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A Note on the Equivalence of Control Systems on Lie Groups

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Abstract. We consider state space equivalence and (a specialization of) feedback equivalence in the context of left-invariant control affine systems. Simple algebraic characterizations of both local and global forms of these equivalence relations are obtained. Several illustrative examples regarding the classification of systems on low-dimensional Lie groups are discussed in some detail.

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1 Introduction

Invariant control systems on (real, finite dimensional) Lie groups have been a topic of interest in mathematical control theory since the early 1970's (see, e.g., [22, 27, 28, 34]). These systems form a natural framework for various (variational) problems in mathematical physics, mechanics, elasticity, and dynamical systems (see, e.g., [3, 20, 27, 32]).

In order to understand the local geometry of control systems, one needs to introduce some natural equivalence relations. The most natural equivalence relation is equivalence up to coordinate changes in the state space (*viz.* state space equivalence). Another weaker equivalence relation often considered is feedback equivalence; here state-dependent transformations of the controls are also allowed (see, e.g., [26, 33]).

In this note we consider state space equivalence and feedback equivalence in the context of left-invariant control affine systems. We adapt Krener's (general) characterization of local state space equivalence [30] to this context. A global analogue is also obtained. Two examples pertaining to classification of systems on the Euclidean group $\mathsf{SE}(2)$ and pseudo-orthogonal group $\mathsf{SO}(2,1)_0$ are provided. We specialize feedback equivalence in the context of left-invariant control affine systems by restricting to transformations compatible with the Lie group structure. This is called detached feedback equivalence. Characterizations of local (resp. global) detached feedback equivalence are obtained in terms of Lie algebra (resp. Lie group) isomorphisms. Further three examples pertaining to the classification of systems on low-dimensional Lie groups (namely $\mathsf{SE}(2)$, $\mathsf{SO}(2,1)_0$ and the oscillator group) are provided. Some remarks conclude the paper. A detailed treatment of these equivalence relations can be found in [18].

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2 Invariant control systems

An ℓ-input *left-invariant control affine system* $\Sigma = (\mathsf{G}, \Xi)$ takes the form

$$\dot{g} = \Xi(g, u) = g(A + u_1 B_1 + \cdots + u_\ell B_\ell), \quad g \in \mathsf{G},\, u \in \mathbb{R}^\ell.$$

Here the state space G is a connected Lie group with Lie algebra $\mathfrak{g}$, and $A, B_1, \ldots, B_\ell \in \mathfrak{g}$. For the sake of simplicity we shall assume that $\mathsf{G} \subseteq \mathsf{GL}\,(n, \mathbb{R})$ is a matrix Lie group. The dynamics $\Xi : \mathsf{G} \times \mathbb{R}^\ell \to T\mathsf{G}$ are invariant under left translations, i.e., $\Xi(g, u) = g\,\Xi(\mathbf{1}, u)$ for all $g \in \mathsf{G}$, $u \in \mathbb{R}^\ell$. The *parametrization map*

$$\Xi(\mathbf{1}, \cdot) : \mathbb{R}^\ell \to \mathfrak{g}, \quad u \mapsto A + u_1 B_1 + \cdots + u_\ell B_\ell$$

is assumed to be injective (i.e., $B_1, \ldots, B_\ell$ are linearly independent). The *trace* $\Gamma = \operatorname{im} \Xi(\mathbf{1}, \cdot) \subset \mathfrak{g}$ of the system is the affine subspace

$$A + \Gamma^0 = A + \langle B_1, \ldots, B_\ell \rangle\,.$$

A system is called *homogeneous* if $A \in \Gamma^0$ and *inhomogeneous* otherwise; a system has *full rank* if its trace Γ generates the whole Lie algebra $\mathfrak{g}$. When G is fixed, we shall specify a system Σ by simply writing

$$\Sigma \;:\; A + u_1 B_1 + \cdots + u_\ell B_\ell.$$

Remark 1. Any controllable system has full rank. On the other hand, any full-rank homogeneous system is controllable. Likewise, full-rank systems evolving on certain Lie groups, such as compact groups and Euclidean groups, are known to be controllable.

