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Interpolating Bézier spline surfaces with local control

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Abstract. This paper presents an approach to construct interpolating spline surfaces over a bivariate network of curves with rectangular patches. Patches of the interpolating spline surface are constructed by means of blending their boundaries with special polynomials. In order to ensure a necessary parametric continuity of the designed surface the polynomials of the corresponding degree are used. The constructed interpolating spline surfaces have local shape control. If the surface frame is determined by means of Bézier curves then patches of the interpolating spline surface are Bézier surfaces.

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1 Introduction

Interpolating spline surfaces play important role in different geometric applications. This paper presents an approach to construction of interpolating spline surfaces which have local shape control. Patches of the interpolating spline surface are constructed by means of blending their boundaries with special polynomials. In order to ensure a necessary parametric continuity of the constructed surface the polynomials of the corresponding degree must be used. The presented approach is aimed at construction of interpolating spline surface over the bivariate network of curves with rectangular patches. Interpolation with Bézier patches over the bivariate network of Bézier curves is considered as application of the general approach. A classification of algorithms for local smooth surface interpolation with piecewise polynomials is given in the paper of Peters [14]. A survey of blending methods that use parametric surfaces can be found in the paper of Vida, Martin, Várady [20]. Construction of surface patches by linear blending of its boundaries was firstly introduced by Coons [5]. Contemporary representation of the patches was given by Forrest [8] and considered by Faux and Pratt [7]. Spline-blended surface interpolation through curve networks was proposed by Gordon [10]. The presented approach can be considered as generalization of the techniques. Another approach to surface interpolation by means of linear blending is considered in the paper of Juhásza and Hoffmann [12].

2 Construction of a rectangular patch by blending its boundaries

Construction of a surface patch by means of linear blending of its boundaries was introduced by Coons [5]. The presented approach can be considered as generalization of the technique.

Consider four parametric curves $\mathbf{p}_0(u)$, $\mathbf{p}_1(u)$, $u \in [0,1]$, and $\mathbf{q}_0(v)$, $\mathbf{q}_1(v)$, $v \in [0,1]$, which have the following boundary points:

$$\mathbf{p}_0(0) = \mathbf{q}_0(0) = \mathbf{r}_{0,0}, \;\; \mathbf{p}_0(1) = \mathbf{q}_1(0) = \mathbf{r}_{1,0}, \tag{1}$$

$$\mathbf{p}_1(0) = \mathbf{q}_0(1) = \mathbf{r}_{0,1}, \;\; \mathbf{p}_1(1) = \mathbf{q}_1(1) = \mathbf{r}_{1,1}. \tag{2}$$

The problem is to construct a rectangular patch $\mathbf{r}(u,v)$, $(u,v) \in [0,1] \times [0,1]$, which has the considered parametric curves as boundaries, that is

$$\mathbf{r}(u,0) = \mathbf{p}_0(u), \;\; \mathbf{r}(u,1) = \mathbf{p}_1(u), \tag{3}$$

$$\mathbf{r}(0,v) = \mathbf{q}_0(v), \;\; \mathbf{r}(1,v) = \mathbf{q}_1(v) \tag{4}$$

and partial derivatives of the patch $\mathbf{r}(u,v)$ satisfy the following conditions at the corner points:

$$\frac{\partial^m \mathbf{r}(u,v)}{\partial u^m}(0,0) = (\mathbf{p}_0^{(m)}(u))(0), \;\; \frac{\partial^m \mathbf{r}(u,v)}{\partial v^m}(0,0) = (\mathbf{q}_0^{(m)}(v))(0), \tag{5}$$

$$\frac{\partial^m \mathbf{r}(u,v)}{\partial u^m}(0,1) = (\mathbf{p}_1^{(m)}(u))(0), \;\; \frac{\partial^m \mathbf{r}(u,v)}{\partial v^m}(0,1) = (\mathbf{q}_0^{(m)}(v))(1), \tag{6}$$

$$\frac{\partial^m \mathbf{r}(u,v)}{\partial u^m}(1,0) = (\mathbf{p}_0^{(m)}(u))(1), \;\; \frac{\partial^m \mathbf{r}(u,v)}{\partial v^m}(1,0) = (\mathbf{q}_1^{(m)}(v))(0), \tag{7}$$

$$\frac{\partial^m \mathbf{r}(u,v)}{\partial u^m}(1,1) = (\mathbf{p}_1^{(m)}(u))(1), \;\; \frac{\partial^m \mathbf{r}(u,v)}{\partial v^m}(1,1) = (\mathbf{q}_1^{(m)}(v))(1), \tag{8}$$

for all $m \in \{1, 2, ..., n\}$ where $s + r = m$ and $n \in \mathbb{N}$. In order to solve the problem define the following parametric surface:

$$\mathbf{r}(u,v) = \mathbf{s}(u,v) - \tilde{\mathbf{r}}(u,v), \;\; u,v \in [0,1], \tag{9}$$

