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Factorizations in the rings of the block matrices

Vasyl' Petrychkovych, Nataliia Dzhaliuk

Abstract. The factorizations in the rings of the block triangular and the block diagonal matrices over an integral domain of finitely generated principal ideals are described. Conditions for existence and uniqueness up to the association of the factorizations in such rings are established. The construction of the factorizations of matrices is reduced to the factorizations of diagonal blocks of the block triangular matrices and the solving of the linear Sylvester matrix equations.

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1 Introduction

Let R be an integral domain of finitely generated principal ideals. We will denote the ring of $n \times n$ matrices by $M(n,R)$, the set of $n \times m$ matrices by $M(n,m,R)$, the group of invertible $n \times n$ matrices over R by $GL(n,R)$, the subring of the block upper triangular matrices

$$T = \operatorname{triang}(T_{11},\dots,T_{kk}) = \begin{bmatrix} T_{11} & T_{12} & \dots & T_{1k} \\ 0 & T_{22} & \dots & T_{2k} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \cdots & T_{kk} \end{bmatrix},$$

where $T_{ii} \in M(n_i,R)$, $i = 1,\dots,k$, by $BT(n_1,\dots,n_k,R)$. Factorizations $T = AB$ and $T = A_1B_1$ of the matrix $T \in M(n,R)$ are called associate if $A_1 = AV$ and $B_1 = V^{-1}B$, where $V \in GL(n,R)$. We will consider the factorizations of matrices in the ring $M(n,R)$ and in its subring $BT(n_1,\dots,n_k,R)$ of the block triangular matrices. We will describe the factorizations of matrices up to the association. We would like to note that the block matrices arise in various problems, such as in [10, 16].

The theory of factorization of the polynomial matrices, which are matrices over the polynomial ring, has been well developed. Such factorizations of the polynomial matrices have been used in the theory of matrix and differential equations [4, 7, 14], in the theory of operator pencils [9] and in other applied problems [8]. In [1], conditions for uniqueness up to the association of the factorizations of matrices over the principal ideal rings have been formulated.

In this article, conditions for existence and uniqueness up to the association of the factorizations in the ring of the block triangular matrices have been obtained. We have established such classes of the block triangular matrices, where each factorization is associated to its factorization in the ring $BT(n_1, \dots, n_k, R)$ of the block triangular matrices. We should note that the block matrices are connected with the matrix linear bilateral equations. It is known that such equation is solvable if and only if the block triangular and the block diagonal matrices composed of the equation coefficients are equivalent [3, 5, 6, 15]. Hence, the factorization of the block triangular matrices is reduced to the factorization of the diagonal blocks and the solving of the matrix linear equations. Similar results for matrices over the ring of polynomials have been obtained in [13].

2 Preliminaries

Let $A \in M(n, m, R), n \le m,\ d_n^A \ne 0$ and the matrix A have the factorization $A = BC,\quad B \in M(n, R),\ C \in M(n, m, R)$. Let us write the matrices in the block form

$$\begin{bmatrix} A_1 \\ \vdots \\ A_k \end{bmatrix} = \begin{bmatrix} B_1 \\ \vdots \\ B_k \end{bmatrix} C, \quad A_i, B_i \in M(n_i, m, R), \quad n_i \ge 1, \quad i = 1, \dots, k. \tag{1}$$

Further, we will denote r-th determinantal divisor of the matrix A by d_r^A, the greatest common divisor of elements a and b by $(a, b) = d$. Let $(d_{n_i}^{A_i}, d_{n_j}^{A_j}) = d^{(A_i, A_j)}$ and $(\det B, d_n^C) = d^{(B,C)}$.

Lemma 1. *Let* $(\det B, d_{n_i}^{A_i}) = \varphi_i, \quad i = 1, \dots, k$. *If*

$$(d^{(B,C)}, d_{n-1}^A) = 1, \tag{2}$$

then $d_{n_i}^{B_i} = \varphi_i$, $i = 1, \dots, k$.

