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Inclusion Properties of Certain Subclass of Univalent Meromorphic Functions Defined by a Linear Operator Associated with the λ-Generalized Hurwitz-Lerch Zeta Function

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Abstract. By using a linear operator associated with the λ-generalized Hurwitz-Lerch zeta function, the authors introduce and investigate several properties of a certain subclass of meromorphically univalent functions in the open unit disk, which is defined here by means of the Hadamard product (or convolution).

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1 Main remarks

Let Σ denote the class of meromorphic functions $f(z)$ normalized by

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the punctured unit disk

$$\mathbb{U}^* = \{z : z \in \mathbb{C} \quad \text{and} \quad 0 < |z| < 1\} = \mathbb{U} \setminus \{0\},$$

$\mathbb{C}$ being (as usual) the set of complex numbers. We denote by $\Sigma\mathcal{S}^*(\beta)$ and $\Sigma\mathcal{K}(\beta)$ ($\beta \geqq 0$) the subclasses of Σ consisting of all meromorphic functions which are, respectively, starlike of order β and convex of order β in $\mathbb{U}^*$ (see also the recent works [1] and [2]).

For functions $f_j(z)$ $(j = 1, 2)$ defined by

$$f_j(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_{k,j} z^k \qquad (j = 1, 2), \tag{2}$$

we denote the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$(f_1 * f_2)(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_{k,1} a_{k,2} z^k. \tag{3}$$

Let us consider the function $\widetilde{\phi}(\alpha,\beta;z)$ defined by

$$\widetilde{\phi}(\alpha,\beta;z)=\frac{1}{z}+\sum_{k=0}^{\infty}\frac{(\alpha)_{k+1}}{(\beta)_{k+1}}a_k z^k \tag{4}$$

$$\left(\beta\in\mathbb{C}\setminus\mathbb{Z}_0^-;\ \alpha\in\mathbb{C}\right),$$

where

$$\mathbb{Z}_0^-=\{0,-1,-2,\cdots\}=\mathbb{Z}^-\cup\{0\}.$$

Here, and in the remainder of this paper, $(\lambda)_\kappa$ denotes the general Pochhammer symbol defined, in terms of the Gamma function, by

$$(\lambda)_\kappa:=\frac{\Gamma(\lambda+\kappa)}{\Gamma(\lambda)}=\begin{cases}\lambda(\lambda+1)\cdots(\lambda+n-1) & (\kappa=n\in\mathbb{N};\ \lambda\in\mathbb{C})\\ 1 & (\kappa=0;\ \lambda\in\mathbb{C}\setminus\{0\}),\end{cases} \tag{5}$$

it being understood *conventionally* that $(0)_0:=1$ and assumed *tacitly* that the Γ-quotient exists (see, for details,[3, p. 21 *et seq.*]), $\mathbb{N}$ being the set of positive integers.

It is easy to see that, in the case when $a_k=1$ $(k=0,1,2,\cdots)$, the following relationship holds true between the function $\widetilde{\phi}(\alpha,\beta;z)$ and the Gaussian hypergeometric function [4]:

$$\widetilde{\phi}(\alpha,\beta;z)=\frac{1}{z}\,{}_2F_1(1,\alpha;\beta;z). \tag{6}$$

Recently, Ghanim ([5]; see also [6] and [7]) made use of the Hadamard product for functions $f(z)\in\Sigma$ in order to introduce a new linear operator $L_a^s(\alpha,\beta)$, which is defined on Σ by

$$\begin{aligned}L_a^s(\alpha,\beta)(f)(z)&=\widetilde{\phi}(\alpha,\beta;z)*G_{s,a}(z)\\&=\frac{1}{z}+\sum_{k=1}^{\infty}\frac{(\alpha)_{n+1}}{(\beta)_{n+1}}\left(\frac{a+1}{a+k}\right)^s a_k z^k \quad \left(z\in\mathbb{U}^*\right),\end{aligned} \tag{7}$$

where

$$\begin{aligned}G_{s,a}(z)&:=(a+1)^s\left[\Phi(z,s,a)-a^s+\frac{1}{z(a+1)^s}\right]\\&=\frac{1}{z}+\sum_{k=1}^{\infty}\left(\frac{a+1}{a+k}\right)^s z^k \qquad \left(z\in\mathbb{U}^*\right)\end{aligned} \tag{8}$$

and the function $\Phi(z,s,a)$ is the well-known Hurwitz-Lerch zeta function defined by (see, for example,[8, p. 121 *et seq.*]; see also [9] and [10, p. 194 *et seq.*])

$$\Phi(z,s,a):=\sum_{n=0}^{\infty}\frac{z^n}{(n+a)^s} \tag{9}$$

$$(a \in \mathbb{C} \setminus \mathbb{Z}_0^-; \; s \in \mathbb{C} \quad \text{when} \quad |z| < 1; \; \Re(s) > 1 \quad \text{when} \quad |z| = 1).$$