where

$$\begin{gathered}\mathbf{s}(u,v) = (1 - w_{n+1}(v))\mathbf{p}_0(u) + w_{n+1}(v)\mathbf{p}_1(u) + \\ +(1 - w_{n+1}(u))\mathbf{q}_0(u) + w_{n+1}(u)\mathbf{q}_1(u),\end{gathered}$$

$$\begin{gathered}\tilde{\mathbf{r}}(u,v) = (1 - w_{n+1}(u))(1 - w_{n+1}(v))\mathbf{r}_{0,0} + w_{n+1}(u)(1 - w_{n+1}(v))\mathbf{r}_{1,0} + \\ +(1 - w_{n+1}(u))w_{n+1}(v)\mathbf{r}_{0,1} + w_{n+1}(u)w_{n+1}(v)\mathbf{r}_{1,1}\end{gathered}$$

and the polynomials $w_n(u)$ are defined as follows:

$$w_n(u) = \sum_{i=n}^{2n-1} b_{2n-1,i}(u), \;\; u \in [0,1],$$

where $b_{n,m}(u)$ denotes a Bernstein polynomial

$$b_{n,m}(u) = \frac{n!}{m!(n-m)!}(1-u)^{n-m}u^m,\ u \in [0,1].$$

Detailed considerations of the polynomials $w_n(u)$ can be found in the paper of Pobegailo [15] where it is shown that the polynomials have the following boundary values:

$$w_n(0) = 0,\ w_n(1) = 1 \tag{10}$$

and satisfy the following boundary conditions:

$$w_n^{(m)}(0) = w_n^{(m)}(1) = 0 \tag{11}$$

for all $m \in \{1, 2, ..., n-1\}$.

Show that the parametric curves $\mathbf{p}_0(u)$, $\mathbf{p}_1(u)$ and $\mathbf{q}_0(v)$, $\mathbf{q}_1(v)$ are boundaries of the patch $\mathbf{r}(u,v)$. Substitution of boundary values of the polynomials $w_{n+1}(u)$ from Equations (10) and parametric curves from Equations (1) and (2) in Equation (9) yields that

$$\mathbf{s}(u,0) = \mathbf{p}_0(u) + (1 - w_{n+1}(u))\mathbf{r}_{0,0} + w_{n+1}(u)\mathbf{r}_{1,0},$$

$$\tilde{\mathbf{r}}(u,0) = (1 - w_{n+1}(u))\mathbf{r}_{0,0} + w_{n+1}(u)\mathbf{r}_{1,0}$$

and therefore

$$\mathbf{r}(u,0) = \mathbf{s}(u,0) - \tilde{\mathbf{r}}(u,0) = \mathbf{p}_0(u).$$

Then

$$\mathbf{s}(0,v) = (1 - w_{n+1}(v))\mathbf{r}_{0,0} + w_{n+1}(v)\mathbf{r}_{0,1} + \mathbf{q}_0(v),$$

$$\tilde{\mathbf{r}}(0,v) = (1 - w_{n+1}(v))\mathbf{r}_{0,0} + w_{n+1}(v)\mathbf{r}_{0,1}$$

and therefore

$$\mathbf{r}(0,v) = \mathbf{s}(u,0) - \tilde{\mathbf{r}}(u,0) = \mathbf{q}_0(v).$$

Analogously it can be shown that

$$\mathbf{r}(u,1) = \mathbf{p}_1(u),\ \mathbf{r}(1,v) = \mathbf{q}_1(v).$$

Thus Equations (3) and (4) are fulfilled.

Show that the patch $\mathbf{r}(u,v)$ has necessary partial derivatives at the corner points, that is Equations (5-8) are also fulfilled. For this purpose compute partial derivatives of the parametric surface $\mathbf{r}(u,v)$. It is obtained that

$$\frac{\partial^m \mathbf{r}(u,v)}{\partial u^m} = \frac{\partial^m \mathbf{s}(u,v)}{\partial u^m} - \frac{\partial^m \tilde{\mathbf{r}}(u,v)}{\partial u^m}$$

where

$$\frac{\partial^m \mathbf{s}(u,v)}{\partial u^m} = (1 - w_{n+1}(v))\mathbf{p}_0^{(m)}(u) + w_{n+1}(v)\mathbf{p}_1^{(m)}(u) +$$

$$+(1-w_{n+1}(u))^{(m)}\mathbf{q}_0(v)+w_{n+1}^{(m)}(u)\mathbf{q}_1(v),$$

$$\frac{\partial^m\tilde{\mathbf{r}}(u,v)}{\partial u^m}=(1-w_{n+1}(u))^{(m)}(1-w_{n+1}(v))\mathbf{r}_{0,0}+w_{n+1}^{(m)}(u)(1-w_{n+1}(v))\mathbf{r}_{1,0}+$$
$$+(1-w_{n+1}(u))^{(m)}w_{n+1}(v)\mathbf{r}_{0,1}+w_{n+1}^{(m)}(u)w_{n+1}(v)\mathbf{r}_{1,1}$$

and analogously

$$\frac{\partial^m\mathbf{r}(u,v)}{\partial v^m}=\frac{\partial^m\mathbf{s}(u,v)}{\partial v^m}-\frac{\partial^m\tilde{\mathbf{r}}(u,v)}{\partial v^m}$$

