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Semi-symmetric isotopic closure of some group varieties and the corresponding identities

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Abstract. Four families of pairwise equivalent identities are given and analyzed. Every identity from each of these families defines one of the following varieties: 1) the semi-symmetric isotopic closure of the variety of all Boolean groups; 2) the semi-symmetric isotopic closure of the variety of all Abelian groups; 3) the semi-symmetric isotopic closure of the variety of all groups; 4) the variety of all semi-symmetric quasigroups. It is proved that these varieties are different and form a chain. Quasigroups belonging to these varieties are described. In particular, quasigroups from 1) and 2) varieties are medial and in addition, they are either groups or non-commutative semi-symmetric quasigroups.

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1 Introduction

It is well known that the class of all semi-symmetric quasigroups is described by

$$xy \cdot x = y. \tag{1}$$

According to A. Sade [26], a groupoid or a quasigroup $(Q; \cdot)$ satisfying the identity (1) for all x, y of Q is called semi-symmetric. He also established properties and structure of semi-symmetric quasigroups. Semi-symmetric quasigroups have also been described as '3-cyclic'. They were studied by J. M. Osborn [21], A. Sade [26–29], N. S. Mendelsohn [19], G. Grätzer and R. Padmanabhan [15], A. Mitschke and H. Werner [20], J. W. DiPaola and E. Nemeth [9]. The use of semi-symmetric quasigroups for reducing homotopies to homomorphisms first appeared in [32], inspired by work of Gvaramiya and Plotkin that interpreted homotopies as homomorphisms of heterogeneous algebras [32]. The classical approach to studying properties of a quasigroup invariant under isotopy was geometrical, through the concept of a 3-net, as presented in A. A. Albert [2], V. D. Belousov [6], H. O. Pflugfelder [23], V. A. Shcherbacov [30], J. D. H. Smith [33] and A. B. Romanowska [34].

F. Sokhatsky [38] proposed a symmetry concept for parastrophes of quasigroup varieties and their quasigroups. This concept is used for the investigation of the parastrophes of quasigroup varieties and, in particular, quasigroups and their parastrophes. F. Sokhatsky's symmetry concept generalizes the symmetry known as triality which was investigated by J. D. H. Smith [31]. If a σ-parastrophe coincides with a quasigroup itself, then σ is called a symmetry of the quasigroup. The set of all symmetries of a binary quasigroup forms a group, which is a subgroup of the symmetry group S_3. According to the symmetry group, there are six classes of quasigroups: commutative (middle symmetric), left-, right-, semi-, totally symmetric and asymmetric (which consists of quasigroups with a unitary symmetry group).

We consider semi-symmetric isotopic closures of some group varieties. The necessary and sufficient conditions for a group isotope to be semi-symmetric are well-known. For example, F. Radó [25] found the necessary and sufficient conditions for existence of the semi-symmetric group isotopes of prime order. The first author of the article [16] established the criterion for the semi-symmetry of group isotopes. The second author [42] gave a variety of Abelian group isotopes containing semi-symmetric medial quasigroups. I. M. H. Etherington [11] and A. Sade [26] showed that every semi-symmetric groupoid is necessarily a semi-symmetric quasigroup. V. V. Iliev [14] studied a construction of the semi-symmetric algebras over a commutative ring with the unit. V. D. Belousov [5] has found a quadratic identity in five variables describing the isotopic closure of all groups. F. M. Sokhatsky [36] has established an identity in four variables which also describes this variety but his identity is not quadratic. The isotopic closure of some group varieties was studied by G. B. Belyavskaya [7], A. Drapal [10], A. Kh. Tabarov [41].

In this article, we have found families of identities: 1) nine quadratic identities in three variables (11); 2) nine quadratic identities in four variables (12); 3) one non-quadratic identity in four variables (15); 4) ten quadratic identities in two variables (Corollary 11). Identities (11) are pairwise equivalent (Lemma 2) and describe the variety $\mathfrak{B}_{\rm ss}$ of the semi-symmetric isotopic closure of all Boolean groups (Corollary 14 from Theorem 8). Identities (12) are pairwise equivalent (Theorem 9) and describe the variety $\mathfrak{A}_{\rm ss}$ of the semi-symmetric isotopic closure of all Abelian groups (Corollary 18). The identity (15) describes the variety $\mathfrak{G}_{\rm ss}$ of the semi-symmetric isotopic closure of all groups (Theorem 10). All identities from Corollary 11 are pairwise equivalent and describe the variety $\mathfrak{S}$ of all semi-symmetric quasigroups (Lemma 1). Every identity from (11), (12), (15) and from Corollary 11 implies semi-symmetry (see corresponding Theorems 6, 7, and Corollaries 12, 22).

The quasigroups belonging to varieties $\mathfrak{B}_{\rm ss}$ and $\mathfrak{A}_{\rm ss}$ are medial (Corollary 19). Moreover, they are either groups or non-commutative semi-symmetric quasigroups (Corollaries 16, 19). All varieties $\mathfrak{B}_{\rm ss}$, $\mathfrak{A}_{\rm ss}$, $\mathfrak{G}_{\rm ss}$ and $\mathfrak{S}$ are totally symmetric, that is every parastrophe of a quasigroup of the variety belongs to this variety (Corollaries 10, 13, 17, 23). It is proved that these varieties are different and form a chain (Theorem 11).

2 Preliminaries

A quasigroup is a natural generalization of the concept of a group. Quasigroups differ from groups in that they need not be associative. A quasigroup is a group if and only if it satisfies the associativity [6].

As usual, whenever unambiguous, a term like $x \cdot y$ is shortened to xy. The word 'iff' stands for 'if and only if'.

An algebra $(Q; \cdot, \overset{\ell}{\cdot}, \overset{r}{\cdot})$ with identities

$$(x \cdot y) \overset{\ell}{\cdot} y = x, \quad (x \overset{\ell}{\cdot} y) \cdot y = x, \quad x \overset{r}{\cdot} (x \cdot y) = y, \quad x \cdot (x \overset{r}{\cdot} y) = y \tag{2}$$

is called a quasigroup [6,12]. In [3], an equational quasigroup is defined as an algebra with three binary operations $(Q; \cdot, \overset{\ell}{\cdot}, \overset{r}{\cdot})$ that fulfill the following six identities: (2) and $x \overset{\ell}{\cdot} (y \overset{r}{\cdot} x) = y$, $(x \overset{\ell}{\cdot} y) \overset{r}{\cdot} x = y$. The triples of identities composed of these six, emphasizing those that axiomatize the variety of quasigroups, are investigated in [22].