We recall that the following new family of the λ-generalized Hurwitz-Lerch zeta functions was introduced and investigated systematically by Srivastava [11] (see also [12–16]):

$$\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,a;b,\lambda) = \frac{1}{\lambda\,\Gamma(s)}$$

$$\cdot \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (\lambda_j)_{n\rho_j}}{(a+n)^s \cdot \prod_{j=1}^{q} (\mu_j)_{n\sigma_j}} \, H^{2,0}_{0,2}\left[(a+n)b^{\frac{1}{\lambda}} \,\middle|\, \begin{matrix} \overline{\qquad\qquad} \\ (s,1),\left(0,\frac{1}{\lambda}\right) \end{matrix}\right] \frac{z^n}{n!} \tag{10}$$

$$\big(\min\{\Re(a),\Re(s)\} > 0; \; \Re(b) > 0; \; \lambda > 0\big)$$

$$\Big(\lambda_j \in \mathbb{C} \;\; (j=1,\cdots,p) \quad \text{and} \quad \mu_j \in \mathbb{C} \setminus \mathbb{Z}_0^- \;\; (j=1,\cdots,q); \; \rho_j > 0 \;\; (j=1,\cdots,p);$$

$$\sigma_j > 0 \;\; (j=1,\cdots,q); \; 1 + \sum_{j=1}^{q} \sigma_j - \sum_{j=1}^{p} \rho_j \geqq 0\Big),$$

where the equality in the convergence condition holds true for suitably bounded values of $|z|$ given by

$$|z| < \nabla := \left(\prod_{j=1}^{p} \rho_j^{-\rho_j}\right) \cdot \left(\prod_{j=1}^{q} \sigma_j^{\sigma_j}\right).$$

Definition 1. The H-function involved in the right-hand side of (10) is the well-known Fox's H-function [17, Definition 1.1] (see also [3, 18]) defined by

$$H^{m,n}_{\mathfrak{p},\mathfrak{q}}(z) = H^{m,n}_{\mathfrak{p},\mathfrak{q}}\left[z \,\middle|\, \begin{matrix} (a_1,A_1),\cdots,(a_{\mathfrak{p}},A_{\mathfrak{p}}) \\ (b_1,B_1),\cdots,(b_{\mathfrak{q}},B_{\mathfrak{q}}) \end{matrix}\right]$$

$$= \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{L}} \Xi(s) z^{-s} \, \mathrm{d}s \qquad \big(z \in \mathbb{C} \setminus \{0\}; \; |\arg(z)| < \pi\big), \tag{11}$$

where

$$\Xi(s) = \frac{\prod_{j=1}^{m} \Gamma(b_j + B_j s) \cdot \prod_{j=1}^{n} \Gamma(1 - a_j - A_j s)}{\prod_{j=n+1}^{\mathfrak{p}} \Gamma(a_j + A_j s) \cdot \prod_{j=m+1}^{\mathfrak{q}} \Gamma(1 - b_j - B_j s)},$$

an empty product is interpreted as 1, $m, n, \mathfrak{p}$ and $\mathfrak{q}$ are integers such that

$$1 \leqq m \leqq \mathfrak{q} \qquad \text{and} \qquad 0 \leqq n \leqq \mathfrak{p},$$

$$A_j > 0 \quad (j = 1, \cdots, \mathfrak{p}) \qquad \text{and} \qquad B_j > 0 \quad (j = 1, \cdots, \mathfrak{q}),$$
$$a_j \in \mathbb{C} \quad (j = 1, \cdots, \mathfrak{p}) \qquad \text{and} \qquad b_j \in \mathbb{C} \quad (j = 1, \cdots, \mathfrak{q})$$

and $\mathcal{L}$ is a suitable Mellin-Barnes type contour separating the poles of the gamma functions

$$\{\Gamma(b_j + B_j s)\}_{j=1}^{m}$$

from the poles of the gamma functions

$$\{\Gamma(1 - a_j + A_j s)\}_{j=1}^{n}.$$

We choose to mention here that, by using the fact that [11, p. 1496, Remark 7]

$$\lim_{b\to 0}\left\{ H_{0,2}^{2,0}\left[(a+n)b^{\frac{1}{\lambda}} \,\middle|\, \begin{matrix} \overline{\qquad\qquad} \\ (s,1),\left(0,\frac{1}{\lambda}\right) \end{matrix} \right] \right\} = \lambda\,\Gamma(s) \qquad (\lambda > 0), \tag{12}$$

the equation (8) reduces to the following form:

$$\begin{aligned} \Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}(z,s,a;0,\lambda) &:= \Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}(z,s,a) \\ &= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(\lambda_j)_{n\rho_j}}{(a+n)^s \cdot \prod_{j=1}^{q}(\mu_j)_{n\sigma_j}} \frac{z^n}{n!}. \end{aligned} \tag{13}$$