where

$$\frac{\partial^m\mathbf{s}(u,v)}{\partial v^m}=(1-w_{n+1}(v))^{(m)}\mathbf{p}_0(u)+w_{n+1}^{(m)}(v)\mathbf{p}_1(u)+$$
$$+(1-w_{n+1}(u))\mathbf{q}_0^{(m)}(v)+w_{n+1}(u)\mathbf{q}_1^{(m)}(v),$$

$$\frac{\partial^m\tilde{\mathbf{r}}(u,v)}{\partial v^m}=(1-w_{n+1}(u))(1-w_{n+1}(v))^{(m)}\mathbf{r}_{0,0}+w_{n+1}(u)(1-w_{n+1}(v))^{(m)}\mathbf{r}_{1,0}+$$
$$+(1-w_{n+1}(u))w_{n+1}^{(m)}(v)\mathbf{r}_{0,1}+w_{n+1}(u)w_{n+1}^{(m)}(v)\mathbf{r}_{1,1}$$

for all $m\in\{1,2,...,n\}$, $n\in\mathbb{N}$. Substituting boundary values of the polynomials $w_{n+1}(u)$ and their derivatives from Equations (10) and (11) in these equations, it is obtained that

$$\frac{\partial^m\mathbf{r}(0,0)}{\partial u^m}=\frac{\partial^m\mathbf{s}(0,0)}{\partial u^m}-\frac{\partial^m\tilde{\mathbf{r}}(0,0)}{\partial u^m}=\mathbf{p}_0^{(m)}(u)$$

and

$$\frac{\partial^m\mathbf{r}(u,v)}{\partial v^m}=\frac{\partial^m\mathbf{s}(u,v)}{\partial v^m}-\frac{\partial^m\tilde{\mathbf{r}}(u,v)}{\partial v^m}=\mathbf{q}_0^{(m)}(v).$$

Thus Equations (5) are fulfilled. Analogously it can be proven that Equations (6)-(8) are also fulfilled.

Now compute mixed partial derivatives of the parametric surface $\mathbf{r}(u,v)$ at the corner points. It is obtained that

$$\frac{\partial^m\mathbf{r}(u,v)}{\partial u^r\partial v^s}=\frac{\partial^m\mathbf{s}(u,v)}{\partial u^r\partial v^s}-\frac{\partial^m\tilde{\mathbf{r}}(u,v)}{\partial u^r\partial v^s}$$

where

$$\frac{\partial^m\mathbf{s}(u,v)}{\partial u^r\partial v^s}=(1-w_{n+1}(v))^{(s)}\mathbf{p}_0^{(r)}(u)+w_{n+1}^{(s)}(v)\mathbf{p}_1^{(r)}(u)+$$
$$+(1-w_{n+1}(u))^{(r)}\mathbf{q}_0^{(s)}(v)+w_{n+1}^{(r)}(u)\mathbf{q}_1^{(s)}(v),$$

$$\frac{\partial^m\tilde{\mathbf{r}}(u,v)}{\partial u^r\partial v^s}=(1-w_{n+1}(u))^{(r)}(1-w_{n+1}(v))^{(s)}\mathbf{r}_{0,0}+w_{n+1}^{(r)}(u)(1-w_{n+1}(v))^{(s)}\mathbf{r}_{1,0}+$$
$$+(1-w_{n+1}(u))^{(r)}w_{n+1}^{(s)}(v)\mathbf{r}_{0,1}+w_{n+1}^{(r)}(u)w_{n+1}^{(s)}(v)\mathbf{r}_{1,1}$$

for all $m \in \{1, 2, ..., n\}$ where $s+r = m$ and $n \in \mathbb{N}$. Substituting values of derivatives which are defined by Equations (11) in these equations, it is obtained that

$$\frac{\partial^m \mathbf{s}(u,v)}{\partial u^r \partial v^s} = 0, \ \frac{\partial^m \tilde{\mathbf{r}}(u,v)}{\partial u^m} = 0$$

and therefore

$$\frac{\partial^m \mathbf{r}(0,0)}{\partial u^r \partial v^s} = \frac{\partial^m \mathbf{s}(0,0)}{\partial u^r \partial v^s} - \frac{\partial^m \tilde{\mathbf{r}}(0,0)}{\partial u^r \partial v^s} = 0.$$

Analogously it can be proven that the other mixed partial derivatives at the corners of the patch $\mathbf{r}(u,v)$ are also equal to zero. Thus it is obtained that

$$\frac{\partial^m \mathbf{r}(u,v)}{\partial u^r \partial v^s}(0,0) = \frac{\partial^m \mathbf{r}(u,v)}{\partial u^r \partial v^s}(0,1) = \frac{\partial^m \mathbf{r}(u,v)}{\partial u^r \partial v^s}(1,0) = \frac{\partial^m \mathbf{r}(u,v)}{\partial u^r \partial v^s}(1,1) = 0 \tag{12}$$

for all $m \in \{1, 2, ..., n\}$ where $s + r = m$ and $n \in \mathbb{N}$. These values of mixed partial derivatives are natural because the patch $\mathbf{r}(u,v)$ is defined only by the boundary curves.

