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Viscous flow through a porous medium filled by liquid with varying viscosity

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Abstract. The paper deals with study of a Stokes-Brinkman system with varying viscosity that describes the fluid flow along an ensemble of partially porous cylindrical particles using the cell approach. We have proved the existence and uniqueness of the solutions as well as derived some uniform estimates.

Mathematics subject classification: 76D03, 76D07, 76D10, 76S99..
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1 Introduction

Pressure driven membrane processes (reverse osmosis, nano-, ultra and micro-filtration), sedimentation, flows of underground water and crude oil are important examples of flow through porous media. Usually porous medium was modeled by a dense set of rigid impermeable (colloid) particles [4]. For now to achieve effective use of a porous medium in the above-mentioned areas, the structure of a porous layer should be viewed from different points of view. For example, it is not necessary that the particles always have a smooth homogeneous surface but also have a rough surface or a surface covered by a porous shell. The hydrodynamic models of colloid particle changed considerably over last decades. The latter attracts itself in terminology too: soft particles [8], i. e. particles with porous hydrodynamically permeable surface layer, draw now more attention than hard impermeable particles [4]. There has been also considerable recent interest in the use of beds of porous particles for biological applications such as perfusion chromatography for purifying proteins and other biomolecules and cell or enzyme immobilization. Therefore a number of technologies require the development of modeling of porous media. The mentioned porous media are frequently modeled as aggregates of particles and/or fibers. The cell model [4] has been very effectively used for investigation of the mentioned above flows. The basic principle of the cell model is to replace a system of randomly oriented particles by a periodic array of spheres or cylinders embedded in a center of spherical or cylindrical liquid cells. Appropriate boundary conditions on the cell boundary are supposed to take into account the influence of surrounding particles on the flow inside the cell and the force applied to the particle in the center of the cell. The four variants of these conditions are known as the Happel (the absence of tangential stresses on the cell surface), Kuwabara (the absence of vortexes - the flow potentiality), Kvashnin (the cell symmetry), and Cunningham (the flow on the surface of cell is assumed to be uniform) models [15]. In the course of filtration processes the structure of the membrane can change due to (i) dissolution of particles, (ii) adsorption of polymers on the surfaces of the particles usually referred to as a poisoning. Both the above mentioned processes result in a formation of a porous shell (in the form of a colloidal layer or a gel layer) on the solid particles surface, which are usually hard to remove. The presence of porous shell on solid particles has a clear impact on the drag force exerted by the flow on the particles. Another situation where the slip velocity is of interest is flow over polymer brushes. Polymer chains attached to the surface of a particle create a porous shell around the particle, effectively increasing its diameter. Penetration of the outer flow into the polymer brush determines the transport of ions and other chemical species between the outer flow and the surface of the particle. Hence, the knowledge of the flow field at the interface between a highly porous medium and a liquid is of a substantial importance. Flow through porous shells is frequently modeled by Brinkman's equation [2], which is a modified form of the Darcy's equation. However, it has been observed that the results obtained based on the Brinkman's equations do not agree with the experimental data for non-homogeneous porous media. A modification of the Brinkman's equation was suggested in [14] for the media having non-homogeneous porosity. To overcome this problem it is possible also to use "variable viscosity model" for the liquid/porous boundary region. We assume below that porous shells under consideration have a uniform porosity but variable liquid viscosity inside porous layer in accordance with power or exponential law. The membranes under investigation below are supposed to be built by either non-porous particles with a rough surface or particles covered by a porous shell. The latter shells also have a rough surface, and a scale of roughness is equal or even bigger than the average pore size inside the shell. The important problem is a correct selection of boundary conditions on surfaces of non-porous but rough surfaces of particles or porous shell of particles. We use bellow the condition of "tangential stresses slippage" which is a jump of tangential stresses at the porous-liquid interface [6, 7]. The aim of this paper is to prove the existence and uniqueness of the solutions of boundary value problems as well as derive some uniform estimates which will be useful for numerical simulations.

2 Statement of the problem

Describe the viscous flow through a porous medium, modeled as a set of parallel composite cylindrical particles, and filled by liquid with varying viscosity by two systems: the Stokes one

$$\begin{cases} \widetilde{\nabla}\widetilde{p}^o = \widetilde{\mu}^0\widetilde{\Delta}\widetilde{\mathbf{v}}^0, \\ \widetilde{\mathrm{div}}\widetilde{\mathbf{v}}^o = 0 \end{cases} \tag{1}$$