The *main operation* of a quasigroup is denoted by $(\cdot)$. A quasigroup operation $(\cdot)$ is often considered together with its inverse operations: left $(\overset{\ell}{\cdot})$ and right $(\overset{r}{\cdot})$ divisions which are defined by: $x \cdot y = z \Leftrightarrow x \overset{r}{\cdot} z = y \Leftrightarrow z \overset{\ell}{\cdot} y = x$. Both inverse operations are also quasigroups.

Such quasigroups are called *equational quasigroups* (*equasigroups*, earlier *primitive quasigroups*). The equational definition of quasigroups is due to T. Evans [13]. The equational definition of twisted quasigroups is due to A. Krapež [18].

The operations (2) and their duals which are defined by

$$x \overset{s}{\cdot} y := y \cdot x, \qquad x \overset{s\ell}{\cdot} y := y \overset{\ell}{\cdot} x, \qquad x \overset{sr}{\cdot} y := y \overset{r}{\cdot} x \tag{3}$$

are called *parastrophes* of $(\cdot)$. The defining identities (2) and (3) are called *primary.*

2.1 On symmetry of an arbitrary proposition

The relationships (3) imply that each identity of the signature $(\cdot, \overset{\ell}{\cdot}, \overset{r}{\cdot}, \overset{s}{\cdot}, \overset{s\ell}{\cdot}, \overset{sr}{\cdot})$ can be written in the signature $(\cdot, \overset{\ell}{\cdot}, \overset{r}{\cdot})$. Nevertheless throughout the article, we consider identities on quasigroups of signature $(\cdot, \overset{\ell}{\cdot}, \overset{r}{\cdot}, \overset{s}{\cdot}, \overset{s\ell}{\cdot}, \overset{sr}{\cdot})$. All parastrophes of $(\cdot)$ can be defined by

$$x_{1\sigma} \overset{\sigma}{\cdot} x_{2\sigma} = x_{3\sigma} :\Leftrightarrow \; x_1 \cdot x_2 = x_3, \tag{4}$$

where $\sigma \in S_3 := \{\iota, \ell, r, s, s\ell, sr\}$, $\ell := (13)$, $r := (23)$, $s := (12)$. It is easy to verify that

$$^{\sigma}\!\left(\overset{\tau}{\cdot}\right) = \left(\overset{\sigma\tau}{\cdot}\right)$$

holds for all $\sigma, \tau \in S_3$.

F. Sokhatsky [38, 39] has shown that a mapping $(\sigma; (\cdot)) \;\mapsto\; (\overset{\sigma}{\cdot})$ is an action on the set Δ of all quasigroup operations defined on Q. A stabilizer $\mathrm{Ps}(\cdot)$ is called a *parastrophic symmetry* of $(\cdot)$. Thus, the number of different parastrophes of a quasigroup operation $(\cdot)$ depends on its group of parastrophic symmetry $\mathrm{Ps}(\cdot)$. Since $\mathrm{Ps}(\cdot)$ is a subgroup of the symmetric group S_3, then there are six classes of quasigroups.

If $\mathrm{Ps}(\cdot) \supseteq A_3$, then a quasigroup is called *semisymmetric*. The class of all semisymmetric quasigroups is described by $x \cdot yx = y$. It means that

$$(\cdot) = (\overset{s\ell}{\cdot}) = (\overset{sr}{\cdot}), \quad (\overset{s}{\cdot}) = (\overset{\ell}{\cdot}) = (\overset{r}{\cdot}). \tag{5}$$

If $\mathrm{Ps}(\cdot) = S_3$, then a quasigroup is called *totally symmetric*. The class of all totally symmetric quasigroups is described by $xy = yx$ and $xy \cdot y = x$, it means that all parastrophes coincide.

Let P be an arbitrary proposition in a class of quasigroups $\mathfrak{A}$. The proposition ${}^{\sigma}P$ is said to be a σ-parastrophe of P, if it can be obtained from P by replacing every $(\overset{\tau}{\cdot})$ with $(\overset{\tau\sigma^{-1}}{\cdot})$; ${}^{\sigma}\mathfrak{A}$ denotes the class of all σ-parastrophes of quasigroups from $\mathfrak{A}$.

Theorem 1 (see [38, 39])**.** *Let $\mathfrak{A}$ be a class of quasigroups, then a proposition P is true in $\mathfrak{A}$ iff ${}^{\sigma}P$ is true in ${}^{\sigma}\mathfrak{A}$.*

Corollary 1 (see [38, 39])**.** *Let P be true in a class of quasigroups $\mathfrak{A}$, then ${}^{\sigma}P$ is true in $\mathfrak{A}$ for all $\sigma \in S_3$.*

Corollary 2 (see [38,39])**.** *Let P be true in a totally symmetric class $\mathfrak{A}$, then ${}^{\sigma}P$ is true in $\mathfrak{A}$ for all σ.*

Definition 1. Transition of the identity $\mathfrak{id}$ to the identity ${}^{\sigma}\mathfrak{id}$ is called a *parastrophic transformation (σ-parastrophic transformation)* if ${}^{\sigma}\mathfrak{id}$ can be obtained by replacing the main operation with its σ^{-1}-parastrophe.

Two identities are called:

1) *equivalent* if they define the same variety;

2) *primarily equivalent* if one of them can be obtained from the other in a finite number of applications of primary identities (2) – (3) (primary equivalent identities are equivalent);

3) *σ-parastrophic* if one of them can be obtained from the other by σ-parastrophic transformation;

4) *σ-parastrophically equivalent* if they define σ-parastrophic varieties (according to Theorem 1, σ-parastrophically equivalent identities define σ-parastrophic varieties);

5) *σ-parastrophically primarily equivalent* if one of them can be obtained in a finite number of applications of primary identities and σ_1 -, σ_2 -, ..., σ_k - parastrophic transformations such that $\sigma_1\sigma_2\ldots\sigma_k = \sigma$ for some $k \in \mathbb{N}$.

In a generalized case σ will be omitted. For example, two identities are called *parastrophically equivalent* if they are σ-parastrophically equivalent for some $\sigma \in S_3$.

2.2 On group isotopes

A groupoid $(Q;\cdot)$ is called an *isotope of a groupoid* $(Q;+)$ iff there exists a triplet of bijections (α,β,γ), which is called an *isotopism*, such that the relationship $x\cdot y := \gamma^{-1}(\alpha x+\beta y)$ holds. An isotope of a group is called a *group isotope*.

Definition 2 (see [36]). Let $(Q;\cdot)$ be a group isotope and 0 be an arbitrary element of Q, then the right side of the formula

$$x\cdot y = \alpha x + a + \beta y \tag{6}$$

is called a 0-*canonical decomposition* if $(Q;+)$ is a group, 0 is its neutral element and $\alpha 0 = \beta 0 = 0$.