3 Construction of spline surfaces by blending frame curves

Spline-blended surface interpolation through curve networks was proposed by Gordon [10]. Then Gregory [11] introduced a smooth interpolation scheme without twist constraints. G^1 smoothness conditions for rectangular and triangular Gregory patches are discussed by Farin and Hansford [6]. Another approach to surface interpolation of control point mesh was proposed by Comninos [4]. The surface is generated by piecewise bicubic interpolation and is derived from a classical Coons patch. This paper presents an approach to interpolating bivariate network of curves by means of patches which are constructed by blending frame curves. The presented approach provides C^n continuity of the constructed surface. Another approach to surface interpolation by means of linear blending is considered in the paper of Juhásza and Hoffmann [12].

Consider a rectangular grid of points $\mathbf{r}_{i,j}$, $i \in \{0, 1, ..., k\}$, $j \in \{0, 1, ..., l\}$, $k, l \in \mathbb{N}$. Suppose that the rectangular grid is framed by parametric curves $\mathbf{p}_{i,j}(u)$, $u \in [0, 1]$, and $\mathbf{q}_{i,j}(v)$, $v \in [0, 1]$, where $i \in \{0, 1, ..., k-1\}$, $j \in \{0, 1, ..., l-1\}$, which satisfy the following boundary conditions:

$$\begin{aligned} \mathbf{p}_{i,j}(0) &= \mathbf{q}_{i,j}(0) = \mathbf{r}_{i,j}, \\ \mathbf{p}_{i,j}(1) &= \mathbf{p}_{i+1,j}(0) = \mathbf{r}_{i+1,j}, \\ \mathbf{q}_{i,j}(1) &= \mathbf{q}_{i,j+1}(0) = \mathbf{r}_{i,j+1}. \end{aligned} \tag{13}$$

Besides the considered parametric curves are C^n continuously joined at the common grid points, that is

$$(\mathbf{p}_{i,j}^{(m)}(u))(1) = (\mathbf{p}_{i+1,j}^{(m)}(u))(0), \ (\mathbf{q}_{i,j}^{(m)}(v))(1) = (\mathbf{q}_{i,j+1}^{(m)}(v))(0) \tag{14}$$

for all $m \in \{1, 2, ..., n\}$, $n \in \mathbb{N}$. The problem is to construct a C^n continuous parametric surface $\mathbf{r}(u, v)$ which interpolates the points of this grid and the parametric curves $\mathbf{p}_{i,j}(u)$ and $\mathbf{q}_{i,j}(v)$ are boundaries of rectangular patches which form the surface. Using Equation (9) define rectangular patches of the surface $\mathbf{r}(u, v)$ as follows:

$$\mathbf{r}_{i,j}(u, v) = \mathbf{s}_{i,j}(u, v) - \tilde{\mathbf{r}}_{i,j}(u, v), \ (u, v) \in [0, 1] \times [0, 1], \tag{15}$$

where

$$\begin{aligned}\mathbf{s}_{i,j}(u, v) = (1 - w_{n+1}(v))\mathbf{p}_{i,j}(u) + w_{n+1}(v)\mathbf{p}_{i,j+1}(u) + \\ +(1 - w_{n+1}(u))\mathbf{q}_{i,j}(u) + w_{n+1}(u)\mathbf{q}_{i+1,j}(u),\end{aligned}$$

$$\begin{aligned}\tilde{\mathbf{r}}_{i,j}(u, v) = (1 - w_{n+1}(u))(1 - w_{n+1}(v))\mathbf{r}_{i,j} + w_{n+1}(u)(1 - w_{n+1}(v))\mathbf{r}_{i+1,j} + \\ +(1 - w_{n+1}(u))w_{n+1}(v)\mathbf{r}_{i,j+1} + w_{n+1}(u)w_{n+1}(v)\mathbf{r}_{i+1,j+1}\end{aligned}$$

for all $i \in \{0, 1, ..., k-1\}$, $j \in \{0, 1, ..., l-1\}$. It follows from Equations (10) and (13) that the parametric curves $\mathbf{p}_{i,j}(u)$, $\mathbf{p}_{i,j+1}(u)$, $\mathbf{q}_{i,j}(v)$ and $\mathbf{q}_{i+1,j}(v)$ are boundaries of the patch $\mathbf{r}_{i,j}(u, v)$.