Definition 2. The function $\Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}(z,s,a)$ involved in (13) is the multi-parameter extension and generalization of the Hurwitz-Lerch zeta function $\Phi(z,s,a)$ introduced by Srivastava *et al.*[16, p. 503, Eq. (6.2)] defined by

$$\Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}(z,s,a) := \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(\lambda_j)_{n\rho_j}}{(a+n)^s \cdot \prod_{j=1}^{q}(\mu_j)_{n\sigma_j}} \frac{z^n}{n!} \tag{14}$$

$$\Bigg(p, q \in \mathbb{N}_0;\ \lambda_j \in \mathbb{C}\ (j = 1, \cdots, p);\ a, \mu_j \in \mathbb{C}\setminus\mathbb{Z}_0^-\ (j = 1, \cdots, q);$$

$$\rho_j, \sigma_k \in \mathbb{R}^+\ (j = 1, \cdots, p;\ k = 1, \cdots, q);$$

$$\Delta > -1 \text{ when } s, z \in \mathbb{C};$$

$$\Delta = -1 \text{ and } s \in \mathbb{C} \text{ when } |z| < \nabla^*;$$

$$\Delta = -1 \text{ and } \Re(\Xi) > \frac{1}{2} \text{ when } |z| = \nabla^* \Bigg)$$

with

$$\nabla^* := \left(\prod_{j=1}^{p} \rho_j^{-\rho_j} \right) \cdot \left(\prod_{j=1}^{q} \sigma_j^{\sigma_j} \right), \tag{15}$$

$$\Delta := \sum_{j=1}^{q} \sigma_j - \sum_{j=1}^{p} \rho_j \quad \text{and} \quad \Xi := s + \sum_{j=1}^{q} \mu_j - \sum_{j=1}^{p} \lambda_j + \frac{p-q}{2}. \tag{16}$$

By applying this new family of the λ-generalized Hurwitz-Lerch zeta functions, Srivastava and Gaboury [19] introduced a new linear operator which provides a generalization of the largely- (and widely-) studied Srivastava-Attiya operator [20] (see also [21–23]). This new operator contains, as its special cases, the operators investigated earlier by Prajapat and Bulboacă [24, p. 571, Eq. (1.8)], Noor and Bukhari [25, p. 2, Eq. (1.3)], Choi *et al.* [26], Cho and Srivastava [27], Jung *et al.* [28], Bernardi [1], Carlson and Shaffer [29], Owa and Srivastava [30] and by Dziok and Srivastava [31,32]. The Dziok-Srivastava convolution operator studied by Dziok and Srivastava [31, 32] is, in turn, a generalization of the Hohlov operator [33] and the Ruscheweyh operator [34]. In fact, the Dziok-Srivastava convolution operator is itself a special case of the Srivastava-Wright operator (see, for details, [35] and [36]; see also the other closely-related works cited in each of these recent publications).

In this paper, we consider the following linear operator:

$$J^{s,a,\lambda,\alpha,\beta}_{(\lambda_p),(\mu_q),b} f(z) : \Sigma \to \Sigma,$$

which is defined by

$$J^{s,a,\lambda,\alpha,\beta}_{(\lambda_p),(\mu_q),b} f(z) = G^{s,a,\lambda}_{(\lambda_p),(\mu_q),b}(z) * \widetilde{\phi}(\alpha,\beta;z),$$

where $*$ denotes the Hadamard product (or convolution) of analytic functions and the function $G^{s,a,\lambda}_{(\lambda_p),(\mu_q),b}(z)$ is given by

$$\begin{aligned} G^{s,a,\lambda}_{(\lambda_p),(\mu_q),b}(z) &:= (a+1)^s \cdot \Bigg[\Phi^{(1,\cdots,1,1,\cdots,1)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,a;b,\lambda) \\ &\qquad\qquad - \frac{a^{-s}}{\lambda\,\Gamma(s)} \Lambda(a,b,s,\lambda) + \frac{(a+1)^{-s}}{z} \Bigg] \\ &= \frac{1}{z} + \sum_{k=1}^{\infty} \frac{\prod_{j=1}^{p} (\lambda_j)_k}{\prod_{j=1}^{q} (\mu_j)_k} \left(\frac{a+1}{a+k} \right)^s \frac{\Lambda(a+k,b,s,\lambda)}{\lambda\,\Gamma(s)} \, \frac{z^k}{k!} \end{aligned} \tag{17}$$

with

$$\Lambda(a,b,s,\lambda) := H^{2,0}_{0,2}\left[ab^{\frac{1}{\lambda}} \,\middle|\, \begin{array}{c} \overline{\qquad\qquad} \\ (s,1),(0,\frac{1}{\lambda}) \end{array} \right].$$