3 State Space Equivalence

Let $\Sigma = (\mathsf{G}, \Xi)$ and $\Sigma = (\mathsf{G}', \Xi')$ be two left-invariant control affine systems with the same input space $\mathbb{R}^\ell$. The systems Σ and Σ' are *locally state space equivalent* (shortly S_{loc}-equivalent) if there exists a diffeomorphism $\phi : N \subseteq \mathsf{G} \to N' \subseteq \mathsf{G}'$ such that $T_g\phi \cdot \Xi(g, u) = \Xi(\phi(g), u)$ for all $g \in \mathsf{G}$ and $u \in \mathbb{R}^\ell$. Here N and N' are some neighbourhoods of the identity elements $\mathbf{1} \in \mathsf{G}$ and $\mathbf{1}' \in \mathsf{G}'$, respectively, and it is assumed that $\phi(\mathbf{1}) = \mathbf{1}'$. Σ and Σ' are state space equivalent (shortly S-equivalent) if this happens globally (i.e., $N = \mathsf{G}$ and $N' = \mathsf{G}'$).

Remark 2. The assumption $\phi(\mathbf{1}) = \mathbf{1}'$ can always be met by composing ϕ with some appropriate left translations.

Krener's result [30] states that full-rank systems Σ and Σ' are S_{loc}-equivalent if and only if there exists a linear isomorphism $\psi : T_\mathbf{1}\mathsf{G} \to T_\mathbf{1}\mathsf{G}'$ such that the equality

$$\psi[\cdots[\Xi_{u_1}, \Xi_{u_2}], \ldots, \Xi_{u_k}](\mathbf{1}) = [\cdots[\Xi'_{u_1}, \Xi'_{u_2}], \ldots, \Xi'_{u_k}](\mathbf{1})$$

holds for any $k \geq 1$ and any $u_1, \ldots, u_k \in \mathbb{R}^\ell$. Here Ξ_u is the vector field specified by $\Xi_u(g) = \Xi(g, u)$. Hence in the context of left-invariant systems we have the following characterization.

Theorem 1. *Two full-rank systems Σ and Σ' are S_{loc}-equivalent if and only if there exists a Lie algebra isomorphism $\psi : \mathfrak{g} \to \mathfrak{g}'$ such that $\psi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}, u)$ for every $u \in \mathbb{R}^\ell$.*

Remark 3. If full-rank systems Σ and Σ' are S_{loc}-equivalent and G and G' are simply connected, then Σ and Σ' are S-equivalent.

On the other hand, we have the following global analogue of this result.

Theorem 2. *Two full-rank systems Σ and Σ' are S-equivalent if and only if there exists a Lie group isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that $T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}, u)$ for every $u \in \mathbb{R}^\ell$.*

Proof. Suppose Σ and Σ' are S-equivalent. Then there exists a diffeomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that $\phi_* \Xi_u = \Xi'_u$. Clearly ϕ satisfies $T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}, u)$. We have $\phi_*[\Xi_u, \Xi_v] = [\phi_*\Xi_u, \phi_*\Xi_v] = [\Xi'_u, \Xi'_v]$. As Σ has full rank, it follows that ϕ preserves left-invariant vector fields and so ϕ is a Lie group isomorphism (see, e.g., [7]). Conversely, suppose $\phi : \mathsf{G} \to \mathsf{G}'$ is a Lie group isomorphism such that $T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}', u)$. By left invariance and as ϕ is an isomorphism we have that $T_g\phi \cdot \Xi(g, u) = T_g\phi \cdot g\, \Xi(\mathbf{1}, u) = T_{\mathbf{1}} L_{\phi(g)} \cdot T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) = \phi(g)\, \Xi'(\mathbf{1}, u) = \Xi'(\phi(g), u)$. $\square$

We conclude the section with some specific examples on the classification, under local state space equivalence, of systems on some three-dimensional Lie groups.

Example 1 (see [1])**.** The Euclidean group

$$\mathsf{SE}\,(2) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ x & \cos z & -\sin z \\ y & \sin z & \cos z \end{bmatrix} \,:\, x, y, z \in \mathbb{R} \right\}$$

has Lie algebra $\mathfrak{se}\,(2)$ given by

$$\left\{ \begin{bmatrix} 0 & 0 & 0 \\ x & 0 & -z \\ y & z & 0 \end{bmatrix} = xE_1 + yE_2 + zE_3 \,:\, x, y, z \in \mathbb{R} \right\}.$$

The nonzero commutator relations for the ordered basis (E_1, E_2, E_3) are $[E_2, E_3] = E_1$ and $[E_3, E_1] = E_2$.