Show that the parametric surface $\mathbf{r}(u, v)$ is C^n continuous. The surface $\mathbf{r}(u, v)$ is C^n continuous at the knot points $\mathbf{r}_{i,j}$ because the frame curves are C^n continuous at the knot points and taking into consideration Equations (12). Then it is necessary to show that the patches of the parametric surface $\mathbf{r}(u, v)$ are smoothly joined along their common boundaries. For this purpose compute partial derivatives of the adjustment patches along their common boundaries. It is obtained taking into account Equations (14) that

$$\begin{aligned}\frac{\partial^m \mathbf{r}_{i,j}(u, v)}{\partial u^m}(1, v) = (1 - w_{n+1}(v))(\mathbf{p}_{i,j}^{(m)}(u))(1) + w_{n+1}(v)(\mathbf{p}_{i,j+1}^{(m)}(u))(1) = \\ = (1 - w_{n+1}(v))(\mathbf{p}_{i+1,j}^{(m)}(u))(0) + w_{n+1}(v)(\mathbf{p}_{i+1,j+1}^{(m)}(u))(0) = \frac{\partial^m \mathbf{r}_{i+1,j}(u, v)}{\partial u^m}(0, v)\end{aligned}$$

and analogously

$$\frac{\partial^m \mathbf{r}_{i,j}(u, v)}{\partial v^m}(u, 1) = \frac{\partial^m \mathbf{r}_{i,j+1}(u, v)}{\partial v^m}(u, 0)$$

for all $m \in \{1, 2, ..., n\}$. Now determine mixed partial derivatives across boundaries of the patches. It is obtained using Equation (14) that

$$\begin{aligned}\frac{\partial^m \mathbf{r}_{i,j}(u, v)}{\partial u^r \partial v^s}(1, v) = (1 - w_{n+1}(v))^{(s)}(\mathbf{p}_{i,j}^{(r)}(u))(1) + w_{n+1}^{(s)}(v)(\mathbf{p}_{i,j+1}^{(m)}(u))(1) = \\ = (1 - w_{n+1}(v))^{(s)}(\mathbf{p}_{i+1,j}^{(m)}(u))(0) + w_{n+1}^{(s)}(v)(\mathbf{p}_{i+1,j+1}^{(m)}(u))(0) = \frac{\partial^m \mathbf{r}_{i,j}(u, v)}{\partial u^r \partial v^s}(0, v)\end{aligned}$$

and analogously

$$\frac{\partial^m \mathbf{r}_{i,j}(u, v)}{\partial u^r \partial v^s}(u, 1) = \frac{\partial^m \mathbf{r}_{i,j+1}(u, v)}{\partial u^r \partial v^s}(u, 0)$$

for all $m \in \{1, 2, ..., n\}$ where $s + r = m$. Thus the spline surface $\mathbf{r}(u, v)$ constructed by means of Equation (19) is C^n continuous.

It is obvious that a shape of the interpolating surface constructed by the proposed method is mainly dependent on boundary curves of the patches. But two features of the interpolating surface shape which are common for all surfaces constructed by the approach can be mentioned.

Firstly it follows from Equations (12) that the twist vector $\mathbf{r}_{u,v}$ is equal to zero at all knot points of the interpolating spline surface. Therefore the proposed method can lead to local flattening of the generated surface near patch corners. There are more elaborated methods which use geometric specifications along the patch boundaries and at the corner points or the surface can be constructed with optimal twist vectors as a tool for interpolating a network of curves with a minimum energy surface, for example, see the paper of Kallay and Ravani [13]. But these methods can be used only for offline processing because it is difficult to adjust additional geometric specifications or global computation procedures for online data point processing. The old problem of specifying the mixed partial derivatives or twist vectors at the grid points for an interpolating surface over a rectangular network of curves is considered in detail by Barnhill, Brown, Klucewicz [1]; Faux, Pratt [7]; Barnhill, Farin, Fayard, Hagen [2].

Secondly it follows from the extremum property of the polynomials $w_n(u)$ that patches of interpolating surfaces are generated with energy minimizing polynomials. It can be seen from profiles of the polynomials that the higher degree of continuity of the interpolating surface the shape of patches closes to shape of frame curves at knot points and an inflection of the shape moves from knot points to a parametric center of the patch.

4 Interpolating Bézier spline surfaces with local control

A translation of the Gordon scheme into a Bézier-like form was carried out by Chiyokura and Kimura [3]. Local surface interpolation with Bézier patches for meshes of cubic curves is described by Shirman, Sequin [18-19]. The method is local and provides G^1 continuity between patches. In this section construction of spline surfaces using blending of Bézier frame curves is presented.

Suppose that frame curves of a rectangular grid are constructed by means of Cn continuous spline Bézier curves. Since the proposed approach is aimed at local interpolation of the framed grid, the Bézier curves must also have a local control. In order to ensure this property Bézier curves, which are segments of the curve net, must have at least 2n+1 order. Such a net of spline curves can be constructed by the approach considered in the paper of Pobegailo [17]. In this case boundaries of the patch $\mathbf{r}_{i,j}(u,v)$ can be described by the following Bézier curves:

$$\begin{aligned}\mathbf{p}_{i,j}(u) &= \sum_{k=0}^{2n+1} b_{2n+1,k}(u)\mathbf{p}_{i,j,k},\ u \in [0,1],\\ \mathbf{q}_{i,j}(v) &= \sum_{l=0}^{2n+1} b_{2n+1,l}(v)\mathbf{q}_{i,j,l},\ v \in [0,1]\end{aligned} \tag{16}$$

where boundary points of the Bézier curves $\mathbf{p}_{i,j}(u)$ and $\mathbf{q}_{i,j}(v)$ are knot points of the grid, that is

$$\begin{aligned}\mathbf{p}_{i,j}(0) = \mathbf{p}_{i,j,0} = \mathbf{r}_{i,j},\ \mathbf{p}_{i,j}(1) = \mathbf{p}_{i,j,2n+1} = \mathbf{r}_{i+1,j},\\ \mathbf{q}_{i,j}(0) = \mathbf{q}_{i,j,0} = \mathbf{r}_{i,j},\ \mathbf{q}_{i,j}(1) = \mathbf{q}_{i,j,2n+1} = \mathbf{r}_{i,j+1}.\end{aligned}$$