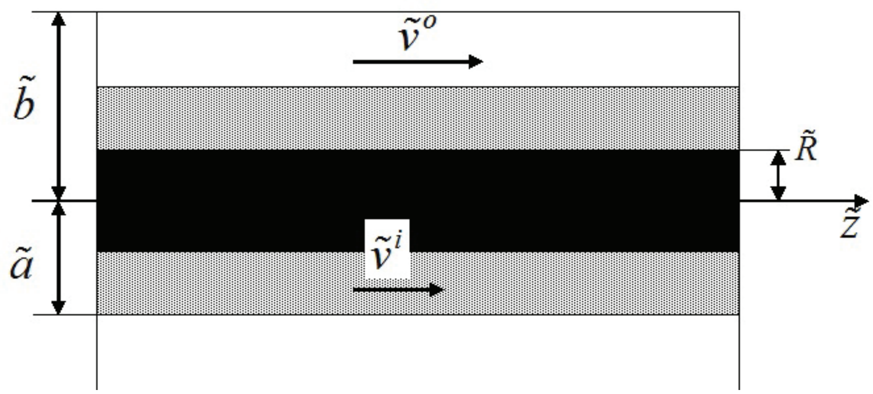

Figure 1. The flow parallel to the cylinders

outside the porous layer $\widetilde{a} \le \widetilde{r} \le \widetilde{b}$ and in the porous layer $\widetilde{R} \le \widetilde{r} \le \widetilde{a}$ by the Brinkman's system

$$\begin{cases} \widetilde{\nabla}\widetilde{p}^i = \widetilde{\mathrm{div}}(\widetilde{\mu}^i\widetilde{D}\widetilde{\mathbf{v}}^i) - \frac{\widetilde{\mu}^o}{\widetilde{k}}\widetilde{\mathbf{v}}^i, \\ \widetilde{\mathrm{div}}\widetilde{\mathbf{v}}^i = 0. \end{cases} \tag{2}$$

Here the tilde denotes dimensional variables, indices o and i refer to the external and porous zones respectively; $\widetilde{\mu}^i$ and $\widetilde{\mu}^o$ are the viscosities of the liquids inside Brinkman's layer and in liquid shell, correspondingly. The variable $\widetilde{k}$ is the specific permeability of the porous layer. We suppose that viscosity of clear liquid $\widetilde{\mu}^o$ is constant over region $\widetilde{a} < \tilde{r} < \tilde{b}$ and viscosity of Brinkmans liquid $\tilde{\mu}^i = \tilde{\mu}^o\left(\frac{\tilde{a}}{\tilde{r}}\right)^\alpha$ increases according to power law from $\tilde{\mu}^o$ at porous media-clear liquid interface to

$$\tilde{\mu}^o\left(\frac{\tilde{a}}{\tilde{R}}\right)^\alpha$$

at the interface between solid core and porous layer. Parameter α is needed in order to get necessary viscosity of Brinkmans liquid in the vicinity of the solid core. The unknown functions are $\widetilde{\mathbf{v}}^o, \widetilde{\mathbf{v}}^i$– the velocity field and the pressure $\widetilde{p}^o, \widetilde{p}^i$.

Also the boundary conditions as follows are set:

$$\widetilde{\mathbf{v}}^i = 0, \text{ as } \widetilde{r} = \widetilde{R}, \tag{3}$$

the continuity condition:

$$\widetilde{\mathbf{v}}^i = \widetilde{\mathbf{v}}^o, \quad \widetilde{\sigma}^o_{rr} = \widetilde{\sigma}^i_{rr}, \text{ as } \widetilde{r} = \widetilde{a}. \tag{4}$$

The condition for a jump of tangential stresses at the interface between porous layer and clear liquid reads,

$$\widetilde{\sigma}^i_{rz} - \widetilde{\sigma}^o_{rz} = \frac{\beta\widetilde{\mu^o}}{\sqrt{\widetilde{k}}}\widetilde{v}^o_z, \quad \text{as } \widetilde{r} = \widetilde{a}. \tag{5}$$

Here $-\infty < \beta < \sqrt{\frac{\tilde{\mu}^i}{\tilde{\mu}^0}}$ is the dimensionless parameter which should be found from a physical experiment [3]. In case of flow which is parallel to the cylinders all four known conditions at the outer cell boundary are reduced to the scalar one [12]:

$$\frac{d\widetilde{v}_z^o}{d\widetilde{r}} = 0, \text{ as } \widetilde{r} = \widetilde{b}. \tag{6}$$

For the convenience of the analysis we pass to the dimensionless operators and variables by the following substitutions:

$$\begin{aligned} &\frac{\tilde{b}}{\tilde{a}} = \frac{1}{\gamma},\ r = \frac{\tilde{r}}{\tilde{a}},\ \ z = \frac{\tilde{z}}{\tilde{a}}, \nabla = \tilde{\nabla} \cdot \tilde{a},\ \ \Delta = \tilde{\Delta} \cdot \tilde{a}^2,\ \delta = \frac{\tilde{\delta}}{\tilde{a}},\ R = \frac{\tilde{R}}{\tilde{a}} = 1 - \delta, \\ &\mathbf{v} = \frac{\tilde{\mathbf{v}}}{\tilde{U}}\ \ p = \frac{\tilde{p}}{\tilde{p}_0},\ \tilde{p}_0 = \frac{\tilde{U} \cdot \tilde{\mu}^o}{\tilde{a}},\ k = \frac{\tilde{k}}{\tilde{a}^2} > 0\,,\ \omega = \frac{dp}{dz}, \end{aligned} \tag{7}$$

where $\widetilde{U}$ is the cell (filtration) velocity $\tilde{U} = -\tilde{L}_{11}\frac{d\tilde{p}}{d\tilde{z}}$, where $\tilde{L}_{11}$ is the hydrodynamic permeability of the membrane [13].