By combining (17) and (4), we obtain

$$J_{(\lambda_p),(\mu_q),b}^{s,a,\lambda,\alpha,\beta} f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{(\alpha)_{k+1} \prod_{j=1}^{p} (\lambda_j)_k}{(\beta)_{k+1} \prod_{j=1}^{q} (\mu_j)_k} \left(\frac{a+1}{a+k}\right)^s \frac{\Lambda(a+k,b,s,\lambda)}{\lambda\,\Gamma(s)}\, a_k\, \frac{z^k}{k!} \tag{18}$$

$$\left(z \in \mathbb{U}^*;\ \alpha, \lambda_j \in \mathbb{C}\ \ (j=1,\cdots,p);\ \beta, \mu_j \in \mathbb{C} \setminus \mathbb{Z}_0^-\ \ (j=1,\cdots,q);\ p \leqq q+1\right)$$

with

$$\min\{\Re(a), \Re(s)\} > 0, \quad \lambda > 0 \quad \text{if} \quad \Re(b) > 0$$

and

$$s \in \mathbb{C} \qquad \text{and} \qquad a \in \mathbb{C} \setminus \mathbb{Z}_0^- \quad \text{if} \quad b = 0.$$

Clearly, upon setting $p-1 = q = 0$ and $\lambda_1 = 1$ in (18) and taking the limit as $b \to 0$, we obtain the operator $L_a^s(\alpha,\beta)(f)(z)$ studied earlier by Ghanim [5].

It is easily observed from (18) that

$$z\left(J_{(\lambda_p),(\mu_q),b}^{s,a,\lambda,\alpha,\beta} f(z)\right)' = \alpha \left(J_{(\lambda_p),(\mu_q),b}^{s,a,\lambda,\alpha+1,\beta} f(z)\right) - (\alpha+1) \left(J_{(\lambda_p),(\mu_q),b}^{s,a,\lambda,\alpha,\beta} f(z)\right) \tag{19}$$

and

$$z\left(J_{(\lambda_p),(\mu_q),b}^{s,a,\lambda,\alpha,\beta+1} f(z)\right)' = \beta \left(J_{(\lambda_p),(\mu_q),b}^{s,a,\lambda,\alpha,\beta} f(z)\right) - (\beta+1) \left(J_{(\lambda_p),(\mu_q),b}^{s,a,\lambda,\alpha,\beta+1} f(z)\right). \tag{20}$$

Now, with the help of the linear operator $J_{(\lambda_p),(\mu_q),b}^{s,a,\lambda,\alpha,\beta} f(z)$, we introduce the following subclass:

$$\Sigma_{(\lambda_p),(\delta_q),b}^{s,a,\lambda,\alpha,\beta}(\mu) = \Sigma(\alpha,\beta,\mu)$$

of meromorphic functions as follows:

Definition 3. For fixed parameters A, B $(-1 \leqq B < A \leqq 1)$ and $0 \leqq \mu < 1$, the function $f(z) \in \Sigma$ is said to be in the class $\Sigma(\alpha,\beta,\mu)$ if it satisfies the following subordination condition:

$$\frac{1}{1-\mu}\left(-\frac{z\left(J_{(\lambda_p),(\mu_q),b}^{s,a,\lambda,\alpha,\beta} f(z)\right)'}{J_{(\lambda_p),(\mu_q),b}^{s,a,\lambda,\alpha,\beta} f(z)} - \mu\right) \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U}^*) \tag{21}$$

or, equivalently,

$$\Sigma(\alpha,\beta,\mu) = \left\{ f : f(z) \in \Sigma \text{ and } \left| \frac{\dfrac{z\left(J_{(\lambda_p),(\mu_q),b}^{s,a,\lambda,\alpha,\beta} f(z)\right)'}{J_{(\lambda_p),(\mu_q),b}^{s,a,\lambda,\alpha,\beta} f(z)} + 1}{B\,\dfrac{z\left(J_{(\lambda_p),(\mu_q),b}^{s,a,\lambda,\alpha,\beta} f(z)\right)'}{J_{(\lambda_p),(\mu_q),b}^{s,a,\lambda,\alpha,\beta} f(z)} + B + (A-B)(1-\mu)} \right| < 1 \right\}. \tag{22}$$

2 A Set of Lemmas

To establish our main results, we shall need each of the following lemmas:

Lemma 1 (see [37]). *If* $-1 \leqq B < A \leqq 1,\ \nu \neq 0$ *and the complex number* τ *satisfies the inequality*:

$$\Re\{\tau\} \geqq -\frac{\nu(1-A)}{1-B},$$

then the following differential equation:

$$q(z) + \frac{zq'(z)}{\nu q(z) + \tau} \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U}),$$

has a univalent solution in $\mathbb{U}$ *given by*

$$q(z) = \begin{cases} \frac{z^{\nu+\tau}(1+Bz)^{\nu(A-B)/B}}{\nu \int_0^z t^{\nu+\tau-1}(1+Bt)^{\nu(A-B)/B}\,dt} - \frac{\tau}{\nu} & (B \neq 0) \\ \\ \frac{z^{\nu+\tau}\exp(\nu Az)}{\nu \int_0^z t^{\nu+\tau-1}\exp(\nu At)\,dt} - \frac{\tau}{\nu} & (B = 0). \end{cases} \tag{23}$$

If the function ϕ *given by*

$$\phi(z) = 1 + c_1 z + c_2 z + \cdots$$

is analytic in $\mathbb{U}$ *and satisfies the following subordination*:

$$\phi(z) + \frac{z\phi'(z)}{\nu\phi(z) + \tau} \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U}), \tag{24}$$

then

$$\phi(z) \prec q(z) \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U})$$

and $q(z)$ *is the best dominant of* (24).

Lemma 2 (see [38]). *Let* v *be a positive measure on* $[0,1]$. *Let* h *be a complex-valued function defined on* $\mathbb{U} \times [0,1]$ *such that* $h(.,t)$ *is analytic in* $\mathbb{U}$ *for each* $t \in [0,1]$ *and* $h(z,.)$ *is* v*-integrable on* $[0,1]$ *for all* $z \in \mathbb{U}$. *Suppose also that* $\Re\{h(z,t)\} > 0$, $h(-r,t)$ *is real and*

$$\Re\left\{\frac{1}{h(z,t)}\right\} \geqq \frac{1}{h(-r,t)} \qquad (|z| \leqq r < 1;\ t \in [0,1]).$$

If

$$\mathfrak{h}(z) = \int_0^1 h(z,t)\,dv(t),$$

then

$$\Re\left\{\frac{1}{\mathfrak{h}(z)}\right\} \geqq \frac{1}{\mathfrak{h}(-r)} \qquad (|z| \leqq r < 1).$$

Lemma 3 (see [39]). *For real numbers a, b and c $(c \neq 0, -1, -2, \cdots)$, it is asserted that*

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,dt = \frac{\Gamma(b)\,\Gamma(c-b)}{\Gamma(c)}\ {}_2F_1(a,b;c;z) \tag{25}$$

$$\big(\Re\{c\} > \Re\{b\} > 0;\ z \in \mathbb{U}\big).$$

Moreover,

$${}_2F_1(a,b;c;z) = {}_2F_1(b,a;c;z) \tag{26}$$

and

$${}_2F_1(a,b;c;z) = (1-z)^{-\alpha}\ {}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right) \tag{27}$$

$$\big(c \neq 0, -1, -2, \cdots;\ |\arg(1-z)| < \pi\big).$$

Inclusion properties of various classes of analytic and meromorphic functions were studied earlier by several different methods (see, for example, [40–43] and [44]). In this paper, we find two inclusion theorems for the meromorphic function class $\Sigma(\alpha, \beta, \mu)$. In particular, we show that, if we increase the parameter α by one, the overall size of the meromorphic function class $\Sigma(\alpha, \beta, \mu)$ would get smaller. On the other hand, by increasing the parameters β by one, the overall size of the meromorphic function class $\Sigma(\alpha, \beta, \mu)$ would get bigger.

3 Main Results

Unless otherwise mentioned, we assume throughout the remainder of the paper that

$$-1 \leqq B < A \leqq 1,\ 0 \leqq \mu < 1,\ \alpha, \beta > 0,\ a \in \mathbb{C} \setminus \mathbb{Z}_0^-,\ s \in \mathbb{C} \quad \text{and} \quad z \in \mathbb{U}.$$

We begin with some inclusion relationships concerning the parameter α of the class $\Sigma(\alpha, \beta, \mu)$.

Theorem 1.

(i) *If $f(z) \in \Sigma(\alpha+1, \beta, \mu)$ and*

$$\alpha - \mu + 1 \geqq \frac{(1-\mu)(1-A)}{(1-B)}, \tag{28}$$

then

$$\frac{1}{1-\mu}\left(-\frac{z\Big(J^{s,a,\lambda,\alpha,\beta}_{(\lambda_p),(\mu_q),b} f(z)\Big)'}{J^{s,a,\lambda,\alpha,\beta}_{(\lambda_p),(\mu_q),b} f(z)} - \mu\right) \prec \frac{1}{1-\mu}\left((\alpha-\mu+1) - \frac{1}{Q_1(z)}\right)$$

$$= q_1(z) \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U}), \tag{29}$$

where

$$Q_1(z) = \begin{cases} \int_0^1 u^{\alpha-1}\left(\frac{1+Bzu}{1+Bz}\right)^{-(1-\mu)(A-B)/B} du & (B \neq 0) \\ \int_0^1 u^{\alpha-1} e^{-(1-\mu)A(u-1)z}\, du & (B = 0) \end{cases}$$

and $q_1(z)$ *is the best dominant of (29). Moreover,*

$$\Sigma\left(\alpha+1,\beta,\mu\right) \subseteq \Sigma\left(\alpha,\beta,\mu\right). \tag{30}$$