Then the patch $\mathbf{r}_{i,j}(u,v)$ can be described using Equations (15) and (16) as follows:

$$\begin{gathered}\mathbf{r}_{i,j}(u,v) = \mathbf{s}_{i,j}(u,v) - \tilde{\mathbf{r}}_{i,j}(u,v) =\\ = \sum_{l=0}^{n} b_{2n+1,l}(v) \sum_{k=0}^{2n+1} b_{2n+1,k}(u)\mathbf{p}_{i,j,k} + \sum_{l=n+1}^{2n+1} b_{2n+1,l}(v) \sum_{k=0}^{2n+1} b_{2n+1,k}(u)\mathbf{p}_{i,j+1,k} +\\ + \sum_{k=0}^{n} b_{2n+1,k}(u) \sum_{l=0}^{2n+1} b_{2n+1,l}(v)\mathbf{q}_{i,j,l} + \sum_{k=n+1}^{2n+1} b_{2n+1,k}(u) \sum_{l=0}^{2n+1} b_{2n+1,l}(v)\mathbf{q}_{i+1,j,l} -\\ - \sum_{k=0}^{n} b_{2n+1,k}(u) \sum_{l=0}^{n} b_{2n+1,l}(v)\mathbf{r}_{i,j} - \sum_{k=n+1}^{2n+1} b_{2n+1,k}(u) \sum_{l=0}^{n} b_{2n+1,l}(v)\mathbf{r}_{i+1,j} -\\ - \sum_{k=0}^{n} b_{2n+1,k}(u) \sum_{l=n+1}^{2n+1} b_{2n+1,l}(v)\mathbf{r}_{i,j+1} - \sum_{k=n+1}^{2n+1} b_{2n+1,k}(u) \sum_{l=n+1}^{2n+1} b_{2n+1,l}(v)\mathbf{r}_{i+1,j+1}.\end{gathered}$$

Combination of the similar terms yields that

$$\begin{aligned}\mathbf{r}_{i,j}(u,v) = &\sum_{k=0}^{n} b_{2n+1,k}(u) \sum_{l=0}^{n} b_{2n+1,l}(v)(\mathbf{p}_{i,j,k} + \mathbf{q}_{i,j,l} - \mathbf{r}_{i,j}) +\\ &+ \sum_{k=0}^{n} b_{2n+1,k}(u) \sum_{l=n+1}^{2n+1} b_{2n+1,l}(v)(\mathbf{p}_{i,j+1,k} + \mathbf{q}_{i,j,l} - \mathbf{r}_{i,j+1}) +\\ &\sum_{k=n+1}^{2n+1} b_{2n+1,k}(u) \sum_{l=0}^{n} b_{2n+1,l}(v)(\mathbf{p}_{i,j,k} + \mathbf{q}_{i+1,j,l} - \mathbf{r}_{i+1,j}) +\\ &\sum_{k=n+1}^{2n+1} b_{2n+1,k}(u) \sum_{l=n+1}^{2n+1} b_{2n+1,l}(v)(\mathbf{p}_{i,j+1,k} + \mathbf{q}_{i+1,j,l} - \mathbf{r}_{i+1,j+1}).\end{aligned}$$

This is a Bézier representation of the patch $\mathbf{r}_{i,j}(u,v)$ for a C^n continuous Bézier spline surface. It can be seen from the last equation that the knot and control points of the Bézier patch $\mathbf{r}_{i,j}(u,v)$ can be arranged in a square block matrix

$$\mathbf{P}_{k,l} = \begin{bmatrix} \mathbf{B}_{0,0} & \mathbf{B}_{0,1} \\ \mathbf{B}_{1,0} & \mathbf{B}_{1,1} \end{bmatrix}.$$

where every internal block corresponds to a term of the patch equation.

In geometric applications surfaces of C^1 and C^2 continuity are usually used. A patch $\mathbf{r}_{i,j}(u,v)$ of the C^1 continuous surface has the following Bézier representations:

$$\mathbf{r}_{i,j}(u,v) = \sum_{k=0}^{3} b_{2n+1,k}(u) \sum_{l=0}^{3} b_{2n+1,l}(v)\mathbf{p}_{k,l}$$

where points $\mathbf{p}_{k,l}$ are corresponding elements of the following matrix:

$$\mathbf{P}_{k,l} = \begin{bmatrix} \mathbf{r}_{i,j} & \mathbf{q}_{i,j,1} & \mathbf{q}_{i,j,2} & \mathbf{r}_{i,j+1} \\ \mathbf{p}_{i,j,1} & \mathbf{p}_{i,j,1} + \mathbf{q}_{i,j,1} - \mathbf{r}_{i,j} & \mathbf{p}_{i,j+1,1} + \mathbf{q}_{i,j,2} - \mathbf{r}_{i,j+1} & \mathbf{p}_{i,j+1,1} \\ \mathbf{p}_{i,j,2} & \mathbf{p}_{i,j,2} + \mathbf{q}_{i+1,j,1} - \mathbf{r}_{i+1,j} & \mathbf{p}_{i,j+1,2} + \mathbf{q}_{i+1,j,2} - \mathbf{r}_{i+1,j+1} & \mathbf{p}_{i,j+1,2} \\ \mathbf{r}_{i+1,j} & \mathbf{q}_{i+1,j,1} & \mathbf{q}_{i+1,j,2} & \mathbf{r}_{i+1,j+1} \end{bmatrix}.$$

A patch $\mathbf{r}_{i,j}(u,v)$ of the C^2 continuous surface has the following Bézier representation:

$$\mathbf{r}_{i,j}(u,v) = \sum_{k=0}^{5} b_{2n+1,k}(u) \sum_{l=0}^{5} b_{2n+1,l}(v)\mathbf{p}_{k,l}$$

where points $\mathbf{p}_{k,l}$ are corresponding elements of the following matrix blocks:

$$\mathbf{B}_{0,0} = \begin{bmatrix} \mathbf{r}_{i,j} & \mathbf{q}_{i,j,1} & \mathbf{q}_{i,j,2} \\ \mathbf{p}_{i,j,1} & \mathbf{p}_{i,j,1} + \mathbf{q}_{i,j,1} - \mathbf{r}_{i,j} & \mathbf{p}_{i,j,1} + \mathbf{q}_{i,j,2} - \mathbf{r}_{i,j} \\ \mathbf{p}_{i,j,2} & \mathbf{p}_{i,j,2} + \mathbf{q}_{i,j,1} - \mathbf{r}_{i,j} & \mathbf{p}_{i,j,2} + \mathbf{q}_{i,j,2} - \mathbf{r}_{i,j} \end{bmatrix},$$

$$\mathbf{B}_{0,1} = \begin{bmatrix} \mathbf{q}_{i,j,3} & \mathbf{q}_{i,j,4} & \mathbf{r}_{i,j+1} \\ \mathbf{p}_{i,j+1,1} + \mathbf{q}_{i,j,3} - \mathbf{r}_{i,j+1} & \mathbf{p}_{i,j+1,1} + \mathbf{q}_{i,j,4} - \mathbf{r}_{i,j+1} & \mathbf{p}_{i,j+1,1} \\ \mathbf{p}_{i,j+1,2} + \mathbf{q}_{i,j,3} - \mathbf{r}_{i,j+1} & \mathbf{p}_{i,j+1,2} + \mathbf{q}_{i,j,4} - \mathbf{r}_{i,j+1} & \mathbf{p}_{i,j+1,2} \end{bmatrix},$$

$$\mathbf{B}_{1,0} = \begin{bmatrix} \mathbf{p}_{i,j,3} & \mathbf{p}_{i,j,3} + \mathbf{q}_{i,j,1} - \mathbf{r}_{i+1,j} & \mathbf{p}_{i,j,3} + \mathbf{q}_{i,j,2} - \mathbf{r}_{i+1,j} \\ \mathbf{p}_{i,j,4} & \mathbf{p}_{i,j,4} + \mathbf{q}_{i+1,j,1} - \mathbf{r}_{i+1,j} & \mathbf{p}_{i,j,4} + \mathbf{q}_{i+1,j,2} - \mathbf{r}_{i+1,j} \\ \mathbf{r}_{i+1,j} & \mathbf{q}_{i+1,j,1} & \mathbf{q}_{i+1,j,2} \end{bmatrix},$$

$$\mathbf{B}_{1,1} = \begin{bmatrix} \mathbf{p}_{i,j+1,3} + \mathbf{q}_{i+1,j,3} - \mathbf{r}_{i+1,j+1} & \mathbf{p}_{i,j+1,3} + \mathbf{q}_{i+1,j,4} - \mathbf{r}_{i+1,j+1} & \mathbf{p}_{i,j+1,3} \\ \mathbf{p}_{i,j+1,4} + \mathbf{q}_{i+1,j,3} - \mathbf{r}_{i+1,j+1} & \mathbf{p}_{i,j+1,4} + \mathbf{q}_{i+1,j,4} - \mathbf{r}_{i+1,j+1} & \mathbf{p}_{i,j+1,4} \\ \mathbf{q}_{i+1,j,3} & \mathbf{q}_{i+1,j,4} & \mathbf{r}_{i+1,j+1} \end{bmatrix}.$$