Denote by B_γ the layer

$$B_\gamma = \{1 \le r \le \frac{1}{\gamma},\ \varphi \in [0, 2\pi],\ z \in [0, \infty)\}$$

and by B_R the set

$$B_R = \{R \le r \le 1,\ \varphi \in [0, 2\pi],\ z \in [0, \infty)\}.$$

In the dimensionless notations the systems (1) and (2) read as

$$\left\{ \begin{array}{l} \nabla p^o = \mu^o \Delta \mathbf{v}^o \text{ in } B_\gamma, \\ \mathrm{div}\mathbf{v}^o = 0 \text{ in } B_\gamma, \\ \frac{dv_z^o}{dr} = 0 \text{ on } r = \frac{1}{\gamma},\ v_z^o = v_z^i \text{ on } r = 1, \end{array} \right. \tag{8}$$

where $\mathbf{v}^i$ is given by

$$\left\{ \begin{array}{l} \nabla p^i = \mathrm{div}(r^{-\alpha} D\mathbf{v}^i) - \frac{\mathbf{v^i}}{k} \text{ in } B_R, \\ \mathrm{div}\mathbf{v}^i = 0 \text{ in } B_R, \\ {v_z}^i = 0 \text{ on } r = R, \\ \frac{dv_z^i}{dr} = \frac{dv_z^o}{dr} + \frac{\beta}{\sqrt{k}} {v_z}^o \text{ as } r = 1. \end{array} \right. \tag{9}$$

The problems (8) and (9) are linked to each other via the boundary condition $\mathbf{v}^i = \mathbf{v}^o$ on the common boundary $r = 1$ which physically means the continuous flow regime.

Our goal is to investigate the qualitative properties of the obtained systems: existence and uniqueness of the solution as well as to derive some apriori estimates.

2.1 The flow parallel to cylinders. The case $\mu^i = \mu^o r^{-\alpha}$

Rewrite the problems (8) and (9) in cylindric coordinates (r, φ, z) with help of formulas

$$\begin{aligned} \nabla p &= \left(\frac{\partial p}{\partial r}, \frac{1}{r}\frac{\partial p}{\partial \varphi}, \frac{\partial p}{\partial z}\right), \\ \mathrm{div} v &= \left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\varphi}{\partial \varphi} + \frac{\partial v_z}{\partial z}\right), \\ \Delta v &= \frac{1}{r}\frac{\partial}{\partial r}(r v) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \varphi^2} + \frac{\partial^2 v}{\partial z^2}, \end{aligned} \tag{10}$$

and consider the case when the flow is parallel to the cylinders, i. e. the components of the solution satisfy $v_r^i = v_r^o = v_\varphi^i = v_\varphi^o = 0$, while nonzero are v_z^i as well as v_z^o. We show now that in such case the divergence free property of the velocity implies independence of velocity and $\frac{\partial p^j}{\partial z}$ on z-variable. Here index j is o or i. Indeed, for $j = o$ or $j = i$ the equation

$$\mathrm{div}\mathbf{v}^j = 0 \Leftrightarrow \left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r^j) + \frac{1}{r}\frac{\partial v_\varphi^j}{\partial \varphi} + \frac{\partial v_z^j}{\partial z}\right) = 0 \Leftrightarrow \frac{\partial v_z^j}{\partial z} = 0$$

implies independence v_z^j on z-variable. For an arbitrary $\mu^i(r)$ when the flow is parallel to z-direction, the term $\mathrm{div}(\mu^i D\mathbf{v})$ becomes

$$\frac{1}{r}\frac{d}{dr}\left(\mu^i\left(r\frac{dv_z}{dr}\right)\right) = \frac{dv_z}{dr}\left(\frac{d\mu^i}{dr} + \frac{\mu^i}{r}\right) + \mu^i\frac{d^2 v_z}{dr^2}$$

in the polar coordinates. Having in mind that $\frac{dp^j}{dz} = const = \omega,\ j = o, i$, we arrive at the following one-dimensional Stokes and Brinkman's equations, where for the simplicity we omit the sub-index z (i. e. the notation v^j should be understood as v_z^j):

$$\frac{d^2 v^o}{dr^2} + \frac{1}{r}\frac{dv^o}{dr} = \omega,\ 1 < r < \frac{1}{\gamma}, \tag{11}$$

$$\frac{d^2 v^i}{dr^2} - \frac{\alpha - 1}{r}\frac{dv^i}{dr} = r^\alpha\left(\frac{v^i}{k} + \omega\right),\ R < r < 1 \tag{12}$$

with boundary conditions

$$\begin{aligned} & v^i = 0 \text{ as } r = R, \\ & v^o = v^i \text{ as } r = 1, \quad \frac{dv^i}{dr} - \frac{dv^o}{dr} = \frac{\beta}{\sqrt{k}}v^o, \quad r = 1. \\ & \frac{dv^o}{dr} = 0 \text{ as } r = \frac{1}{\gamma}. \end{aligned} \tag{13}$$