In this case, we say: the element 0 defines the *canonical decomposition*; $(Q;+)$ is its *decomposition group*; α, β are its *coefficients* and a is its *free member*.

Theorem 2 (see [36]). *An arbitrary element of a group isotope uniquely defines a canonical decomposition of the isotope.*

Corollary 3 (see [36]). *The isotopic closure of the variety of all groups is a variety of quasigroups which is described by the following identity:*

$$(x(u\overset{r}{\cdot} y)\overset{\ell}{\cdot} u)z = x(u\overset{r}{\cdot}(y\overset{\ell}{\cdot} u)z). \tag{7}$$

Corollary 4 (see [35]). *If a group isotope* $(Q;\cdot)$ *satisfies the identity*

$$w_1(x)\cdot w_2(y) = w_3(y)\cdot w_4(x)$$

and the variables x, y *are quadratic, then* $(Q;\cdot)$ *is isotopic to a commutative group.*

Recall that a variable is *quadratic* in an identity if it has exactly two appearances in this identity. An identity is called *quadratic* if all variables are quadratic. If a quasigroup $(Q;\cdot)$ is isotopic to a parastrophe of a quasigroup $(Q;\circ)$, then $(Q;\cdot)$ and $(Q;\circ)$ are called *isostrophic*.

Theorem 3 (see [37]). *Let four pairwise isostrophic operations connected by a quadratic identity satisfy the conditions:*

1) an arbitrary subterm of the length two has two different variables;

2) an arbitrary subterm of the length three has three different variables.

Then all these operations are isotopic to the same group.

Belousov's theorem on four quasigroups [1, 4, 40] implies the following corollary.

Corollary 5. *If four quasigroups are connected by the generalized associativity law, then each of these quasigroups is isotopic to the same group.*

Theorem 4 and its Corollary 7 below are well known and can be found in many articles, for example, in [6, 36].

Theorem 4. *A triple (α, β, γ) of permutations of a set Q is an autotopism of a group $(Q, +)$ iff there exists an automorphism θ of $(Q, +)$ and elements $b, c \in Q$ such that*

$$\alpha x = c + \theta x - b, \quad \beta x = b + \theta x, \quad \gamma x = c + \theta x.$$

Corollary 6. *(6) is a canonical decomposition of a group iff $\alpha = \beta = \iota$.*

Proof. Let (6) be a canonical decomposition of a group $(Q; \cdot)$. Therefore, the groups $(Q; +)$ and $(Q; \cdot)$ are isotopic, consequently they are isomorphic and let φ be the corresponding isomorphism. Then

$$\varphi(\varphi^{-1}x + \varphi^{-1}y) = \alpha x + a + \beta y$$

holds. Theorem 4 implies the existence of an automorphism θ and an element b from $(Q; +)$ such that $\varphi x = b + \theta x$. Therefore,

$$x - b + y = \alpha x + a + \beta y$$

holds. The left and the right sides of the equality are canonical decomposition of the same group isotope. Its uniqueness implies $\alpha = \beta = \iota$. □

Corollary 7. *Let α, β_1, β_2, β_3, β_4 be permutations of a set Q. Besides, α is a unitary transformation of a group $(Q, +)$ and let*

$$\alpha(\beta_1 x + \beta_2 y) = \beta_3 u + \beta_4 v,$$

where $\{x, y\} = \{u, v\}$ holds for all $x, y \in Q$. Then the following statements are true:

1) *α is an automorphism of $(Q, +)$ if $u = x,\ v = y$;*

2) *α is an anti-automorphism of $(Q, +)$ if $u = y,\ v = x$.*

Systematizing all criteria on symmetry, the first author [16] gave a classification of group isotopes according to their groups of parastrophic symmetry and formulated the corollary on the classification of isotopes of Abelian groups.

Theorem 5 (see [16])**.** *Let $(Q; \cdot)$ be a group isotope and (6) be its canonical decomposition, then $(Q; \cdot)$ is*

1) *commutative iff $(Q; +)$ is Abelian and $\beta = \alpha$;*

2) *left symmetric iff $(Q; +)$ is Abelian and $\beta = -\iota$;*

3) *right symmetric iff $(Q; +)$ is Abelian and $\alpha = -\iota$;*

4) *semi-symmetric iff α is an anti-automorphism of $(Q; +)$, $\beta = \alpha^{-1}$, $\alpha^3 = -I_a^{-1}$, $\alpha a = -a$, where $I_a(x) := -a + x + a$;*

5) *totally symmetric iff* $(Q;+)$ *is Abelian and* $\alpha = \beta = -\iota$;

6) *asymmetric iff* $(Q;+)$ *is not Abelian or* $-\iota \neq \alpha \neq \beta \neq -\iota$ *and at least one of the following conditions is true:* α *is not an anti-automorphism,* $\beta \neq \alpha^{-1}$, $\alpha^3 \neq -I_a^{-1}$, $\alpha a \neq -a$.

Theorem 5 implies Corollary 8.

Corollary 8 (see [16])**.** *Let* $(Q;\cdot)$ *be an isotope of an Abelian group and (6) be its canonical decomposition, then* $(Q;\cdot)$ *is*

1) *commutative iff* $\beta = \alpha$;

2) *left symmetric iff* $\beta = -\iota$;

3) *right symmetric iff* $\alpha = -\iota$;

4) *semi-symmetric iff* α *is an automorphism of* $(Q;+)$, $\beta = \alpha^{-1}$, $\alpha^3 = -\iota$, $\alpha a = -a$;

5) *totally symmetric iff* $\alpha = \beta = -\iota$;

6) *asymmetric iff* $-\iota \neq \alpha \neq \beta \neq -\iota$ *and at least one of the following conditions is true:* α *is not an automorphism,* $\beta \neq \alpha^{-1}$, $\alpha^3 \neq -\iota$, $\alpha a \neq -a$.

3 Identities implying semi-symmetry

In this section, we find the relations among identities specifying semi-symmetric quasigroups. We systematize some well-known results for identities in two variables for using them in our further investigation. A semi-symmetry can be defined by different conditions. We consider some of them. We find nine quadratic identities in three variables and nine quadratic identities in four variables each of them implies semi-symmetry.

3.1 Identities in two variables

A quasigroup $(Q;\cdot)$ is called *s*emi-symmetric if the identity (1) holds for all x, y from Q. Using the definition of the left division, we have the equivalent identity $y \overset{\ell}{\cdot} x = xy$. We apply the definition of s-parastrophe to the left and to the right sides of the identity separately:

$$x \overset{s\ell}{\cdot} y = xy, \qquad y \overset{\ell}{\cdot} x = y \overset{s}{\cdot} x. \tag{8}$$

These identities mean that $(\overset{s\ell}{\cdot}) = (\cdot)$ and $(\overset{\ell}{\cdot}) = (\overset{s}{\cdot})$ hold. That is why each identity from (8) is equivalent to (1). The equality $(\overset{s\ell}{\cdot}) = (\cdot)$ means that $s\ell \in \mathrm{Ps}(\cdot)$.