Any two-input inhomogeneous full-rank system on $\mathsf{SE}\,(2)$ is S_{loc}-equivalent to exactly one of the following full-rank systems:

$$\begin{aligned}
\Sigma^{(2,1)}_{1,\alpha\beta\boldsymbol{\gamma}} &: \ \alpha E_3 + u_1(E_1 + \gamma_1 E_2) + u_2(\beta E_2) \\
\Sigma^{(2,1)}_{2,\alpha\beta\boldsymbol{\gamma}} &: \ \beta E_1 + \gamma_1 E_2 + \gamma_2 E_3 + u_1(\alpha E_3) + u_2 E_2 \\
\Sigma^{(2,1)}_{3,\alpha\beta\boldsymbol{\gamma}} &: \ \beta E_1 + \gamma_1 E_2 + \gamma_2 E_3 + u_1(E_2 + \gamma_3 E_3) + u_2(\alpha E_3)
\end{aligned}$$

or, in *matrix form*

$$\Sigma^{(2,1)}_{1,\alpha\beta\boldsymbol{\gamma}} : \left[\begin{array}{c|cc} 0 & 1 & 0 \\ 0 & \gamma_1 & \beta \\ \alpha & 0 & 0 \end{array}\right], \qquad \Sigma^{(2,1)}_{2,\alpha\beta\boldsymbol{\gamma}} : \left[\begin{array}{c|cc} \beta & 0 & 0 \\ \gamma_1 & 0 & 1 \\ \gamma_2 & \alpha & 0 \end{array}\right], \qquad \Sigma^{(2,1)}_{3,\alpha\beta\boldsymbol{\gamma}} : \left[\begin{array}{c|cc} \beta & 0 & 0 \\ \gamma_1 & 1 & 0 \\ \gamma_2 & \gamma_3 & \alpha \end{array}\right].$$

Here $\alpha > 0, \beta \neq 0$ and $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}$ parametrize families of class representatives, each different values yielding distinct (non-equivalent) class representatives.

The group of automorphisms $\mathsf{Aut}(\mathfrak{se}\,(2))$ is given by

$$\left\{ \begin{bmatrix} x & y & v \\ -\varsigma y & \varsigma x & w \\ 0 & 0 & \varsigma \end{bmatrix} \; : \; \begin{array}{c} x, y, v, w \in \mathbb{R}, \\ x^2 + y^2 \neq 0, \; \varsigma = \pm 1 \end{array} \right\}.$$

Let $\Sigma \, : \, \sum a_i E_i + u_1 \sum b_i E_i + u_2 \sum c_i E_i$ be a two-input inhomogeneous full-rank system; in matrix form

$$\Sigma : \quad \left[\begin{array}{c|cc} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{array} \right].$$

It straightforward to show that there exists an automorphism $\psi \in \mathsf{Aut}\,(\mathfrak{se}\,(2))$ such that

$$\psi \cdot \left[\begin{array}{c|cc} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{array} \right] = \left[\begin{array}{c|cc} 0 & 1 & 0 \\ 0 & \gamma_1 & \beta \\ \alpha & 0 & 0 \end{array} \right] \quad \text{if } b_3 = 0 \text{ and } c_3 = 0$$

$$\psi \cdot \left[\begin{array}{c|cc} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{array} \right] = \left[\begin{array}{c|cc} \beta & 0 & 0 \\ \gamma_1 & 0 & 1 \\ \gamma_2 & \alpha & 0 \end{array} \right] \quad \text{if } b_3 \neq 0 \text{ and } c_3 = 0$$

$$\psi \cdot \left[\begin{array}{c|cc} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{array} \right] = \left[\begin{array}{c|cc} \beta & 0 & 0 \\ \gamma_1 & 1 & 0 \\ \gamma_2 & \gamma_3 & \alpha \end{array} \right] \quad \text{if } c_3 \neq 0.$$

Thus Σ is S_{loc}-equivalent to $\Sigma_{1,\alpha\beta\boldsymbol{\gamma}}$, $\Sigma_{2,\alpha\beta\boldsymbol{\gamma}}$, or $\Sigma_{3,\alpha\beta\boldsymbol{\gamma}}$. It is a simple matter to verify that no two of the class representatives are equivalent.