3 On the existence and uniqueness of the solution

3.1 Preliminaries

We recall some basic definitions of Sobolev spaces. The Sobolev space $H^1(\Omega)$ is defined as the completion of the set of functions from the space $C^\infty(\overline{\Omega})$ by the norm $\|u\|_{H^1(\Omega)} = \sqrt{\int\limits_\Omega (u^2 + |\nabla u|^2)\, dx}$; the space $H^{-1}(\Omega)$ denotes the dual space to H^1, i. e. the set of functionals defined on the elements in $H^1(\Omega)$. Following the traditions, we denote by H the set of functions u from $H^1(\Omega)$ such that $\mathrm{div} u = 0$. Finally, $\overset{\circ}{L}_2(\Omega)$ consists of functions $u \in L_2$ satisfying the condition $\int\limits_\Omega u\, dx = 0$. In our analysis the following classical theorem will be used (see [1] and [5]):

Theorem 1 (Lions-Lax-Milgram Lemma)**.** *Let U and V be two real Hilbert spaces and let $B : U \times V \to \mathbb{R}$ be a continuous bilinear functional, where V is continuously embedded in U ($\|u\|_U \le c\|u\|_V$). Suppose also that B is coercive in the following sense: for some constant $c > 0$ and all $u \in U$, $|B[u, u]| \ge c\|u\|^2_U$. Then, for all $f \in V^*$, there exists a unique solution $u = u_f \in U$ to the weak problem $B[u_f, v] = \langle f, v\rangle$ for all $v \in V$. Moreover, the solution depends continuously on the given datum: $\|u_f\|_U \le \frac{1}{c}\|f\|_{V^*}$.*

3.2 The weak solution

Multiplying equations (11), (12) by v^o, v^i respectively and integrating the result over the corresponding domains, we can define the weak solutions v^o and v^i.

Definition 1. The function $v^o \in H^1(B_R)$ is called the weak solution to (11) if the following integral identity holds:

$$-\int\limits_1^{\frac{1}{\gamma}} \left(\frac{dv^o}{dr}\right)^2 dr + \int\limits_1^{\frac{1}{\gamma}} \frac{v^o}{r}\frac{dv^o}{dr}\, dr = \left.\frac{dv^o}{dr}\right|_{r=1} v^o(1) + \omega \int\limits_1^{\frac{1}{\gamma}} v^o\, dr. \tag{14}$$

The function $v^i \in H^1(B_R)$ is called the weak solution to (12) if it satisfies

$$\int\limits_R^1 \left(\frac{dv^i}{dr}\right)^2 dr + (\alpha - 1)\int\limits_R^1 \frac{1}{r} v^i \frac{dv^i}{dr}\, dr + \frac{1}{k}\int\limits_R^1 r^\alpha (v^i)^2\, dr + \omega \int\limits_R^1 r^\alpha v^i\, dr = \frac{1}{2}\left.\frac{d(v^i)^2}{dr}\right|_{r=1}. \tag{15}$$

Here we used integration by parts, the boundary conditions for v^i and observation that

$$v^i \frac{dv^i}{dr} = \frac{1}{2}\frac{d(v^i)^2}{dr}.$$

By using the boundary conditions

$$v^o = v^i, \quad \frac{dv^i}{dr} - \frac{dv^0}{dr} = \frac{\beta}{\sqrt{k}} v^o, \quad r = 1,$$

one can rewrite the identity (15) to the form

$$\begin{aligned}\int_R^1 \left(\frac{dv^i}{dr}\right)^2 dr + (\alpha-1)\int_R^1 \frac{1}{r}v^i\frac{dv^i}{dr}\, dr + \frac{1}{k}\int_R^1 r^\alpha (v^i)^2\, dr + \omega\int_R^1 r^\alpha v^i\, dr = \\ = \frac{1}{2}\left.\frac{d(v^o)^2}{dr}\right|_{r=1} + \frac{\beta}{\sqrt{k}}(v^o(1))^2.\end{aligned} \tag{16}$$

Remark 1. Exactly in the same way one can define the weak solution to (8) and (9) for an arbitrary viscosity $\mu^i(r) = \mu^o\mu(r)$. The integral identities will replace $r^{-\alpha}$ by the function $\mu(r)$.

3.3 The main result

Let us prove the existence and uniqueness of the weak solution. The following theorem gives such result.