Similarly, one can show that the identity

$$x \cdot yx = y \tag{9}$$

is equivalent to
$$y \overset{sr}{\cdot} x = yx, \qquad x \overset{r}{\cdot} y = x \overset{s}{\cdot} y. \tag{10}$$

Therefore, $(\overset{sr}{\cdot}) = (\cdot)$ and $(\overset{r}{\cdot}) = (\overset{s}{\cdot})$ hold and the equality $(\overset{sr}{\cdot}) = (\cdot)$ means that $sr \in \mathrm{Ps}(\cdot)$. As a result, we obtain the following lemma.

Lemma 1. *In an arbitrary quasigroup $(Q;\cdot)$ the following statements are equivalent:*

1) $(Q;\cdot)$ is semi-symmetric;

2) A_3 is a subgroup of $\mathrm{Ps}(\cdot)$;

3) $(Q;\cdot)$ satisfies (9).

Proof. *1)* $\Leftrightarrow$ *2).* As we have shown above, (1) is equivalent to $s\ell \in \mathrm{Ps}(\cdot)$. But $s\ell$ generates the group A_3, then A_3 is a subgroup of $\mathrm{Ps}(\cdot)$. The inverse statement is evident. *2)* $\Leftrightarrow$ *3)* can be proved in the same way. □

Corollary 9. *If a semi-symmetric variety contains s-parastrophe of each of its quasigroups, then it is totally symmetric.*

Proof. The proof follows from item *1)* of Lemma 1. □

Corollary 10. *The variety of all semi-symmetric quasigroups is totally symmetric.*

Proof. Let $\mathfrak{S}$ be the variety of semi-symmetric quasigroups. Therefore, $\mathfrak{S}$ contains $s\ell$-parastrophe of an arbitrary quasigroup from $\mathfrak{S}$. s-Parastrophe of a quasigroup from $\mathfrak{S}$ satisfies s-parastrophe of the identity (1), i.e., $(x \overset{s}{\cdot} y) \overset{s}{\cdot} x = y$. The identity is equivalent to $x \cdot yx = y$ which defines $\mathfrak{S}$. Thus, $s\ell$ and s belong to the group $\mathrm{Ps}(\mathfrak{S})$, that is why $\mathrm{Ps}(\mathfrak{S}) = S_3$. It means that $\mathfrak{S}$ is totally symmetric. □

Corollary 11. *The identities (1), (8), (9), (10) and $x(x \overset{\ell}{\cdot} y) = y$, $(x \overset{r}{\cdot} y)y = x$, $x \overset{\ell}{\cdot} xy = y$, $xy \overset{r}{\cdot} y = x$, $x \overset{\ell}{\cdot} y = yx$, $x \overset{r}{\cdot} y = yx$ are equivalent.*

Proof. Using the definitions of the left and right divisions, the proof is evident. □

The equivalency of the identities (1), (9) and the last two identities from Corollary 11 is shown in [31, Proposition 1.2]. The equivalency of the identities (1), (8), (9), (10) and the last two identities from Corollary 11 are established in [8, 24].

Thus, we have the variety of all semi-symmetric quasigroups, defined by one of ten equivalent axioms from Corollary 11.

Corollary 12. *The identities from Corollary 11 imply semi-symmetry.*

3.2 Identities in three variables

In this subsection, nine quadratic identities in three variables are investigated, namely

$$\begin{array}{llllll} (x\cdot yz)\cdot z = yx, & (i_1) & x\cdot(xy\cdot z) = zy, & (i_2) & xy\cdot yz = zx, & (i_3) \\ x(y(yx\cdot z)) = z, & (i_4) & xy\cdot(y\cdot xz) = z, & (i_5) & x(xy\cdot yz) = z, & (i_6) \\ ((x\cdot yz)z)y = x, & (i_7) & (xy\cdot z)\cdot zy = x, & (i_8) & (xy\cdot yz)z = x. & (i_9) \end{array} \tag{11}$$

In this form, these identities were among 100 identities without squares, which were listed in [17]. We establish relations among identities (11), namely, relations of equivalency and parastrophically primary equivalency. Each quasigroup satisfying one of the identities from (11) is semi-symmetric (Theorem 6).

Proposition 1. *The identities* (i_4), (i_5), (i_6) *are equivalent.*

Proof. Multiply (i_4) by yx from the left: $yx\cdot(x\cdot(y\cdot(yx\cdot z))) = yx\cdot z$. Replacing $yx\cdot z$ with z, we have $yx\cdot(x\cdot yz) = z$. Mutually relabeling x and y, we obtain (i_5). Since applied transformations are invertible, then (i_4) and (i_5) are equivalent. Multiplying (i_5) by x from the left and replacing xz with z, we obtain equivalency of (i_5) and (i_6). □

Proposition 2. *The identities* (i_7), (i_8), (i_9) *are equivalent.*

Proof. Multiply (i_7) by yz from the right: $(((x\cdot yz)\cdot z)\cdot y)\cdot yz = x\cdot yz$. Replacing $x\cdot yz$ with x, we obtain $(xz\cdot y)\cdot yz = x$. Mutually relabeling z and y, we obtain (i_8). Since applied transformations are invertible, then (i_7) and (i_8) are equivalent.

Multiplying (i_8) by y from the right and replacing xy with x, we have $(xz\cdot zy)\cdot y = x$. Mutually relabeling z and y, we obtain the equivalency of (i_8) and (i_9). □

Theorem 6. *Every identity from (11) implies semi-symmetry.*

Proof. Let $(Q,\cdot)$ be a quasigroup. Replacing z with x in identities (i_1) and (i_2), we have

$$(x\cdot yx)\cdot x = y\cdot x, \qquad x\cdot(xy\cdot x) = x\cdot y.$$

Canceling out x in both sides of these identities, we obtain semi-symmetric identity in both cases.

We put $z = y \overset{r}{\cdot} x$ in (i_3), $z = yx \overset{r}{\cdot} x$ in (i_4):

$$xy\cdot y(y\overset{r}{\cdot} x) = (y\overset{r}{\cdot} x)x, \qquad x\cdot y(yx\cdot(yx\overset{r}{\cdot} x)) = yx\overset{r}{\cdot} x.$$

Apply (2):

$$xy\cdot x = (y\overset{r}{\cdot} x)\cdot x, \qquad x\cdot yx = yx\overset{r}{\cdot} x.$$

Canceling out x in the first identity and replacing yx with x in the second identity, we obtain $xy = y\overset{r}{\cdot} x$ in both cases. According to the right division, we obtain semi-symmetric identity $y\cdot xy = x$.