Proof. Let $k = 2$. From $A_1 = B_1 C$ and $(\det B, d_{n_1}^{A_1}) = \varphi_1$ it follows that $d_{n_1}^{B_1} | d_{n_1}^{A_1}$ and $d_{n_1}^{B_1} | \varphi_1$, that is $\varphi_1 = d_{n_1}^{B_1} g$. We assume that $d_{n_1}^{B_1} \ne \varphi_1$. This means that $g \notin U(R)$, where $U(R)$ is the group of units of the ring R.

Let p be an irreducible element from the ring R such that $p | g$. We suppose that $p | d^{(B,C)}$. The matrix B_1 can be written as

$$B_1 = G F_1, \quad G \in M(n_1, R), \quad F_1 \in M(n_1, n, R), \quad \det G = d_{n_1}^{B_1}, \quad d_{n_1}^{F_1} = 1.$$

Hence, $A_1 = G H_1, \quad H_1 \in M(n_1, n, R)$. So, from (1) we obtain

$$\begin{bmatrix} H_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ B_2 \end{bmatrix} C.$$

For matrix F_1 there exists such a matrix $W \in GL(n,R)$ that $F_1 W = \begin{bmatrix} I_{n_1} & 0 \end{bmatrix}$, where I_{n_1} is an identity matrix. Therefore

$$\begin{bmatrix} H_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} I_{n_1} & 0 \\ \widetilde{B}_{21} & \widetilde{B}_{22} \end{bmatrix} \widetilde{C}, \quad \widetilde{C} = W^{-1}C, \widetilde{B}_{21} \in M(n_2, n_1, R), \widetilde{B}_{22} \in M(n_2, R).$$

So, for the matrix

$$V = \begin{bmatrix} I_{n_1} & 0 \\ -\widetilde{B}_{21} & I_{n_2} \end{bmatrix}$$

we obtain

$$V \begin{bmatrix} H_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} = \begin{bmatrix} I_{n_1} & 0 \\ 0 & \widetilde{B}_{22} \end{bmatrix} \begin{bmatrix} \widetilde{C}_{11} & \widetilde{C}_{12} \\ \widetilde{C}_{21} & \widetilde{C}_{22} \end{bmatrix} = \widetilde{B}\widetilde{C}.$$

Since $p | \det \widetilde{B}_{22}$, we obtain $p | d_{n_2}^{H_2}$. On the other hand $p | d_{n_1}^{H_1}$, hence $p | d_{n-1}^{A}$, which contradicts the condition (2) of the lemma. Thus, $g \in U(R)$ and $d_{n_1}^{B_1} = \varphi_1$.

In the same way, we can prove that $d_{n_2}^{B_2} = \varphi_2$.

If $p \nmid d^{(B,C)}$ (does not divide), the proof of the lemma is similar.

For an arbitrary k, we prove the lemma by induction. □

Lemma 2. *Let* $((d_{n_l}^{A_l}, d_{n_{l+1}}^{A_{l+1}}), d^{(B,C)}) = 1, \quad l = 1, \dots, k-1.$ *If* $d_n^A = d_{n_1}^{A_1} \cdots d_{n_k}^{A_k}$, *then* $\det B = d_{n_1}^{B_1} \cdots d_{n_k}^{B_k}$.

Proof. Following the same procedure as in the proof of Lemma 1, we obtain that $(\det B, d_{n_i}^{A_i}) = d_{n_i}^{B_i}$, $i = 1, \dots, k$. We suppose that $\det B = d_{n_1}^{B_1} \cdots d_{n_k}^{B_k} f$, $f \notin U(R)$. The matrices B_i and A_i from (1) can be written as $B_i = G_i F_i$, $A_i = G_i H_i$, $G_i \in M(n_i, R)$, F_i, $H_i \in M(n_i, m, R)$, $\det G_i = d_{n_i}^{B_i}$, $d_{n_i}^{F_i} = 1$, $i = 1, \dots, k$.