Theorem 2. *The unique solution $v^i \in H^1(B_R)$ to (15) does exist and satisfies the estimates*

$$\begin{aligned}\|r^{\frac{\alpha}{2}}v^i\|^2_{L_2(R,1)} &\le |v^o(1)|\frac{k\omega}{2}\left(1-\frac{1}{\gamma^2}\right) + \frac{\beta}{\sqrt{k}}(v^o(1))^2, \\ \left\|\frac{dv^i}{dr}\right\|^2_{L_2(R,1)} &\le |v^o(1)|\frac{k\omega}{2}\left(1-\frac{1}{\gamma^2}\right) + \frac{\beta}{\sqrt{k}}(v^o(1))^2,\end{aligned} \tag{17}$$

where v^o is the unique solution satisfying (11).

Proof. Let us analyze first the solvability of equation (11). It is easy to find the analytical solution to (11), which evidently coincides with the solution in the weak sense. Indeed,

$$\frac{d}{dr}\left(r\frac{dv^o}{dr}\right) = r\omega \Leftrightarrow \frac{dv^o}{dr} = \frac{r\omega}{2} + \frac{C}{r}.$$

Boundary condition $\frac{dv^o}{dr} = 0$ at $r = \frac{1}{\gamma}$ implies that $C = -\frac{\omega}{2\gamma^2}$. Integrating the equation once more, one derives that

$$v^o = C_1 - \frac{\omega}{2\gamma^2}\ln r + \frac{\omega r^2}{4}, \quad \text{where } C_1 = v^i(1) - \frac{\omega}{4} \tag{18}$$

due to the condition $v^o = v^i$ at $r = 1$. The uniqueness of v^o follows directly from formula (18) or can be derived from equations (11), assuming the existence of two different functions $v^o_1 \neq v^o_2$. This technique is quite standard so we skip the full details.

Denote by $B[v^i, v^i] : H^1(B_R) \times H^1(B_R) \to \mathbb{R}$ the bilinear form

$$B[v^i, v^i] = \int_R^1 \left(\frac{dv^i}{dr}\right)^2 dr + (\alpha-1)\int_R^1 \frac{1}{r}v^i\frac{dv^i}{dr}\, dr + \frac{1}{k}\int_R^1 r^\alpha (v^i)^2\, dr - \frac{1}{2}\left.\frac{d(v^i)^2}{dr}\right|_{r=1}.$$

Define the functional on the space $H^1(B_R)$:

$$\langle f, v^i \rangle = -\omega \int\limits_R^1 r^\alpha v^i \, dr,$$

then the question on the existence and uniqueness of the solution (15) is reduced to solvability of

$$B[v^i, v^i] = \langle f, v^i \rangle$$

for any $f \in H^{-1}(B_R)$. Let us establish the coerciveness of $B[v^i, v^i]$ (see Theorem 1). Evaluating the boundary conditions

$$\frac{1}{2} \frac{d(v^i)^2}{dr}\bigg|_{r=1} = \frac{dv^o}{dr}\bigg|_{r=1} v^o(1) + \frac{\beta}{\sqrt{k}}(v^o(1))^2 = v^o(1)\frac{\omega}{2}\left(1 - \frac{1}{\gamma^2}\right) + \frac{\beta}{\sqrt{k}}(v^o(1))^2$$

and using the evident inequalities

$$(v^o)^2(1) \le \int\limits_R^1 \frac{1}{r} v^i \frac{dv^i}{dr} \, dr \le \frac{1}{2R}(v^o)^2(1), \quad \frac{1}{k}\int\limits_R^1 r^\alpha (v^i)^2 \, dr \ge \frac{R^\alpha}{k}\int\limits_R^1 (v^i)^2 \, dr,$$

$$(v^o(1))^2 = (v^i(1))^2 = \left(\int\limits_R^1 \frac{dv^i}{dr} \, dr\right)^2 \le \delta \int\limits_R^1 \left(\frac{dv^i}{dr}\right)^2 \, dr,$$

we conclude that

$$\begin{aligned} |B[v^i, v^i]| \ge \bigg| \int\limits_R^1 \left(\frac{dv^i}{dr}\right)^2 dr + \frac{1}{k}\int\limits_R^1 r^\alpha (v^i)^2 \, dr + \left(\frac{\alpha - 1}{2} - \frac{\beta}{\sqrt{k}}\right)(v^o)^2(1) \\ - v^o(1)\frac{\omega}{2}\left(1 - \frac{1}{\gamma^2}\right)\bigg| \ge C(\alpha, \beta, \gamma, R, k, v^o(1))\|v^i\|^2_{H_1(B_R)}, \end{aligned} \tag{19}$$

where

$$C(\alpha, \beta, \gamma, R, k, v^o(1)) \text{ is a constant and } \|v^i\|^2_{H_1(B_R)} = \int\limits_R^1 \left(\frac{dv^i}{dr}\right)^2 dr + \int\limits_R^1 (v^i)^2 \, dr.$$

Hence, the unique solution v^i exists due to Lions-Lax-Milgram Lemma.