Example 2 (see [19])**.** The pseudo-orthogonal group

$$\mathsf{SO}\,(2,1) = \{ g \in \mathbb{R}^{3\times 3} \; : \; g^\top J g = g, \, \det g = 1 \}$$

is a three-dimensional simple Lie group. Here $J = \operatorname{diag}(1,1,-1)$. The identity component of $\mathsf{SO}\,(2,1)$ is $\mathsf{SO}\,(2,1)_0 = \{ g \in \mathsf{SO}\,(2,1) \, : \, g_{33} > 0 \}$. Its Lie algebra $\mathfrak{so}\,(2,1)$ is given by

$$\left\{ \begin{bmatrix} 0 & z & y \\ -z & 0 & x \\ y & x & 0 \end{bmatrix} = xE_1 + yE_2 + zE_3 \; : \; x, y, z \in \mathbb{R} \right\}$$

and has commutator relations $[E_2, E_3] = E_1$, $[E_3, E_1] = E_2$, and $[E_1, E_2] = -E_3$.

Any two-input homogeneous full-rank system on $\mathsf{SO}\,(2,1)_0$ is S_{loc}-equivalent to exactly one of the following full-rank systems (displayed in matrix form):

$$\Sigma^{(2,0)}_{1,\boldsymbol{\alpha\gamma}} : \left[\begin{array}{c|cc} \gamma_3 & \alpha_2 & 0 \\ 0 & 0 & 0 \\ \gamma_2 & \gamma_1 & \alpha_1 \end{array}\right], \qquad \Sigma^{(2,0)}_{2,\beta\boldsymbol{\gamma}} : \left[\begin{array}{c|cc} \gamma_3 & \beta+\gamma_1 & 1 \\ 0 & 0 & 0 \\ \gamma_2 & \gamma_1 & 1 \end{array}\right],$$
$$\Sigma^{(2,0)}_{3,\boldsymbol{\alpha}\beta\boldsymbol{\gamma}} : \left[\begin{array}{c|cc} (\beta+\frac{1}{4})\gamma_2 & \beta+\frac{1}{4} & 0 \\ \gamma_3 & \gamma_1 & \alpha_1 \\ (\beta-\frac{1}{4})\gamma_2 & \beta-\frac{1}{4} & 0 \end{array}\right].$$

Here $\alpha_i > 0, \beta \neq 0$ and $\gamma_i \in \mathbb{R}$ parametrize families of class representatives, each different values yielding distinct (non-equivalent) class representatives.

The group $\mathsf{Aut}(\mathfrak{so}\,(2,1))$ of automorphisms of $\mathfrak{so}\,(2,1)$ is exactly $\mathsf{SO}\,(2,1)$. The (Lorentzian) product $\odot$ on $\mathfrak{so}\,(2,1)$ is given by $A \odot B = a_1b_1 + a_2b_2 - a_3b_3$; here $A = \sum a_iE_i$ and $B = \sum b_iE_i$. Any automorphism ψ preserves $\odot$, i.e., $(\psi \cdot A) \odot (\psi \cdot B) = A \odot B$. Furthermore, the group $\mathsf{Aut}(\mathfrak{so}\,(2,1))$ acts transitively on each of the hyperboloids (and punctured cone) $\mathcal{H}_\alpha = \{A \in \mathfrak{so}\,(2,1) \,:\, A \odot B = \alpha,\, A \neq 0\}$. Hence for every $A \in \mathfrak{so}\,(2,1)$, there exists $\psi \in \mathsf{Aut}(\mathfrak{so}\,(2,1))$ such that $\psi \cdot A$ equals αE_2, αE_3, or $E_1 + E_3$. The subgroup of automorphisms fixing these elements are $\{\exp(tE_2), \varsigma \circ \exp(tE_2) \,:\, t \in \mathbb{R}\}$, where

$$\varsigma = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$\{\exp(tE_3) \,:\, t \in \mathbb{R}\}$, and $\{\exp(t(E_1 + E_3)) \,:\, t \in \mathbb{R}\}$, respectively. Moreover, any automorphism fixing at least two of E_1, E_2, E_3, and $E_1 + E_3$ is the identity automorphism.