By Proposition 1, the identities (i_4), (i_5), (i_6) are equivalent. Then the identities (i_5), (i_6) imply semi-symmetry.

We replace x with $x \overset{\ell}{\cdot} yz$ in (i_7): $((x \overset{\ell}{\cdot} yz) \cdot yz)z \cdot y = x \overset{\ell}{\cdot} yz$.

Apply (2): $xz \cdot y = x \overset{\ell}{\cdot} yz$.

Putting $x = y$, we obtain semi-symmetric law $yz \cdot y = z$. Proposition 2 implies that semi-symmetric law follows from (i_8) and (i_9). $\square$

3.3 Identities in four variables

In this subsection, nine quadratic identities in four variables

$$\begin{array}{llll} (xy \cdot u) \cdot xv = y \cdot uv, & (m_1) & xy \cdot (u \cdot vy) = xu \cdot v, & (m_2) \\ (x \cdot (yu \cdot v)) \cdot y = xu \cdot v, & (m_3) & x \cdot ((y \cdot ux) \cdot v) = y \cdot uv, & (m_4) \\ xy \cdot (ux \cdot vy) = uv, & (m_5) & (xy \cdot uv) \cdot xu = yv, & (m_6) \\ xy \cdot (ux \cdot v) = u \cdot yv, & (m_7) & (x \cdot yu) \cdot vy = xv \cdot u, & (m_8) \\ x \cdot ((y \cdot xu) \overset{\ell}{\cdot} v) = uv \cdot y & (m_9) & & \end{array} \tag{12}$$

are considered. It is proved that each of these identities implies semi-symmetry.

Theorem 7. *Every identity from* (12) *implies semi-symmetry.*

Proof. Put $u = x$ in (m_1) and $u = y$ in (m_2):

$$(xy \cdot x) \cdot xv = y \cdot xv, \qquad xy \cdot (y \cdot vy) = xy \cdot v.$$

Canceling out xv in the first identity and xy in the second one, we receive semi-symmetry from each of these identities.

When we put $v = y$ in (m_3) and $y = x$ in (m_4), then

$$(x \cdot (yu \cdot y)) \cdot y = xu \cdot y, \qquad x \cdot ((x \cdot ux) \cdot v)) = x \cdot uv.$$

Cancel out y in the first identity and x in the second one:

$$x \cdot (yu \cdot y) = xu, \qquad (x \cdot ux) \cdot v = uv.$$

Canceling out x and v respectively in these identities, we receive semi-symmetry in both cases.

Put $v = x$ in (m_5) and $u = y$ in (m_6):

$$xy \cdot (ux \cdot xy) = ux, \qquad (xy \cdot yv) \cdot xy = yv.$$

Replace xy with y and ux with u in the first identity, xy with x and yv with v in the second one. We obtain semi-symmetric law in both cases.

Putting $ux = y$ and $u = y \overset{\ell}{\cdot} x$ in (m_7), we have $xy \cdot yv = (y \overset{\ell}{\cdot} x) \cdot yv$. Canceling out yv in both sides of the identity and using the definition of the left division, we receive the semi-symmetric identity.

Put $yu = v$ and $u = y \overset{r}{\cdot} v$ in (m_8), then $xv \cdot vy = xv \cdot (y \overset{r}{\cdot} v)$. Divide both sides of this identity by xv. According to Corollary 11, the obtained identity is equivalent to semi-symmetry.

Put $xu = v$ and $x = v \overset{\ell}{\cdot} u$ in (m_9):

$$(v \overset{\ell}{\cdot} u) \cdot (yv \overset{\ell}{\cdot} v) = uv \cdot y.$$

According to the first identity from (2), we have $(v \overset{\ell}{\cdot} u) \cdot y = uv \cdot y$. Divide both sides of this identity by y on the right. According to Corollary 11, the obtained identity is equivalent to semi-symmetry. □

4 The varieties of semi-symmetric isotopic closures of some groups

V. D. Belousov [5] has found a quadratic identity in five variables describing the isotopic closure of all groups:

$$(x(y \overset{r}{\cdot} z) \overset{\ell}{\cdot} u)v = x(y \overset{r}{\cdot} (z \overset{\ell}{\cdot} u)v).$$

F. M. Sokhatsky [36] has established an identity (7) in four variables, which also describes isotopic closure of all groups, but it is not quadratic.

In this section, we find the semi-symmetric isotopic closure of all Boolean groups, the semi-symmetric isotopic closure of all Abelian groups and the semi-symmetric isotopic closure of all groups.

4.1 The variety of semi-symmetric isotopes of all Boolean groups

In this subsection, we consider the semi-symmetric isotopic closure of Boolean groups. We find nine identities (11) which describe the variety of semi-symmetric isotopes of all Boolean groups. This variety is totally symmetric, that is every parastrophe of a quasigroup from the variety belongs to it. These quasigroups are medial and they are either groups or non-commutative semi-symmetric quasigroups.

Lemma 2. *The identities (11) are equivalent and define a totally symmetric variety.*

Proof. To obtain (i_3) we use semi-symmetry law:

- multiply (i_9) by z from the left;
- multiply (i_6) by x from the right;
- replace z with yz in (i_2) and multiply the obtained identity by x from the right;
- replace x with xy in (i_1) and multiply the obtained identity by z from the left.

Taking into account Proposition 1 and Proposition 2, we obtain equivalency of all identities from (11).

Consider s-parastrophe of (i_1): $(x \overset{s}{\cdot} (y \overset{s}{\cdot} z)) \overset{s}{\cdot} z = y \overset{s}{\cdot} x$. By the definition of s-parastrophe of the operation $(\cdot)$, we obtain $z \cdot (zy \cdot x) = xy$. This identity coincides with (i_2) after mutual relabeling of x and z. This means that s-parastrophe of (i_1) defines the same variety. Since the variety is semi-symmetric, then it is totally symmetric. □

Theorem 8. *In an arbitrary quasigroup $(Q; \cdot)$ the following statements are equivalent:*

1) $(Q; \cdot)$ *is a semi-symmetric isotope of a Boolean group;*

2) $(Q; \cdot)$ *satisfies an arbitrary identity from (11);*

3) there exists a Boolean group $(Q; +)$, its automorphism α and an element $a \in Q$ such that
$$x \cdot y = \alpha x + a + \alpha^2 y, \quad \alpha^3 = \iota, \quad \alpha a = a. \tag{13}$$

Proof. Since all identities from (11) are equivalent by virtue of Lemma 2, then they define the same variety. Therefore, it is enough to prove the theorem for one of them.