Observe also that the identity (16) imply the estimates

$$\begin{aligned} \|r^{\frac{\alpha}{2}} v^i\|^2_{L_2(R,1)} \le \frac{k}{2}\left| \frac{d(v^o)^2}{dr}\bigg|_{r=1}\right| + \frac{\beta}{\sqrt{k}}(v^o(1))^2, \\ \left\|\frac{dv^i}{dr}\right\|^2_{L_2(R,1)} \le \frac{k}{2}\left| \frac{d(v^o)^2}{dr}\bigg|_{r=1}\right| + \frac{\beta}{\sqrt{k}}(v^o(1))^2. \end{aligned} \tag{20}$$

Coming back to estimates (20) and evaluating

$$\frac{1}{2}\left.\frac{d(v^o)^2}{dr}\right|_{r=1} = v^o(1)\frac{\omega}{2}\left(1-\frac{1}{\gamma^2}\right),$$

we derive the asymptotics

$$\|r^{\frac{\alpha}{2}}v^i\|^2_{L_2(R,1)} \le |v^o(1)|\frac{k\omega}{2}\left(1-\frac{1}{\gamma^2}\right)+\frac{\beta}{\sqrt{k}}(v^o(1))^2,$$

$$\left\|\frac{dv^i}{dr}\right\|^2_{L_2(R,1)} \le |v^o(1)|\frac{k\omega}{2}\left(1-\frac{1}{\gamma^2}\right)+\frac{\beta}{\sqrt{k}}(v^o(1))^2.$$

□

4 Exponential viscosity

Assume now that

$$\widetilde{\mu}^i = \widetilde{\mu}^o e^{-\alpha\left(\frac{\tilde{r}}{\tilde{a}}-1\right)}, \quad \alpha > 0 \tag{21}$$

and again the flow is parallel to z-axis. Making an analogous steps to come to dimensionless form of the Brinkman's equation, one gets the equation

$$\frac{d^2v^i}{dr^2}+\left(-\alpha+\frac{1}{r}\right)\frac{dv^i}{dr} = e^{\alpha\left(\frac{r}{a}-1\right)}\left(\frac{v^i}{k}+\omega\right), \ R<r<1 \tag{22}$$

with boundary conditions

$$\begin{aligned} &v^i = 0 \text{ as } r = R,\\ &v^i = v^o \text{ as } r = 1,\\ &\frac{dv^i}{dr}-\frac{dv^o}{dr} = \frac{\beta}{\sqrt{k}}v^o, \quad r = 1. \end{aligned} \tag{23}$$

The weak solution to (22), (23) satisfies

$$\begin{aligned} &\int\limits_R^1\left(\frac{dv^i}{dr}\right)^2 dr+\frac{1}{k}\int\limits_R^1 e^{\alpha\left(\frac{r}{a}-1\right)}(v^i)^2\,dr+\frac{\alpha}{2}\int\limits_R^1\frac{d}{dr}(v^i)^2\,dr+\omega\int\limits_R^1 e^{\alpha\left(\frac{r}{a}-1\right)}v^i\,dr = \\ &= \frac{1}{2}\left.\frac{d(v^i)^2}{dr}\right|_{r=1}+\frac{1}{2}\int\limits_R^1\frac{1}{r}\frac{d}{dr}(v^i)^2\,dr. \end{aligned} \tag{24}$$

Applying the Newton- Leibnitz formula and taking into account the boundary conditions for v^i, the identity (24) can be rewritten as follows:

$$\begin{gathered}\int_R^1 \left(\frac{dv^i}{dr}\right)^2 dr + \frac{1}{k}\int_R^1 e^{\alpha\left(\frac{r}{a}-1\right)}(v^i)^2\,dr + \frac{\alpha+1}{2}(v^o(1))^2 - \\ -\frac{1}{2}\int_R^1 \frac{(v^i)^2}{r^2}\,dr + \omega\int_R^1 e^{\alpha\left(\frac{r}{a}-1\right)}v^i\,dr = \frac{1}{2}\left.\frac{d(v^o)^2}{dr}\right|_{r=1} + \frac{\beta}{\sqrt{k}}(v^o(1))^2.\end{gathered} \tag{25}$$

Similarly, one can prove the following theorem on the existence and uniqueness of the solution to (24).