Suppose $\Sigma : A + u_1B_1 + u_2B_2$ is a two-input homogeneous full-rank system on $\mathsf{SO}\,(2,1)_0$. Then there exists an automorphism $\psi \in \mathsf{Aut}(\mathfrak{so}\,(2,1))$ such that $\psi \cdot B_2$ equals αE_2, αE_3 or $E_1 + E_3$. Hence Σ is equivalent to $\Sigma' : A' + u_1B_1' + u_2(\alpha E_3)$, $\Sigma' : A' + u_1B_1' + u_2(E_1 + E_3)$, or $\Sigma' : A' + u_1B_1' + u_2(\alpha E_2)$. In each case we then further reduce the system by considering the action of the subgroup of automorphisms fixing E_3, $E_1 + E_3$, or E_2, respectively, on the system.

4 Detached Feedback Equivalence

Two systems Σ and Σ' are (globally) feedback equivalent if there exists a diffeomorphism $\Phi : \mathsf{G} \times \mathbb{R}^\ell \to \mathsf{G}' \times \mathbb{R}^{\ell'}$, $(g, u) \mapsto (\phi(g), \varphi(g, u))$ transforming the first system into the second, i.e., $T_g\phi \cdot \Xi(g, u) = \Xi'(\phi(g), \varphi(g, u))$. We specialize feedback equivalence, by requiring that the transformation $u' = \varphi(g, u)$ is constant over the state space; such transformations are exactly those that are compatible with the Lie group structure (cf. [7]). More precisely, Σ and Σ' are called *locally detached feedback equivalent* (shortly DF_{loc}-equivalent) if there exist diffeomorphims $\phi : N \subseteq \mathsf{G} \to N' \subseteq \mathsf{G}'$ and $\varphi : \mathbb{R}^\ell \to \mathbb{R}^{\ell'}$ such that $T_g\phi \cdot \Xi(g, u) = \Xi'(\phi(g), \varphi(u))$ for

$g \in N$, $u \in \mathbb{R}^\ell$. Here N and N' are some neighbourhoods of the identity elements $\mathbf{1} \in \mathsf{G}$ and $\mathbf{1}' \in \mathsf{G}'$ and it is assumed that $\phi(\mathbf{1}) = \mathbf{1}'$. On the other hand, Σ and Σ' are called *detached feedback equivalent* (shortly DF-equivalent) if this happens globally (i.e., $N = \mathsf{G}$ and $N' = \mathsf{G}'$).

Theorem 3. *Two full-rank systems Σ and Σ' are DF_{loc}-equivalent if and only if there exists a Lie algebra isomorphism $\psi : \mathfrak{g} \to \mathfrak{g}'$ such that $\psi \cdot \Gamma = \Gamma'$.*

Proof. Suppose Σ and Σ' are DF_{loc}-equivalent, i.e., there exist diffeomorphisms $\phi : N \subseteq \mathsf{G} \to N' \subseteq \mathsf{G}'$ and $\varphi : \mathbb{R}^\ell \to \mathbb{R}^{\ell'}$ such that $T_g\phi \cdot \Xi(g,u) = \Xi'(\phi(g), \varphi(u))$. Then $T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}', \varphi(u))$ and so $T_{\mathbf{1}}\phi \cdot \Gamma = \Gamma'$. It remains to be shown that $T_{\mathbf{1}}\phi$ preserves the Lie bracket. We have that $\phi_*[\Xi_u, \Xi_v] = [\phi_*\Xi_u, \phi_*\Xi_v]$ for left-invariant vector fields $\Xi_u = \Xi(\cdot, u)$ and $\Xi_v = \Xi(\cdot, v)$. Hence, $T_{\mathbf{1}}\phi \cdot [\Xi_u(\mathbf{1}), \Xi_v(\mathbf{1})] = [\Xi'_{\varphi(u)}(\mathbf{1}'), \Xi'_{\varphi(v)}(\mathbf{1}')] = [T_{\mathbf{1}}\phi \cdot \Xi_u(\mathbf{1}), T_{\mathbf{1}}\phi \cdot \Xi_v(\mathbf{1})]$. Likewise $T_{\mathbf{1}}\phi \cdot [\Xi_u(\mathbf{1}), [\Xi_u(\mathbf{1}), \Xi_w(\mathbf{1})]] = [T_{\mathbf{1}}\phi \cdot \Xi_u(\mathbf{1}), T_{\mathbf{1}}\phi \cdot [\Xi_v(\mathbf{1}), \Xi_w(\mathbf{1})]]$ and similarly for higher order commutators. As the elements $\Xi_u(\mathbf{1})$, $u \in \mathbb{R}^\ell$ generate $\mathfrak{g}$, it follows that $T_{\mathbf{1}}\phi$ is a Lie algebra isomorphism.