1) $\Leftrightarrow$ *3).* Let $(Q; \cdot)$ be a semi-symmetric isotope of a Boolean group $(G; *)$. Then all groups being isotopic to $(Q; \cdot)$ are Boolean. Therefore, according to item *5)* of Theorem 5, item *1)* and item *3)* of the theorem are equivalent.

2) $\Rightarrow$ *1).* Let $(Q; \cdot)$ be a quasigroup satisfying the identity (i_1) from (11). By Theorem 6, $(Q; \cdot)$ is a semi-symmetric quasigroup. According to Theorem 3 and Corollary 5, this quasigroup is isotopic to a group, so $(Q; \cdot)$ is a semi-symmetric group isotope.

3) $\Rightarrow$ *2).* Let (13) hold for a quasigroup $(Q; \cdot)$. Prove that the identity (i_1) is true. Indeed,
$$(x \cdot yz) \cdot z = \alpha(\alpha x + a + \alpha^2(\alpha y + a + \alpha^2 z)) + a + \alpha^2 z.$$
Because α is an automorphism, then
$$(x \cdot yz) \cdot z = \alpha^2 x + \alpha a + \alpha y + a + \alpha^2 z + a + \alpha^2 z.$$
Since $(Q; +)$ is a Boolean group and $\alpha a = a$, then $2a = 0$ and $2\alpha^2 z = 0$. Consequently,
$$(x \cdot yz) \cdot z = \alpha y + a + \alpha^2 x = y \cdot x.$$
□

Theorem 8 implies several corollaries.

Corollary 13. *The variety of quasigroups being defined by one of the identities (11) is totally symmetric.*

Proof. The proof follows from Lemma 2 and from Theorem 8. □

Corollary 14. *The semi-symmetric isotopic closure of all Boolean groups is defined by pairwise equivalent identities* (11)*.*

Proof. The proof is evident, taking into account Theorem 8. □

Corollary 15. *The semi-symmetric isotopic closure of all Boolean groups is the intersection of the variety of all semi-symmetric quasigroups and the variety of all Boolean groups.*

Proof. The proof immediately follows from Theorem 8 and Corollary 14. □

Corollary 16. *Every quasigroup satisfying one of the identities (11) is either a Boolean group or a non-commutative semi-symmetric quasigroup.*

Proof. Let $(Q;\cdot)$ be a quasigroup satisfying the identity (i_1). Then by Theorem 8, its canonical decomposition has the form (13), where α is some automorphism and $a \in Q$.

If $(Q;\cdot)$ is commutative, then according to Theorem 5, $\alpha^2 = \alpha$, i.e., $\alpha = \iota$. The equality $x \cdot y = x + a + y$ means that L_a is an isomorphism between $(Q;\cdot)$ and $(Q;+)$. Thus, $(Q;\cdot)$ is a Boolean group.

If $(Q;\cdot)$ is non-commutative, then according to Theorem 5 $\alpha^2 \neq \alpha$. Therefore, $\alpha \neq \iota$ and according to Corollary 6, $(Q;\cdot)$ is not a group, but by Theorem 6, it is semi-symmetric. □

Example 1. Consider the group $\mathbb{Z}_2^2 := \mathbb{Z}_2 \times \mathbb{Z}_2$. Define the transformation α of the set $\mathbb{Z}_2^2$:

$$\alpha(x) := x \cdot \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Since $\alpha^3 = \iota$, then α is an automorphism of the group $\mathbb{Z}_2^2$. By Theorem 8, a quasigroup $(\mathbb{Z}_2^2;\circ)$ defined by the equation $x \circ y := \alpha x + \alpha^2 y$ satisfies the identity (i_1). Because $\alpha \neq \alpha^2$, then $(Q,\circ)$ is non-commutative. By Corollary 16, the quasigroup $(Q,\circ)$ is semi-symmetric and not a group.

4.2 The variety of semi-symmetric isotopes of all Abelian groups

In this subsection, the variety being defined by identities (12) is considered. Each of these identities determines the totally symmetric variety of all semi-symmetric medial quasigroups. This variety is the semi-symmetric isotopic closure of all Abelian groups. Quasigroups belonging to this variety are either Boolean groups or non-Boolean totally symmetric quasigroups or non-commutative semi-symmetric quasigroups.

Theorem 9. *The identities* (12) *are equivalent and define the variety of all medial semi-symmetric quasigroups.*

Proof. According to Theorem 7, semi-symmetry follows from any identity in (12). Using semi-symmetry, further it will be shown that each of the identities from (12) is equivalent to mediality.

Put yx instead of y in (m_1) and yv instead of v in (m_2):

$$((x \cdot yx) \cdot u) \cdot xv = yx \cdot uv, \qquad xy \cdot (u \cdot (yv \cdot y)) = xu \cdot yv.$$

Using semi-symmetry, we receive mediality in both cases.

Replace y with uy in (m_3) and use semi-symmetry: $(x \cdot yv) \cdot uy = xu \cdot v$. Replacing v with vy and applying semi-symmetry to the last identity, we get mediality.

Change x by xu in (m_4) and apply the semi-symmetric identity: $xu \cdot (yx \cdot v) = y \cdot uv$. Put y instead of xy in the obtained identity. Using semi-symmetry, we receive $xu \cdot yv = xy \cdot uv$, that is the medial law holds for every x, u, y, v.

Multiply (m_6) by xu on the left and (m_5) by xy on the right:

$$xu \cdot ((xy \cdot uv) \cdot xu) = xu \cdot yv, \qquad (xy \cdot (ux \cdot vy)) \cdot xy = uv \cdot xy.$$

Applying semi-symmetry to these identities, we obtain medial identity in the first case and $ux \cdot vy = uv \cdot xy$ in the second one. The last identity means that the mediality holds for all u, x, v, y.

Substitute u with xu in (m_7), u with uy in (m_8) and apply semi-symmetry to the received identities, as a result we obtain mediality in both cases.

Consider (m_9). Since $(\cdot)$ is semi-symmetric, then $(\overset{\ell}{\cdot}) = (\overset{s}{\cdot})$, that is $x \overset{\ell}{\cdot} y = yx$. Then (m_9) can be written as follows: $x \cdot (v \cdot (y \cdot xu)) = uv \cdot y$. Replace x with ux in this identity and use semi-symmetry: $ux \cdot (v \cdot yx) = uv \cdot y$. Substituting y with xy and using semi-symmetry law, we have $ux \cdot vy = uv \cdot xy$. It means that the mediality holds for all u, x, v, y.