From (1) we obtain

$$\begin{bmatrix} H_1 \\ \vdots \\ H_k \end{bmatrix} = \begin{bmatrix} F_1 \\ \vdots \\ F_k \end{bmatrix} C \tag{3}$$

or else $H = FC$. Let q be an irreducible element from the ring R such that $q | f$. It is obvious that $q | \det H$. Since $\det H = d_{n_1}^{H_1} \cdots d_{n_k}^{H_k}$, $q | d_{n_i}^{H_i}$ for a certain i. We assume that $q | d_{n_1}^{H_1}$. Then from (3) we have $H_1 = F_1 C$. Since $d_{n_1}^{F_1} = 1$, $q | d_n^C$. So, from $H_j = F_j C$ we obtain $q | d_{n_j}^{H_j}$, $j = 1, \dots, k$. Thus $q | d^{(A_l, A_{l+1})}$ for all $l = 1, \dots, k-1$.

Hence, we get $q | d^{(B,C)}$. Since $((d_{n_l}^{A_l}, d_{n_{l+1}}^{A_{l+1}}), d^{(B,C)}) = 1$, $l = 1, \dots, k-1$, $q = 1$. So $f \in U(R)$ and thus, $\det B = d_{n_1}^{B_1} \cdots d_{n_k}^{B_k}$. □

Corollary 1. *Let* $A \in M(n,R)$ *and* $\det A = \varphi_1 \cdots \varphi_k$. *Then the matrix* A *is the right equivalent to the block diagonal matrix, that is* $AV = \operatorname{diag}(D_1, \dots, D_k)$, $D_i \in$

$M(n_i, R)$, $\det D_i = \varphi_i$, $i = 1, \dots k$, *if and only if the matrix* A *can be written in the form*

$$A = \begin{bmatrix} A_1 \\ \vdots \\ A_k \end{bmatrix}, \quad A_i \in M(n_i, m, R), \ d_{n_i}^{A_i} = \varphi_i, \quad i = 1, \dots, k.$$

Lemma 3. *Let* $C \in M(n, m, R)$, $n \le m$ *and* $d_n^C \ne 0$. *Let* $A = \begin{bmatrix} \mathbf{c}_{j_1} & \dots & \mathbf{c}_{j_n} \end{bmatrix}$ *be a submatrix which is composed of* $j_1, \dots, j_n$ *columns of the matrix* C *and such that* $\det A = d_n^C$. *Then there exists a matrix* $Q \in GL(m, R)$ *such that* $CQ = \begin{bmatrix} A & 0 \end{bmatrix}$.

Proof. Using the elementary column operations, we reduce the matrix C to the form $CP = \begin{bmatrix} A & B \end{bmatrix} = C_1$, where $P \in GL(m, R)$. For the matrices A and B there exist matrices $V_1 \in GL(n, R)$ and $V_2 \in GL(m - n, R)$ such that $AV_1 = A_1$, $BV_2 = B_1$ and they are lower triangular matrices.

Put $m - n \ge n$. Then

$$\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \end{bmatrix} =$$

$$\begin{bmatrix} a_1 & 0 & \cdots & 0 & b_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ a_{21} & a_2 & \cdots & 0 & b_{21} & b_2 & \cdots & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_n & b_{n1} & b_{n2} & \cdots & b_n & 0 & \cdots & 0 \end{bmatrix} = C_2.$$

It is obvious that $d_n^{C_2} = \det A_1 = \det A$. Therefore all the n-th order minors of the matrix C_2 are divided by $\det A_1 = a_1 \cdots a_n$. Hence, the element b_i of the matrix C_2 is divided by a_i for all $i = 1, \dots, n$.

Using the elementary column operations, we reduce the matrix C_2 to the form

$$\begin{bmatrix} a_1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ a_{21} & a_2 & \cdots & 0 & b'_{21} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ a_{n1} & a_{n2} & \cdots & a_n & b'_{n1} & b'_{n2} & \cdots & b'_{n,n-1} & 0 & 0 & \cdots & 0 \end{bmatrix} = C_3.$$

Continuing this way, we obtain that $C_1 W = \begin{bmatrix} A_1 & 0 \end{bmatrix}$, where $W \in GL(m, R)$. Hence, the matrix C is the right equivalent to the matrix $\begin{bmatrix} A & 0 \end{bmatrix}$.