Conversely, suppose ψ is a Lie algebra isomorphism such that $\psi \cdot \Gamma = \Gamma'$. Then there exist neighbourhoods N and N' of $\mathbf{1}$ and $\mathbf{1}'$, respectively, and a local group isomorphism $\phi : N \to N'$ such that $T_{\mathbf{1}}\phi = \psi$ (see, e.g., [29]). Also, there exists a unique affine isomorphism $\varphi : \mathbb{R}^\ell \to \mathbb{R}^{\ell'}$ such that $\psi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}, \varphi(u))$. Therefore, (locally) we get $T_g\phi \cdot \Xi(g,u) = T_{\mathbf{1}}L_{\phi(g)} \cdot \psi \cdot \Xi(\mathbf{1}, u) = T_{\mathbf{1}}L_{\phi(g)} \cdot \Xi'(\mathbf{1}', \varphi(u)) = \Xi'(\phi(g), \varphi(u))$. Hence Σ and Σ' are DF_{loc}-equivalent. $\square$

The global analogue of the characterization for detached feedback equivalence follows similarly (and so the proof is omitted).

Theorem 4. *Two full-rank systems Σ and Σ' are DF-equivalent if and only if there exists a Lie group isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that $T_{\mathbf{1}}\phi \cdot \Gamma = \Gamma'$.*

We conclude the section with some specific examples on the classification, under local detached feedback equivalence, of systems on some low-dimensional Lie groups.

Example 3 (see [12])**.** Any two-input inhomogeneous full-rank system on the Euclidean group $\mathsf{SE}(2)$ is DF_{loc}-equivalent to exactly one of the following full-rank systems:

$$\begin{aligned} \Sigma_1 &: E_1 + u_1 E_2 + u_2 E_3, \\ \Sigma_{2,\alpha} &: \alpha E_3 + u_1 E_1 + u_2 E_2. \end{aligned}$$

Here $\alpha > 0$ parametrizes a family of class representatives, each different value corresponding to a distinct non-equivalent representative.

Let Σ be an inhomogeneous system with trace $\Gamma = \sum a_i E_i + \langle \sum b_i E_i, \sum c_i E_i \rangle$. If $E^3(\Gamma^0) \neq \{0\}$, then $\Gamma = a'_1 E_1 + a'_2 E_2 + \langle b'_1 E_1 + b'_2 E_2, c'_1 E_1 + c'_2 E_2 + E_3 \rangle$. (Here E^3 denotes the corresponding element of the dual basis.) As $(b'_1)^2 + (b'_2)^2 \neq 0$, the equation

$$\begin{bmatrix} b'_2 & -b'_1 \\ b'_1 & b'_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} a'_2 \\ a'_1 \end{bmatrix}$$

has a unique solution (with $v_2 \neq 0$). Therefore

$$\psi = \begin{bmatrix} v_2 b_2' & v_2 b_1' & c_1' \\ -v_2 b_1' & v_2 b_2' & c_2' \\ 0 & 0 & 1 \end{bmatrix}$$

is an automorphism such that $\psi \cdot \Gamma_1 = \psi \cdot (E_1 + \langle E_2, E_3 \rangle) = \Gamma$. Thus Σ is DF_{loc}-equivalent to Σ_1. On the other hand, suppose $E^3(\Gamma^0) = \{0\}$. Then $\Gamma = a_3 E_3 + \langle E_1, E_2 \rangle$. Hence $\psi = \operatorname{diag}(1, 1, \operatorname{sgn}(a_3))$ is an automorphism such that $\psi \cdot \Gamma = \alpha E_3 + \langle E_1, E_2 \rangle$ with $\alpha > 0$. Thus Σ is DF_{loc}-equivalent to $\Sigma_{2,\alpha}$. As the subspace $\langle E_1, E_2 \rangle$ is invariant (under automorphisms), Σ_1 and $\Sigma_{2,\alpha}$ cannot be DF_{loc}-equivalent. It is easy to show that $\Sigma_{2,\alpha}$ and $\Sigma_{2,\alpha'}$ are DF_{loc}-equivalent only if $\alpha = \alpha'$.