5 Rational Bézier spline surfaces with local control

Now suppose that frame curves of the rectangular grid are constructed by means of C^n continuous rational spline Bézier curves with a local shape control. In order to ensure this property rational Bézier curves, which are segments of the net, must have at least 2n+1 order. In this case boundaries of the patch $\mathbf{r}_{i,j}(u,v)$ can be described by the following rational Bézier curves:

$$\mathbf{p}_{i,j}(u) = \frac{\sum\limits_{k=0}^{2n+1} b_{2n+1,k}(u) w_{i,j,k} \mathbf{p}_{i,j,k}}{\sum\limits_{k=0}^{2n+1} b_{2n+1,k}(u) w_{i,j,k}}, \ u \in [0,1],$$

$$\mathbf{q}_{i,j}(v) = \frac{\sum\limits_{l=0}^{2n+1} b_{2n+1,l}(u) w_{i,j,l} \mathbf{q}_{i,j,l}}{\sum\limits_{l=0}^{2n+1} b_{2n+1,l}(u) w_{i,j,l}}, \ v \in [0,1]$$

where boundary points of the rational Bézier curves $\mathbf{p}_{i,j}(u)$ and $\mathbf{q}_{i,j}(u)$ are knot points of the grid. Such a net of spline curves can be constructed by the approach considered in the paper of Pobegailo [16]. Introduce the following denotations for numerators and denominators of the rational Bézier curves $\mathbf{p}_{i,j}(u)$ and $\mathbf{q}_{i,j}(u)$:

$$\mathbf{P}_{i,j}(u) = \sum_{k=0}^{2n+1} b_{2n+1,k}(u) w_{i,j,k} \mathbf{p}_{i,j,k}, \ u \in [0,1],$$

$$\mathbf{Q}_{i,j}(v) = \sum_{l=0}^{2n+1} b_{2n+1,l}(v) w_{i,j,l} \mathbf{q}_{i,j,l}, \ v \in [0,1],$$

$$P_{i,j}(u) = \sum_{k=0}^{2n+1} b_{2n+1,k}(u) w_{i,j,k}, \ u \in [0,1],$$

$$Q_{i,j}(v) = \sum_{l=0}^{2n+1} b_{2n+1,l}(v) w_{i,j,l}, \ v \in [0,1].$$

Then by analogy with non-rational case, see Equation (15), define the following rational patches:

$$\mathbf{r}_{i,j}(u,v) = \frac{\mathbf{S}_{i,j}(u,v) - \tilde{\mathbf{R}}_{i,j}(u,v)}{S_{i,j}(u,v) - \tilde{R}_{i,j}(u,v)}, \ (u,v) \in [0,1] \times [0,1]$$

where

$$\mathbf{S}_{i,j}(u,v) = (1 - w_{n+1}(v)) \mathbf{P}_{i,j}(u) + w_{n+1}(v) \mathbf{P}_{i,j+1}(u) +$$
$$+ (1 - w_{n+1}(u)) \mathbf{Q}_{i,j}(u) + w_{n+1}(u) \mathbf{Q}_{i+1,j}(u),$$

$$\begin{gathered}\tilde{\mathbf{R}}_{i,j}(u,v) = (1 - w_{n+1}(u))(1 - w_{n+1}(v))w_{i,j}\mathbf{r}_{i,j}+\\ w_{n+1}(u)(1 - w_{n+1}(v))w_{i+1,j}\mathbf{r}_{i+1,j}+\\ +(1 - w_{n+1}(u))w_{n+1}(v)w_{i,j+1}\mathbf{r}_{i,j+1}+\\ w_{n+1}(u)w_{n+1}(v)w_{i+1,j+1}\mathbf{r}_{i+1,j+1},\end{gathered}$$

$$\begin{gathered}S_{i,j}(u,v) = (1 - w_{n+1}(v))P_{i,j}(u) + w_{n+1}(v)P_{i,j+1}(u)+\\ +(1 - w_{n+1}(u))Q_{i,j}(u) + w_{n+1}(u)Q_{i+1,j}(u),\end{gathered}$$

$$\begin{gathered}\tilde{R}_{i,j}(u,v) = (1 - w_{n+1}(u))(1 - w_{n+1}(v))w_{i,j}+\\ w_{n+1}(u)(1 - w_{n+1}(v))w_{i+1,j}+\\ +(1 - w_{n+1}(u))w_{n+1}(v)w_{i,j+1}+\\ w_{n+1}(u)w_{n+1}(v)w_{i+1,j+1}\end{gathered}$$

for all $i \in \{0, 1, ..., k-1\}$, $j \in \{0, 1, ..., l-1\}$. By transition to homogeneous coordinates and using Grassmann algebra of weighted points, see Goldman [9], it can be proven that the constructed rational spline surfaces are C^n continuous. It should be noted that rational spline surface provides more opportunities for modification of its shape by changing weights of knot points.

6 Conclusions

The approach to construction of C^n continuous interpolating spline surfaces by means of blending boundaries of the surface patches is introduced. The considered spline surfaces are constructed locally over bivariate networks of curves. This approach ensures local control of the interpolating surface shape. If the surface frame is determined by means of Bézier curves then patches of the interpolating spline surface are represented by Bézier surfaces. General properties of the interpolating surface shape are considered. The proposed approach can be used for sketching and fast prototyping of spline surfaces in geometric design. Besides local control of the constructed interpolating surfaces makes the approach useful in on-line geometric applications.

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