Theorem 3. *The solution to (24) does exist, is unique and satisfies estimates*

$$\begin{gathered}\|e^{\frac{\alpha}{2}\left(\frac{r}{a}-1\right)}v^i\|^2_{L_2(R,1)} \le \left(1+\frac{2R^2(1+\delta\beta k^{-\frac{1}{2}})}{2R^2-\delta}\right)\left|v^o(1)\frac{\omega k}{2}\left(1-\frac{1}{\gamma^2}\right)\right|, \\ \left\|\frac{dv^i}{dr}\right\|^2_{L_2(R,1)} \le \frac{2R^2}{2R^2-\delta}\left|v^o(1)\frac{\omega k}{2}\left(1-\frac{1}{\gamma^2}\right)\right|.\end{gathered} \tag{26}$$

Proof. All steps of the proof are identical to the previously considered case in Lemma 2. We introduce the bilinear form $B[v^i,v^i] : H^1(B_R)\times H^1(B_R)\to\mathbb{R}$:

$$B[v^i,v^i] = \int_R^1\left(\frac{dv^i}{dr}\right)^2 dr - \frac{1}{2}\int_R^1\frac{(v^i)^2}{r^2}\,dr + \frac{1}{k}\int_R^1 e^{\frac{\alpha}{2}\left(\frac{r}{a}-1\right)}(v^i)^2\,dr -$$

$$-\frac{1}{2}\left.\frac{d(v^i)^2}{dr}\right|_{r=1} + \frac{1}{2}\int_R^1\frac{1}{r}\frac{d}{dr}(v^i)^2\,dr$$

and functional

$$\langle f, v^i\rangle = \left.\frac{dv^o}{dr}\right|_{r=1} v^o(1) - \left(\frac{\alpha+1}{2}-\frac{\beta}{\sqrt{k}}\right)(v^o(1))^2 - \omega\int_R^1 e^{\frac{\alpha}{2}\left(\frac{r}{a}-1\right)}v^i\,dr.$$

In order to use the Lions-Lax-Milgram Lemma on the existence and uniqueness of the solution, it is required to get the estimate $|B[v^i,v^i]| \ge C\|v^i\|^2_{H_1(B_R)}$. In view of the inequality

$$-\frac{1}{2}\int_R^1\frac{(v^i)^2}{r^2}\,dr \ge -\frac{1}{2}\int_R^1 (v^i)^2\,dr,$$

we get the desired bound

$$|B[v^i,v^i]| \geq \int\limits_R^1 \left(\frac{dv^i}{dr}\right)^2 dr + \frac{1}{k}\int\limits_R^1 \left(e^{\alpha\left(\frac{r}{a}-1\right)} - \frac{k}{2}\right)(v^i)^2\, dr \geq C(\alpha,R,k)\|v^i\|^2_{H_1(B_R)}. \tag{27}$$

Here the constant $C(\alpha,R,k) = \min\{1, |\frac{e^{\alpha\left(\frac{R}{a}-1\right)}}{k} - \frac{1}{2}|\}$. We note that the classical Friedrich's inequality

$$\int\limits_R^1 (v^i)^2\, dr \leq \delta \int\limits_R^1 \left(\frac{dv^i}{dr}\right)^2 dr \tag{28}$$

is valid for function v^i since it vanishes on the boundary $r = R$. Moreover, the constant is equal to the square of the strip $\{R \leq r \leq 1\} \times 1 = \delta$. To obtain the estimates (26) we use the integral identity in the form

$$\begin{gathered}\int\limits_R^1 \left(\frac{dv^i}{dr}\right)^2 dr + \frac{1}{k}\int\limits_R^1 e^{\alpha\left(\frac{r}{a}-1\right)}(v^i)^2\, dr + \frac{\alpha+1}{2}(v^o(1))^2 + \omega\int\limits_R^1 e^{\alpha\left(\frac{r}{a}-1\right)} v^i\, dr = \\ \frac{1}{2}\left.\frac{d(v^o)^2}{dr}\right|_{r=1} + \frac{\beta}{\sqrt{k}}(v^o(1))^2 + \frac{1}{2}\int\limits_R^1 \frac{(v^i)^2}{r^2}\, dr.\end{gathered} \tag{29}$$

It implies the following estimates:

$$\begin{gathered}\int\limits_R^1 \left(\frac{dv^i}{dr}\right)^2 dr \leq \frac{1}{2}\left.\frac{d(v^o)^2}{dr}\right|_{r=1} + \frac{\beta}{\sqrt{k}}(v^o(1))^2 + \frac{1}{2}\int\limits_R^1 \frac{(v^i)^2}{r^2}\, dr, \\ \int\limits_R^1 e^{\alpha\left(\frac{r}{a}-1\right)}(v^i)^2\, dr \leq \frac{1}{2}\left.\frac{d(v^o)^2}{dr}\right|_{r=1} + \frac{\beta}{\sqrt{k}}(v^o(1))^2 + \frac{1}{2}\int\limits_R^1 \frac{(v^i)^2}{r^2}\, dr.\end{gathered} \tag{30}$$

The first term in the right-hand side is bounded as in (17):

$$\left|\frac{1}{2}\left.\frac{d(v^o)^2}{dr}\right|_{r=1}\right| \leq \left|v^o(1)\frac{\omega k}{2}\left(1-\frac{1}{\gamma^2}\right)\right|. \tag{31}$$