Example 4 (see [10])**.** Any two-input homogeneous full-rank system on the pseudo-orthogonal group $\mathsf{SO}(2,1)$ is DF_{loc}-equivalent to exactly one of the following full-rank systems:

$$\begin{aligned} \Sigma_1 &: u_1 E_1 + u_2 E_2, \\ \Sigma_2 &: u_1 E_2 + u_2 E_3. \end{aligned}$$

Let Σ be a two-input homogeneous full-rank system with trace $\Gamma = \langle A, B \rangle$. The *sign* $\sigma(\Gamma)$ of Γ is given by

$$\sigma(\Gamma) = \operatorname{sgn}\left(\begin{vmatrix} A \odot A & A \odot B \\ A \odot B & B \odot B \end{vmatrix} \right).$$

(It is easy to show that $\sigma(\Gamma)$ does not depend on the parametrization.) As $\odot$ is preserved by automorphisms, it follows that $\sigma(\psi \cdot \Gamma) = \sigma(\Gamma)$. A straightforward computation shows that if $\sigma(\Gamma) = 0$, then Σ does not have full rank.

Suppose $\sigma(\Gamma) = -1$. Then we may assume that $a_3 \neq 0$. Hence $\Gamma = \langle a_1' E_1 + a_2' E_2 + E_3, r \sin\theta E_1 + r \cos\theta E_2 \rangle$. Thus

$$\psi = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is an automorphism such that $\psi \cdot \Gamma = \langle a_1'' E_1 + E_3, E_2 \rangle$. Now, as $\sigma(\psi \cdot \Gamma) = -1$, we have $(a_1'')^2 - 1 < 0$ and so $\psi \cdot \Gamma = \langle \sinh\vartheta E_1 + \cosh\vartheta E_3, E_2 \rangle$. Therefore

$$\psi' = \begin{bmatrix} \cosh\vartheta & 0 & -\sinh\vartheta \\ 0 & 1 & 0 \\ -\sinh\vartheta & 0 & \cosh\vartheta \end{bmatrix}$$

is an automorphism such that $\psi' \cdot \psi \cdot \Gamma = \langle E_3, E_2 \rangle$. Thus Σ is DF_{loc}-equivalent to Σ_1. If $\sigma(\Gamma) = 1$, then a similar argument shows that there exists an automorphism ψ such that $\psi \cdot \Gamma = \langle E_1, E_2 \rangle$ (and so Σ is DF_{loc}-equivalent to Σ_2). Lastly, as $\sigma(\Gamma_1) = 1$ and $\sigma(\Gamma_2) = -1$, it follows that Σ_1 and Σ_2 are not equivalent.

Example 5 (see [15])**.** The (four-dimensional) oscillator Lie group has parametrization

$$\mathsf{Osc}: \begin{bmatrix} 1 & -y\cos\theta - z\sin\theta & z\cos\theta - y\sin\theta & -2x \\ 0 & \cos\theta & \sin\theta & z \\ 0 & -\sin\theta & \cos\theta & y \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where $x, y, z, \theta \in \mathbb{R}$. Its Lie algebra likewise has parametrization

$$\mathfrak{osc}: \begin{bmatrix} 0 & -y & z & -2x \\ 0 & 0 & \theta & z \\ 0 & -\theta & 0 & y \\ 0 & 0 & 0 & 0 \end{bmatrix} = xE_1 + yE_2 + zE_3 + \theta E_4$$

where $x, y, z, \theta \in \mathbb{R}$. The nonzero commutator relations are $[E_2, E_3] = E_1$, $[E_2, E_4] = -E_3$, and $[E_3, E_4] = E_2$. Osc decomposes as a semidirect product $\mathsf{H}_3 \rtimes \mathsf{SO}(2)$ of the Heisenberg group H_3 and orthogonal group $\mathsf{SO}(2)$; furthermore, it is a nontrivial central extension of the Euclidean group $\mathsf{SE}(2)$ ([25]). The oscillator group was first studied by Streater [35]; it is associated with the harmonic oscillator problem, from whence it gets its name. This group (and its higher dimensional analogues) have been studied by several authors in both differential geometry and mathematical physics (see, e.g., [21, 23, 24, 31]).