Thus, a quasigroup satisfying an arbitrary identity from (12) is semi-symmetric and medial simultaneously. This means that identities (12) define the same variety of semi-symmetric medial quasigroups. □

Corollary 17. *The variety of all semi-symmetric medial quasigroups is totally symmetric.*

Proof. It is well known that the variety of all medial quasigroups is totally symmetric, according to Corollary 10, the variety of all semi-symmetric quasigroups is totally symmetric as well. Therefore, the variety of all semi-symmetric medial quasigroups is totally symmetric, since it is the intersection of two totally symmetric varieties. □

Corollary 18. *The semi-symmetric isotopic closure of all Abelian groups is defined by pairwise equivalent identities* (12)*.*

Proof. By virtue of Theorem 9, all identities from (12) are equivalent, then it is enough to prove this theorem for one of them. Let $(Q;\cdot)$ be an arbitrary quasigroup. Let us prove that $(Q;\cdot)$ is semi-symmetric isotope of Abelian groups iff it satisfies the identity (m_1).

Let $(Q;\cdot)$ satisfy (m_1), then according to Theorem 9, $(Q;\cdot)$ is medial and Toyoda-Bruck theorem implies that $(Q;\cdot)$ is an isotope of an Abelian group. By Theorem 7, $(Q;\cdot)$ is semi-symmetric. Thus, $(Q;\cdot)$ is semi-symmetric isotope of an Abelian group.

Vice versa, let $(Q;\cdot)$ be an arbitrary semi-symmetric isotope of an Abelian group. Then by item *4)* of Corollary 8, its canonical decomposition is the following:

$$x\cdot y=\alpha x+a+\alpha^{-1}y,\quad \alpha^3=-\iota,\quad \alpha a=-a, \tag{14}$$

where $(Q;+)$ is an Abelian group, α is its automorphism and an element $a\in Q$. Let us show that conditions (14) satisfy the identity (m_1).

$$(xy\cdot u)\cdot xv\overset{(14)}{=}\alpha(\alpha(\alpha x+a+\alpha^{-1}y)+a+\alpha^{-1}u)+a+\alpha^{-1}(\alpha x+a+\alpha^{-1}v).$$

Because α and α^{-1} are automorphisms, then

$$(xy\cdot u)\cdot xv=\alpha^3x+\alpha^2a+\alpha y+\alpha a+u+a+x+\alpha^{-1}a+\alpha^{-2}v.$$

Since $(Q;+)$ is an Abelian group and $\alpha^3=-\iota$, $\alpha a=-a$, then

$$(xy\cdot u)\cdot xv=-x+a+\alpha y-a+u+a+x+\alpha^{-1}a+\alpha^{-2}v=$$

$$=\alpha y+a+\alpha^{-1}\alpha u+\alpha^{-1}a+\alpha^{-2}v=\alpha y+a+\alpha^{-1}(\alpha u+a+\alpha^{-1}v)=y\cdot uv.$$

□

Corollary 19. *The semi-symmetric isotopic closure of all Abelian groups is the intersection of the variety of semi-symmetric quasigroups and the variety of all medial semi-symmetric quasigroups.*

Proof. The proof immediately follows from Theorem 9 and Corollary 18. □

Corollary 20. *Every quasigroup satisfying one of the identities* (12) *is either a Boolean group or a non-Boolean totally symmetric quasigroup, or a non-commutative semi-symmetric quasigroup.*

Proof. Let $(Q;\cdot)$ be a quasigroup satisfying the identity (m_1), then according to the proof of Corollary 18, (14) is its canonical decomposition.

If $\alpha=\iota$, then Corollary 6 implies that $(Q;\cdot)$ is a Boolean group.

If $\alpha=-\iota$, then according to item *5)* of Corollary 8, the quasigroup $(Q;\cdot)$ is totally symmetric. There is at least one totally symmetric quasigroup which is non-Boolean group. For example, the quasigroup $(\mathbb{Z}_3;\bullet)$ defined by $x\bullet y:=-x+1-y$ is totally symmetric quasigroup and is a non-Boolean group, since $2\cdot(-1)=-2\neq 0$.

Consider the case $\alpha\neq\iota$ and $\alpha\neq-\iota$. Since condition $\alpha^3=-\iota$ from (14) implies $\alpha\neq\alpha^{-1}$, then quasigroup $(Q;\cdot)$ is non-commutative. But canonical decomposition (14) satisfies semi-symmetry. Indeed,

$$x\cdot yx\overset{(14)}{=}\alpha x+a+\alpha^{-1}(\alpha y+a+\alpha^{-1}x)=\alpha x+a+y+\alpha^{-1}a+\alpha^{-2}x.$$

Since conditions $\alpha^3=-\iota$ and $\alpha a=-a$ imply $\alpha^{-2}=-\alpha$ and $\alpha^{-1}a=-a$, then $x\cdot yx=\alpha x+a+y-a-\alpha x=y$. The corollary has been proved. □

Example 2. The quasigroup $(\mathbb{Z}_9;*)$, $x*y = 2x+3+5y$, belongs to the variety $\mathfrak{A}_{\mathrm{ss}}$ and does not belong to the variety $\mathfrak{B}_{\mathrm{ss}}$. Indeed, this quasigroup satisfies the canonical decomposition (14), since $\alpha^3 = 2^3 = -\iota$, $\alpha a = 2\cdot 3 = 6 = -3$ and does not satisfy conditions (13), because $\alpha^3 \neq \iota$. Thus, taking into account Corollary 20, $(\mathbb{Z}_9;*)$ is a non-commutative semi-symmetric quasigroup.

4.3 The variety of semi-symmetric isotopes of all groups

In this subsection, we find an identity which describes the semi-symmetric isotopic closure of all groups.

Theorem 10. *In an arbitrary quasigroup $(Q;\cdot)$ the following statements are equivalent:*

1) *$(Q;\cdot)$ is a semi-symmetric group isotope;*
2) *$(Q;\cdot)$ satisfies*
$$u(x\cdot yu) = z(x\cdot(uy\cdot z)u); \tag{15}$$
3) *there exists a group $(Q;+)$, its anti-automorphism α, an element $a\in Q$ such that $x\cdot y = \alpha x + a + \alpha^{-1}y$ and $\alpha^3 = -I_a^{-1}$, $\alpha a = -a$, where $I_a(x) := -a+x+a$.*

Proof. *2) $\Rightarrow$ 1).* Let a quasigroup $(Q;\cdot)$ satisfy (15). Put $z=u$ in (15):
$$u(x\cdot yu) = u(x\cdot(uy\cdot u)u).$$
Cancelling out u, x, u, we obtain identity (1). Hence, $(Q;\cdot)$ is semi-symmetric.