(ii) *If the additional constraints* $0 < B < 1$ *and*

$$\alpha + 1 \geqq \frac{(1-\mu)(A-B)}{B} \tag{31}$$

are satisfied, then

$$\frac{1-|A|}{1-|B|} < \frac{1}{1-\mu}\left(-\Re\left\{\frac{z\left(J^{s,a,\lambda,\alpha,\beta}_{(\lambda_p),(\mu_q),b} f(z)\right)'}{J^{s,a,\lambda,\alpha,\beta}_{(\lambda_p),(\mu_q),b} f(z)}\right\} - \mu\right) < \rho_1, \tag{32}$$

where

$$\rho_1 = \frac{1}{1-\mu}\left\{(\alpha-\mu+1) - \frac{\alpha}{{}_2F_1\left(1, \frac{(1-\mu)(A-B)}{B}; \alpha+1; \frac{B}{B-1}\right)}\right\}. \tag{33}$$

The bound ρ_1 *is the best possible.*

Proof. Let $f(z) \in \Sigma(\alpha+1,\beta,\mu)$ and set

$$\phi(z) = \frac{1}{1-\mu}\left(-\frac{z\left(J^{s,a,\lambda,\alpha,\beta}_{(\lambda_p),(\mu_q),b} f(z)\right)'}{J^{s,a,\lambda,\alpha,\beta}_{(\lambda_p),(\mu_q),b} f(z)} - \mu\right). \tag{34}$$

Then it is clear that $\phi(z)$ is analytic in $\mathbb{U}$ and $\phi(0) = 1$. An application of the identity (19) in (34) yields

$$-(1-\mu)\phi(z) + (\alpha-\mu+1) = \alpha\,\frac{J^{s,a,\lambda,\alpha+1,\beta}_{(\lambda_p),(\mu_q),b} f(z)}{J^{s,a,\lambda,\alpha,\beta}_{(\lambda_p),(\mu_q),b} f(z)}. \tag{35}$$

By using the logarithmic differentiation of both sides of (35) with respect to z, we obtain

$$\phi(z) + \frac{z\phi'(z)}{(\alpha-\mu+1)-(1-\mu)\phi(z)} = \frac{1}{1-\mu}\left(-\frac{z\left(J^{s,a,\lambda,\alpha+1,\beta}_{(\lambda_p),(\mu_q),b} f(z)\right)'}{J^{s,a,\lambda,\alpha+1,\beta}_{(\lambda_p),(\mu_q),b} f(z)} - \mu\right)$$
$$\prec \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U}).$$

Therefore, by applying Lemma 1 with

$$\nu = -(1-\mu) \qquad \text{and} \qquad \tau = \alpha - \mu + 1,$$

we have

$$\phi(z) \prec q_1(z) \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U}),$$

where the best dominant $q_1(z)$ is defined by (29). The proof of Theorem 1 (i) is completed.

In order to establish (32) of Theorem 1 (ii), we observe that an application of the principle of subordination in (21) gives

$$\frac{1-|A|}{1-|B|} < \frac{1}{1-\mu}\left(-\Re\left\{\frac{z\left(J^{s,a,\lambda,\alpha,\beta}_{(\lambda_p),(\mu_q),b}f(z)\right)'}{J^{s,a,\lambda,\alpha,\beta}_{(\lambda_p),(\mu_q),b}f(z)}\right\}-\mu\right),$$

which is precisely the left-hand inequality in (32). Also, by the principle of subordination in (29), we have

$$\begin{aligned}\frac{1}{1-\mu}\left(-\Re\left\{\frac{z\left(J^{s,a,\lambda,\alpha,\beta}_{(\lambda_p),(\mu_q),b}f(z)\right)'}{J^{s,a,\lambda,\alpha,\beta}_{(\lambda_p),(\mu_q),b}f(z)}\right\}-\mu\right) &\leqq \sup_{z\in\mathbb{U}^*}\Re\{q_1(z)\}\\ &= \sup_{z\in\mathbb{U}}\left[\frac{1}{1-\mu}\left(\alpha-\mu+1-\Re\left\{\frac{1}{Q_1(z)}\right\}\right)\right]\\ &= \frac{1}{1-\mu}\left(\alpha-\mu+1-\inf_{z\in\mathbb{U}}\Re\left\{\frac{1}{Q_1(z)}\right\}\right).\end{aligned} \tag{36}$$