Applying the inequality (28), we consider the third term:

$$\frac{1}{2}\int\limits_R^1 \frac{(v^i)^2}{r^2}\, dr \leq \frac{1}{2R^2}\int\limits_R^1 (v^i)^2\, dr \leq \frac{\delta}{2R^2}\int\limits_R^1 \left(\frac{dv^i}{dr}\right)^2 dr. \tag{32}$$

Now we can use this result in the first inequality of (30):

$$\int\limits_R^1 \left(\frac{dv^i}{dr}\right)^2 dr \leq \left|v^o(1)\frac{\omega k}{2}\left(1-\frac{1}{\gamma^2}\right)\right| + \frac{\delta}{2R^2}\int\limits_R^1 \left(\frac{dv^i}{dr}\right)^2 dr. \tag{33}$$

Consequently,

$$\int_R^1 \left(\frac{dv^i}{dr}\right)^2 dr \leq \frac{2R^2}{2R^2-\delta}\left|v^o(1)\frac{\omega k}{2}\left(1-\frac{1}{\gamma^2}\right)\right|. \tag{34}$$

Finally, the term $\frac{\beta}{\sqrt{k}}(v^o(1))^2$ can also be estimated by $\int_R^1 \left(\frac{dv^i}{dr}\right)^2 dr$ with help of Friedrichs inequality (28) and Hölder inequality

$$\int_\Omega |f(x)g(x)|\, dx \leq \left(\int_\Omega |f(x)|^p\, dx\right)^{\frac{1}{p}} \left(\int_\Omega |g(x)|^q\, dx\right)^{\frac{1}{q}}, \ \frac{1}{p}+\frac{1}{q}=1$$

if one apply it for $\Omega=(R,1)$, $f=\frac{dv^i}{dr}$, $g=1$ and $p=q=2$:

$$\begin{aligned}\frac{\beta}{\sqrt{k}}(v^o(1))^2 = \frac{\beta}{\sqrt{k}}(v^i(1))^2 = \frac{\beta}{\sqrt{k}}\left(\int_R^1 \frac{dv^i}{dr}\, dr\right)^2 \leq \frac{\delta\beta}{\sqrt{k}}\int_R^1\left(\frac{dv^i}{dr}\right)^2 dr \leq \\ \leq \frac{2R^2\delta\beta k^{-\frac{1}{2}}}{2R^2-\delta}\left|v^o(1)\frac{\omega k}{2}\left(1-\frac{1}{\gamma^2}\right)\right|.\end{aligned} \tag{35}$$

The results in (33), (34) and (35) can be directly used to estimate the second line in (30):

$$\begin{aligned}\int_R^1 e^{\alpha\left(\frac{r}{a}-1\right)}(v^i)^2\, dr \leq \left|v^o(1)\frac{\omega k}{2}\left(1-\frac{1}{\gamma^2}\right)\right| + \frac{\delta}{2R^2}\int_R^1\left(\frac{dv^i}{dr}\right)^2 dr \leq \\ \left|v^o(1)\frac{\omega k}{2}\left(1-\frac{1}{\gamma^2}\right)\right|\left(1+\frac{2R^2(1+\delta\beta k^{-\frac{1}{2}})}{2R^2-\delta}\right).\end{aligned} \tag{36}$$

$\square$

5 Concluding remarks

Let us observe that estimates (17) and (26) show the continuous dependence of the solution v^i on initial data $k, \omega, \beta, \gamma, \delta, R$ as well as on the solution v^o at the common boundary $r = 1$. Note also that factor $r^{\frac{\alpha}{2}}$ in the estimate (17) and similarly $e^{\alpha\left(\frac{r}{a}-1\right)}$ in (26) means the following asymptotical behaviour of v^i : $v^i \sim C(v^o(1), k, \omega, \beta, \gamma)r^{-\frac{\alpha}{2}}$, where the constant $C(v^o(1), k, \omega, \beta, \gamma)$ depends on $v^o(1), k, \omega, \beta, \gamma$. Analogously, $v^i \sim C(v^o, k, \omega, \beta, \gamma, \delta, R)e^{-\frac{\alpha}{2}\left(\frac{r}{a}-1\right)}$ in the second case. Roughly speaking, the solution v^i is proportional to square root of the viscosity. If one apply the estimate (35) to the second term in right-hand side of (17), we get the upper bounds in the form which involves R as well:

$$\begin{aligned} \|r^{\frac{\alpha}{2}} v^i\|^2_{L_2(R,1)} &\le |v^o(1)| \frac{k\omega}{2} \left(1 - \frac{1}{\gamma^2}\right) \left(1 + \frac{2R^2\delta\beta k^{-\frac{1}{2}}}{2R^2 - \delta}\right), \\ \left\| \frac{dv^i}{dr} \right\|^2_{L_2(R,1)} &\le |v^o(1)| \frac{k\omega}{2} \left(1 - \frac{1}{\gamma^2}\right) \left(1 + \frac{2R^2\delta\beta k^{-\frac{1}{2}}}{2R^2 - \delta}\right). \end{aligned} \tag{37}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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