Multiply (15) by z from the right and use the identity of semi-symmetry:
$$u(x\cdot yu)\cdot z = x\cdot(uy\cdot z)u. \tag{16}$$
Since $(Q;\cdot)$ is semi-symmetric, then (5) hold. Replacing the operation $(\cdot)$ with its patasrophes in (16), we have (7). Corollary 3 implies that $(Q;\cdot)$ is isotopic to a group.

1) $\Rightarrow$ 2). Let $(Q;\cdot)$ be a semi-symmetric group isotope, then the equalities (5) are true and (7) can be written as (16). Multiply (16) by z from the left and apply the identity (1). As a result we obtain (15).

3) $\Leftrightarrow$ 1). It follows from item *5)* of Theorem 5. □

Corollary 21. *The semi-symmetric isotopic closure of all groups is defined by* (15).

Proof. It is evident from Theorem 10. □

Corollary 22. *The identity* (15) *implies semi-symmetry.*

Proof. The proof follows from Theorem 10. □

Corollary 23. *The variety of quasigroups being defined by (15) is totally symmetric.*

Proof. Let $\mathfrak{Q}$ be the variety defined by (15). It means that each quasigroup $(Q;\cdot)$ from $\mathfrak{Q}$ satisfies the identity $x{\cdot}yx = y$. This identity is equivalent to $xy{\cdot}x = y$. Define the operation $(\circ) := (\overset{s}{\cdot})$. Then the last identity can be written as $x \circ (y \circ x) = y$, i.e., s-parastrophe of an arbitrary quasigroup from $\mathfrak{Q}$ is in $\mathfrak{Q}$. Thus, for all $\sigma \in S_3$ the relation ${}^{\sigma}\mathfrak{Q} = \mathfrak{Q}$. Therefore, this variety is totally symmetric. □

5 Main results

In this article, we have found families (11), (12) and (15) of identities. Namely:

1) identities (11) are pairwise equivalent and describe the variety $\mathfrak{B}_{\rm ss}$ of the semi-symmetric isotopic closure of all Boolean groups;

2) identities (12) are pairwise equivalent and describe the variety $\mathfrak{A}_{\rm ss}$ of the semi-symmetric isotopic closure of all Abelian groups;

3) the identity (15) describes the variety $\mathfrak{G}_{\rm ss}$ of the semi-symmetric isotopic closure of all groups;

4) the identities from Corollary 11 are pairwise equivalent and describe the variety $\mathfrak{S}$ of all semi-symmetric quasigroups.

To establish a relationship among these varieties we give the following examples.

Example 3. In the symmetric group $(S_3;\cdot)$, where $(\cdot)$ denotes the composition of permutations, we define a transformation α by $\alpha(x) := s\ell \cdot x^{-1} \cdot sr$. Here α is anti-automorphism of the group $(S_3;\cdot)$ and $\alpha^3 = I$. Indeed,

$$\alpha(x \cdot y) = s\ell \cdot (xy)^{-1} \cdot sr = s\ell \cdot y^{-1}x^{-1} \cdot sr = s\ell y^{-1}sr \cdot s\ell x^{-1}sr = \alpha(y) \cdot \alpha(x),$$

$$\alpha^3(x) = \alpha\left(s\ell(s\ell x^{-1}sr)^{-1}sr\right) = \alpha(s\ell s\ell xsrsr) = (s\ell)^3x^{-1}(sr)^3 = I.$$

According to item *5)* of Theorem 5, the groupoid $(S_3;\circ)$ is defined by

$$x \circ y := \alpha(x) \cdot \alpha^{-1}(y)$$

and it is a semi-symmetric group isotope. Therefore, S_3 is a semi-symmetric isotope of a non-commutative group.

Example 4. Let $Q := \{1,2,3,4,5\}$. On the set Q we define the operation $(\cdot)$:

$(\cdot)$	1	2	3	4	5
1	1	4	5	3	2
2	5	2	4	1	3
3	4	5	3	2	1
4	2	3	1	5	4
5	3	1	2	4	5

$(\circ)$	1	2	3	4	5
1	1	2	3	4	5
2	2	4	5	3	1
3	3	5	4	1	2
4	4	1	2	5	3
5	5	3	1	2	4

It is easy to verify that $(Q;\cdot)$ is a semi-symmetric quasigroup. Permuting rows by the cycle (2534) and columns by the cycle (2435), we obtain the loop $(Q;\circ)$. Suppose, the quasigroup $(Q;\cdot)$ is isotopic to a group $(G;\diamond)$. $(Q;\circ)$ and $(Q;\cdot)$ are isotopic according to construction of $(Q;\circ)$. Then the loop $(Q;\circ)$ and the group $(G;\diamond)$ are isotopic, therefore they are isomorphic. $(Q;\circ)$ is commutative as a prime order group. But this statement is false, because $4\circ 2 = 1 \neq 3 = 2\circ 4$. Consequently, the assumption is false and the quasigroup $(Q;\cdot)$ is not a group isotope.

Theorem 11. *The varieties $\mathfrak{B}_{\mathrm{ss}}$, $\mathfrak{A}_{\mathrm{ss}}$, $\mathfrak{G}_{\mathrm{ss}}$ and $\mathfrak{S}_{\mathrm{ss}}$ are different and form the following chain: $\mathfrak{B}_{\mathrm{ss}} \subset \mathfrak{A}_{\mathrm{ss}} \subset \mathfrak{G}_{\mathrm{ss}} \subset \mathfrak{S}$.*

Proof. Nonstrict inclusion of these varieties follows from their definitions. To prove strict inclusion, we consider some examples of quasigroups which belong to a wider variety and do not belong to the smaller variety. The total symmetry of each of the varieties $\mathfrak{B}_{\mathrm{ss}}$, $\mathfrak{A}_{\mathrm{ss}}$, $\mathfrak{G}_{\mathrm{ss}}$, $\mathfrak{S}$ is provided by Corollaries 10, 13, 17, 23.

In Example 2, the groupoid $(\mathbb{Z}_9;*)$ is a semi-symmetric quasigroup and it is isotopic to the cyclic group $(\mathbb{Z}_9;+)$, which is not Boolean. Hence, $(\mathbb{Z}_9;*)$ belongs to the variety $\mathfrak{A}_{\mathrm{ss}}$ and does not belong to $\mathfrak{B}_{\mathrm{ss}}$.

The quasigroup $(S_3;\circ)$ from Example 3 belongs to the variety $\mathfrak{G}_{\mathrm{ss}}$ and does not belong to $\mathfrak{A}_{\mathrm{ss}}$, because the group S_3 is non-commutative.

The quasigroup $(Q;\cdot)$ from Example 4 belongs to the variety $\mathfrak{S}_{\mathrm{ss}}$ and does not belong to $\mathfrak{G}_{\mathrm{ss}}$, because the quasigroup $(Q;\cdot)$ is not isotopic to a group. $\square$

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