The rest of the proof is devoted to find

$$\inf_{z\in\mathbb{U}}\Re\left\{\frac{1}{Q_1(z)}\right\}.$$

By hypothesis, $B \neq 0$. Therefore, by using (29), we have

$$Q_1(z) = (1+Bz)^{\delta}\int_0^1 u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1+Bzu)^{-\delta}\,du,$$

where

$$\delta = \frac{(1-\mu)(A-B)}{B} \qquad \text{and} \qquad \gamma = \alpha+1.$$

Also, since $\gamma > \alpha > 0$, by successively using (25) to (27) of Lemma 3, we obtain

$$Q_1(z) = \frac{\Gamma(\alpha)}{\Gamma(\gamma)}\,{}_2F_1\left(1,\delta;\gamma;\frac{Bz}{Bz+1}\right). \tag{37}$$

Furthermore, the condition:

$$\alpha + 1 > \frac{(1-\mu)(A-B)}{B} \qquad (0 < B < 1)$$

implies that $\gamma > \delta > 0$. Another application of (27) of Lemma 3 to (37) gives

$$Q_1(z) = \int_0^1 h(z,u)\,dv(u),$$

where

$$h(z,u) = \frac{1+Bz}{1+(1-u)Bz} \qquad (0 \leqq u \leqq 1)$$

and

$$dv(u) = \frac{\Gamma(\alpha)}{\Gamma(\delta)\,\Gamma(\gamma-\delta)} u^{\delta-1}\,(1-u)^{\gamma-\delta-1}\,du$$

is a positive measure on $u \in [0,1]$. We note that

$$\Re\{h(z,u)\} > 0 \qquad \text{and} \qquad h(-r,u)$$

is real for $0 \leqq r < 1$ and $u \in [0,1]$. Therefore, by using Lemma 2, we get

$$\Re\left\{\frac{1}{Q_1(z)}\right\} \geqq \frac{1}{Q_1(-r)} \qquad (|z| \leqq r < 1),$$

so that

$$\begin{aligned}
\inf_{z\in\mathbb{U}} \Re\left\{\frac{1}{Q_1(z)}\right\} &= \sup_{0\leqq r<1} \frac{1}{Q_1(-r)} \\
&= \sup_{0\leqq r<1} \frac{1}{\int_0^1 h(-r,u)\,dv} = \frac{1}{\int_0^1 h(-1,u)\,dv} = \frac{1}{Q_1(-1)} \\
&= \frac{\alpha}{{}_2F_1\left(1, \frac{(1-\mu)(A-B)}{B}, \alpha+1, \frac{B}{B-1}\right)}. \qquad (38)
\end{aligned}$$

Hence, in view of (36), the right-hand inequality of (32) follows from (38).

The result is the best possible as the function $q_1(z)$ is the best dominant of (29). This completes the proof of Theorem 1. □

The next theorem gives the corresponding results involving the parameter β.

Theorem 2.

(i) *If* $f(z) \in \Sigma(\alpha,\beta,\mu)$ *and*

$$\beta - \mu + 1 \geqq \frac{(1-\mu)(1-A)}{(1-B)}, \qquad (39)$$

then

$$\frac{1}{1-\mu}\left(-\frac{z\left(J^{s,a,\lambda,\alpha,\beta+1}_{(\lambda_p),(\mu_q),b}f(z)\right)'}{J^{s,a,\lambda,\alpha,\beta+1}_{(\lambda_p),(\mu_q),b}f(z)}-\mu\right)\prec\frac{1}{1-\mu}\left((\beta-\mu+1)-\frac{1}{Q_2(z)}\right)$$

$$=q_2(z)\prec\frac{1+Az}{1+Bz}\qquad(z\in\mathbb{U}),\tag{40}$$

where

$$Q_2(z)=\begin{cases}\int_0^1 u^{\beta-1}\left(\frac{1+Bzu}{1+Bz}\right)^{-(1-\mu)(A-B)/B}du & (B\neq 0)\\ \int_0^1 u^{\beta-1}\,e^{-(1-\mu)A(u-1)z}du & (B=0)\end{cases}$$

and $q_2(z)$ *is the best dominant of* (40). *It is also asserted that*

$$\Sigma(\alpha,\beta,\mu)\subseteq\Sigma(\alpha,\beta+1,\mu).\tag{41}$$

(ii) *If the additional constraints* $0<B<1$ *and*

$$\beta+1\geqq\frac{(1-\mu)(A-B)}{B},\tag{42}$$

are satisfied, then

$$\frac{1-|A|}{1-|B|}<\frac{1}{1-\mu}\left(-\Re\left\{\frac{z\left(J^{s,a,\lambda,\alpha,\beta+1}_{(\lambda_p),(\mu_q),b}f(z)\right)'}{J^{s,a,\lambda,\alpha,\beta+1}_{(\lambda_p),(\mu_q),b}f(z)}\right\}-\mu\right)<\rho_2,\tag{43}$$

where

$$\rho_2=\frac{1}{1-\mu}\left((\beta+1-\mu)-\frac{\beta}{{}_2F_1\left(1,\frac{(1-\mu)(A-B)}{B};\beta+1;\frac{B}{B-1}\right)}\right).\tag{44}$$

