)^-$, we have $(a * x)^- \in F$. Now let $(a * x)^- \in F$. Thus $(a * x) \to 0 \in F$ and $0 \to (a * x) = 1 \in F$. Therefore $[a] * [x] = [0]$ and by item (i), $[a] = [0]$ or

$[x^n] = [x]^n = [0]$, for some $n \in \mathbb{N}$. So $a \in F^-$ or $x^n \in F^-$, i.e. $a \in F^-$ or $x \in \sqrt{F^-}$. Hence as $x \notin \sqrt{F^-}$, we have $a \in F^-$.
$(ii) \Rightarrow (iii)$ Assume that X is a subset of A such that $X \nsubseteq \sqrt{F^-}$. It is clear that $F^- \subseteq \{a \in A : (a * x)^- \in F, \text{ for all } x \in X\}$. Now let $(a * x)^- \in F$, for all $x \in X$. Since $X \nsubseteq \sqrt{F^-}$, there exists $y \in X - \sqrt{F^-}$. So $(a * y)^- \in F$ and by item (ii), $a \in F^-$.
$(iii) \Rightarrow (i)$ Let $[x] * [y] = [0]$, for $x, y \in A$. Thus $(x * y)^- \in F$. If there exists $n \in \mathbb{N}$ such that $[y]^n = [0]$, then A/F is a semi-integral BL-algebra. Otherwise for all $n \in \mathbb{N}$, $[y]^n \neq [0]$. So for all $n \in \mathbb{N}$, $(y^n)^- \notin F$. Hence $y \notin \sqrt{F^-}$. So $\{y\} \nsubseteq \sqrt{F^-}$ and $(x * y)^- \in F$. Therefore by item (iii), $x \in F^-$. So $x^- \in F$ and $[x] = [0]$. □

Lemma 5. *Let $f : A \to B$ be a BL-homomorphism, F be a filter of B and $G = f^{-1}(F)$ be a filter of A. Then the following conditions hold:*
(i) If F is an m-semi-integral filter, then G is an $f^{-1}(m)$-semi-integral filter of A.
(ii) Let f be a BL-epimorphism. Then F is an m-semi-integral filter of B if and only if G is an $f^{-1}(m)$-semi-integral filter of A.

Proof. (i) Let F be a semi-integral filter of B and $(a * b)^- \in G$, for $a, b \in A$. Then

$$\begin{aligned}
f((a * b)^-) \in F \quad &\Rightarrow \quad (f(a) * f(b))^- \in F, \\
&\Rightarrow \quad (f(a))^- \in F \text{ or } (f(b)^n)^- \in F, \text{ for some } n \in \mathbb{N}, \\
&\Rightarrow \quad f(a^-) \in F \textit{ or } f((b^n)^-) \in F, \text{ for some } n \in \mathbb{N}, \\
&\Rightarrow \quad (a)^- \in G \text{ or } (b^n)^- \in G, \text{ for some } n \in \mathbb{N}, \\
&\Rightarrow \quad G \text{ is a semi-integral filter of } A.
\end{aligned}$$

Now let $Rad(F) = m$. Then $f^{-1}(Rad(F)) = f^{-1}(m)$ and so by Theorem 4.5 [9], $f^{-1}(Rad(F)) = Rad(f^{-1}(F))$, i.e. $Rad(G) = f^{-1}(m)$. Therefore G is a $f^{-1}(m)$-semi-integral filter of A.

(ii) Let G be a semi-integral filter of A and $(a * b)^- \in F$, for $a, b \in B$. Since f is onto, then there exist $c, d \in A$, such that $a = f(c)$ and $b = f(d)$ and so $f((c * d)^-) = (f(c) * f(d))^- = (a * b)^- \in F$. Thus

$$\begin{aligned}
(c * d)^- \in f^{-1}(F) = G \quad &\Rightarrow \quad c^- \in G \text{ or } (d^n)^- \in G, \text{ for some } n \in \mathbb{N}, \\
&\Rightarrow \quad (f(c))^- \in F \text{ or } (f(d)^n)^- \in F, \text{ for some } n \in \mathbb{N}, \\
&\Rightarrow \quad (a)^- \in F \text{ or } (b^n)^- \in F, \text{ for some } n \in \mathbb{N}, \\
&\Rightarrow \quad F \text{ is a semi-integral filter of } B.
\end{aligned}$$

Let $Rad(G) = f^{-1}(m)$. Then by Theorem 4.5 [9],

$$Rad(F) = f(f^{-1}(Rad(F))) = f(Rad(f^{-1}(F))) = f(f^{-1}(m)) = m.$$

Therefore F is an m-semi-integral filter of B. □

Let F be a filter of BL-algebra A and $x \in A$. From [8], $(F : x) = \{r \in A : r \vee x \in F\}$ is a filter of A which contains F and $(F : x)$ is a proper filter of A, when $x \notin F$.

By Theorem 4.5(1) [8] and Proposition 5, we have:

Proposition 12. *Let F be an m-semi-integral filter of BL-algebra A and $x \in A - F$. Then*
(i) $Rad(F) = Rad((F : x))$.
(ii) $(F : x)$ is an m-semi-integral filter.

Open Problem. Is intersection of two m-semi-integral filters an m-semi-integral filter?

The following example shows that if F is an m_1-semi-integral filter and G is an m_2-semi-integral filter, then $F \cap G$ is not a semi-integral filter.

Example 9. Let $A = \{0, a, b, c, d, e, f, g, 1\}$, where $0 < a < b, d < e < 1$ and $0 < c < d, f < g < 1$. Define $*$ and $\rightarrow$ as follow:

$*$	0	a	b	c	d	e	f	g	1
0	0	0	0	0	0	0	0	0	0
a	0	0	a	0	0	a	0	0	a
b	0	a	b	0	a	b	0	a	b
c	0	0	0	0	0	0	c	c	c
d	0	0	a	0	0	a	c	c	d
e	0	a	b	0	a	b	c	d	e
f	0	0	0	c	c	c	f	f	f
g	0	0	a	c	c	d	f	f	g
1	0	a	b	c	d	e	f	g	1

$\rightarrow$	0	a	b	c	d	e	f	g	1
0	1	1	1	1	1	1	1	1	1
a	g	1	1	g	1	1	g	1	1
b	f	g	1	f	g	1	f	g	1
c	e	e	e	1	1	1	1	1	1
d	d	e	e	g	1	1	g	1	1
e	c	d	e	f	g	1	f	g	1
f	b	b	b	e	e	e	1	1	1
g	a	b	b	d	e	e	g	1	1
1	0	a	b	c	d	e	f	g	1

Then $(A, \wedge, \vee, *, \rightarrow, 0, 1)$ is a BL-algebra. We have $F = \{e, b, 1\}$ and $G = \{f, g, 1\}$

are semi-integral filters and also $Rad(F) = F$ and $Rad(G) = G$. We get that $F \cap G$ is not a semi-integral filter, since $(b * f)^- \in F \cap G$ while $b^- \notin F \cap G$ and $(f^n)^- \notin F \cap G$, for all $n \in \mathbb{N}$.

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Somayeh Motamed
Department of Mathematics, Bandar Abbas Branch
Islamic Azad University, Bandar Abbas, Iran
E-mail: *somayeh.motamed@iauba.ac.ir; s.motamed63@yahoo.com*

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