Any homogeneous full-rank system on Osc is DF_{loc}-equivalent to exactly one of the following full rank systems:

$$\begin{aligned} \Sigma^{(2,0)} &: u_1E_2 + u_2E_4 \\ \Sigma_1^{(3,0)} &: u_1E_1 + u_2E_2 + u_3E_4 \\ \Sigma_2^{(3,0)} &: u_1E_2 + u_2E_3 + u_3E_4 \\ \Sigma^{(4,0)} &: u_1E_1 + u_2E_2 + u_3E_3 + u_4E_4. \end{aligned}$$

The group of automorphisms takes the form

$$\mathsf{Aut}(\mathfrak{osc}): \begin{bmatrix} \sigma\left(x^2+y^2\right) & wy - \sigma vx & -wx - \sigma vy & u \\ 0 & x & y & v \\ 0 & -\sigma y & \sigma x & w \\ 0 & 0 & 0 & \sigma \end{bmatrix}$$

where $x, y, u, v, w \in \mathbb{R}$, $x^2 + y^2 \neq 0$, and $\sigma = \pm 1$. Clearly no single-input homogeneous system has full rank. Suppose Σ is a two-input full-rank system with trace $\Gamma = \langle \sum a_iE_i, \sum b_iE_i \rangle$. As Σ has full rank, it follows that $E^4(\Gamma) \neq \{0\}$. Hence $\Gamma = \langle a_1'E_1 + a_2'E_2 + a_3'E_3 + E_4, b_1'E_1 + b_2'E_2 + b_3'E_3 \rangle$. Therefore,

$$\psi = \begin{bmatrix} 1 & a_2' & a_3' & -a_1 - (a_2')^2 - (a_3')^2 \\ 0 & 1 & 0 & -a_2' \\ 0 & 0 & 1 & -a_3' \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is an automorphism such that $\psi \cdot \Gamma = \langle E_4, b_1'' E_1 + r\cos\theta E_2 + r\sin\theta E_3\rangle$ with $r > 0$. (We have that $r \neq 0$ as Σ has full rank.) Accordingly,

$$\psi' = \begin{bmatrix} \frac{1}{r^2} & -\frac{b_1''\cos\theta}{r^3} & -\frac{b_1''\sin\theta}{r^3} & 0 \\ 0 & \frac{\cos\theta}{r} & \frac{\sin\theta}{r} & \frac{b_1''}{r^2} \\ 0 & -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is an automorphism such that $(\psi' \circ \psi) \cdot \Gamma = \left\langle E_2, \frac{b_1''}{r^2} E_2 + E_4 \right\rangle = \langle E_2, E_4\rangle$. Consequently Σ is DF_{loc}-equivalent to $\Sigma^{(2,0)}$. The three-input case is similar, although somewhat more involved. (The four-input case is trivial.)

Remark 4. The examples discussed in this note deal only with the local case. The approach for the global case is very similar; however, one needs to first determine the subgroup $d\,\mathsf{Aut}(\mathsf{G})$ of $\mathsf{Aut}(\mathfrak{g})$. For $\mathsf{SE}\,(2)$ and $\mathsf{SO}\,(2,1)_0$ it turns out that $d\,\mathsf{Aut}(\mathsf{G}) = \mathsf{Aut}(\mathfrak{g})$ (see, e.g., [16, 19]). For the oscillator group, this does not hold true.

5 Closing Remarks

State space equivalence is a very strong equivalence relation. Hence, any general classification leads to a large number of equivalence classes and so is of little use (except perhaps in low dimensions, e.g., [1, 19]). On the other hand, detached feedback equivalence is noticeably weaker, and so leads to far fewer equivalence classes. On three-dimensional Lie groups, a full classification (both local and global) of systems under detached feedback equivalence has been achieved ([8, 10–12], and [16]; see also [13, 17]). In the same vein, on several other low-dimensional (matrix) Lie groups, important classification results have also been obtained (cf. [2, 5, 15]).

Detached feedback equivalence has a natural extension to invariant optimal control problems (cf. [9, 14]). Two optimal control problems are *cost equivalent* if the underlying control systems are detached feedback equivalent and the change of controls φ is compatible with the costs. (Such a perspective was used to classify the corresponding sub-Riemannian structures on the Heisenberg groups [6]; see also [4, 5].)

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