The bound ρ_2 *is the best possible.*

Proof. Let $f(z)\in\Sigma(\alpha,\beta,\mu)$ and set

$$\phi(z)=\frac{1}{1-\mu}\left(-\frac{z\left(J^{s,a,\lambda,\alpha,\beta+1}_{(\lambda_p),(\mu_q),b}f(z)\right)'}{J^{s,a,\lambda,\alpha,\beta+1}_{(\lambda_p),(\mu_q),b}f(z)}-\mu\right).\tag{45}$$

Then, by using (17) and logarithmic differentiation for (45) with respect to z, we get

$$\phi(z)+\frac{z\phi'(z)}{-(1-\mu)\phi(z)+(\beta+1-\mu)}=\frac{1}{1-\mu}\left(-\frac{z\left(J^{s,a,\lambda,\alpha,\beta}_{(\lambda_p),(\mu_q),b}f(z)\right)'}{J^{s,a,\lambda,\alpha,\beta}_{(\lambda_p),(\mu_q),b}f(z)}-\mu\right)$$

$$\prec \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U}).$$

Therefore, by an application of Lemma 1 with

$$\nu = -(1-\mu) \qquad \text{and} \qquad \tau = \beta - \mu + 1,$$

we have

$$\phi(z) \prec q_2(z) \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U}),$$

where the best dominant $q_2(z)$ is defined by (40). The proof of Theorem 2 (i) is completed.

In order to establish (43) of Theorem 2 (ii), we apply the principle of subordination in (21) and use the same technique which was used in the proof of Theorem 1. We thus find that

$$\begin{aligned} Q_2(z) &= (1+Bz)^{\delta} \int_0^1 u^{\beta-1} (1-u)^{\gamma-\beta-1} (1+Bzu)^{-\delta} \, du \\ &= \frac{\Gamma(\beta)}{\Gamma(\gamma)} \, {}_2F_1\left(1, \delta; \gamma; \frac{Bz}{Bz+1}\right) \end{aligned} \tag{46}$$

where $\delta = \frac{(1-\mu)(A-B)}{B}$ and $\gamma = \beta + 1$.

Furthermore, the condition:

$$\beta + 1 > \frac{(1-\mu)(A-B)}{B} \qquad (0 < B < 1)$$

implies that $\gamma > \delta > 0$. Another application of (27) of Lemma 3 to (46) gives

$$Q_2(z) = \int_0^1 h(z,u) \, dv(u),$$

where

$$h(z,u) = \frac{1+Bz}{1+(1-u)Bz}, \qquad (0 \leqq u \leqq 1)$$

and

$$dv(u) = \frac{\Gamma(\beta)}{\Gamma(\gamma)\Gamma(\gamma-\delta)} \, u^{\delta-1} (1-u)^{\gamma-\delta-1} \, du.$$

Using Lemma 2 implies that

$$\inf_{z \in \mathbb{U}} \Re\left\{\frac{1}{Q_2(z)}\right\} = \frac{\beta}{{}_2F_1\left(1, \frac{(1-\mu)(A-B)}{B}; \beta+1; \frac{B}{B-1}\right)}. \tag{47}$$

The right-hand inequality of (43) follows from (47).

The bound ρ_2 is sharp by the principle of subordination. The proof of Theorem 2 is thus completed. $\square$

4 Concluding Remarks and Observations

In our present sequel to an earlier work (see [5, 6, 14] and [15]), we have investigated several further properties of the linear operator defined by (18), which is associated with Hurwitz-Lerch zeta function:

$$J^{s,a,\lambda,\alpha,\beta}_{(\lambda_p),(\mu_q),b} f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{(\alpha)_{k+1} \prod\limits_{j=1}^{p} (\lambda_j)_k}{(\beta)_{k+1} \prod\limits_{j=1}^{q} (\mu_j)_k} \left(\frac{a+1}{a+k}\right)^s \frac{\Lambda(a+k,b,s,\lambda)}{\lambda\,\Gamma(s)}\, a_k\, \frac{z^k}{k!},$$

as given by (8) and with the notation used with (17). The various properties and results, which we have presented in this paper, are related to a certain subclass of the class of (normalized) meromorphically univalent functions in the punctured unit disk $\mathbb{U}^*$, which is defined here by means of the Hadamard product (or convolution). Many interesting results (asserted by Theorems 1 and 2 above) have also been deduced in this paper. In addition, there are more extensions and ideas that can be found based on these results.

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