If $m - n < n$, the proof of the lemma is similar. □

Corollary 2. *Let* $C = \begin{bmatrix} A & B \end{bmatrix}$, $C \in M(n, m, R)$, $A \in M(n, R)$, $d_n^C \ne 0$. *If* $\det A = d_n^C$, *then there exists such a unitriangular matrix* $S = \begin{bmatrix} I_n & S_{12} \\ 0 & I_{m-n} \end{bmatrix}$ *that*

$$\begin{bmatrix} A & B \end{bmatrix} S = \begin{bmatrix} A & 0 \end{bmatrix}.$$

3 Factorizations of the block matrices

We suppose that the nonsingular matrix $T = \mathrm{triang}(T_{11}, \ldots, T_{kk})$ has the factorization in the ring $BT(n_1, \ldots, n_k, R)$:

$$T = BC = \begin{bmatrix} B_{11} & B_{12} & \ldots & B_{1k} \\ 0 & B_{22} & \ldots & B_{2k} \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \cdots & B_{kk} \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} & \ldots & C_{1k} \\ 0 & C_{22} & \ldots & C_{2k} \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \cdots & C_{kk} \end{bmatrix}, \tag{4}$$

where $B_{ii}, C_{ii} \in M(n_i, R)$, $B_{ij}, C_{ij} \in M(n_i, n_j, R)$, $i, j = 1, \ldots, k,\ i < j$. Then the diagonal blocks T_{ii} and their determinants $\det T_{ii}$ of the matrix T have such factorizations

$$T_{ii} = B_{ii} C_{ii}, \quad i = 1, \ldots, k, \tag{5}$$

and

$$\det T_{ii} = \varphi_i \psi_i, \quad \varphi_i = \det B_{ii},\ \psi_i = \det C_{ii}, \quad i = 1, \ldots, k. \tag{6}$$

Definition 1. We will call the factorization (4) of the matrix T the corresponding one to the factorization (5) of its diagonal blocks T_{ii} and the parallel one to the factorization (6) of the determinants $\det T_{ii}$ of their diagonal blocks or briefly, the parallel factorization of the matrix T in the ring $BT(n_1, \ldots, n_k, R)$.

It should be highlighted that there does not exist the corresponding factorization of the matrix T, that is its factorization in the ring $BT(n_1, \ldots, n_k, R)$, for every factorization (5) of the diagonal blocks T_{ii}.

For each factorization

$$\det T = \varphi\psi,\ \varphi = \prod_{i=1}^{k} \varphi_i,\ \psi = \prod_{i=1}^{k} \psi_i,\ i = 1, \ldots, k, \tag{7}$$

of the determinant of the matrix T there exists the parallel factorization $T = BC$ of the matrix T in the ring $M(n, R)$, that is the factorization is such that $\det B = \varphi,\ \det C = \psi$. However, there does not exist the parallel factorization (4) in the ring $BT(n_1, \ldots, n_k, R)$ for every factorization $\det T_{ii} = \varphi_i \psi_i$ of the determinants of the diagonal blocks T_{ii} of the matrix T.

Further, we describe the factorizations of the matrices in the ring $BT(n_1, \ldots, n_k, R)$. We have established some conditions, under which the factorizations of the matrices $T \in BT(n_1, \ldots, n_k, R)$ are the same block triangular form up to the association, that is when they are the factorizations in the ring $BT(n_1, \ldots, n_k, R)$. We have proved the uniqueness criteria of such factorizations.

Theorem 1. *Let $T \in BT(n_1, \ldots, n_k, R)$ be a nonsingular matrix and its diagonal blocks T_{ii}, $i = 1, \ldots, k$, have the factorizations of the form (5). Then there exists a unique up to the association factorization of the matrix T in the ring $BT(n_1, \ldots, n_k, R)$, that is $T = \mathrm{triang}(B_{11}, \ldots, B_{kk})\mathrm{triang}(C_{11}, \ldots, C_{kk})$ if and only if*

$$(\det B_{ss}, \det C_{s+t,s+t}) = 1, \quad \textit{for all} \quad s = 1, \ldots, k-1, \quad t = 1, \ldots, k-s. \tag{8}$$

Proof. The matrix T has the factorization (4) corresponding to the factorizations (5) of the diagonal blocks T_{ii}, $i = 1, \dots, k$, if and only if the system of the linear matrix equations

$$B_{ii}X_{ij} + Y_{ij}C_{jj} + \sum_{l=i+1}^{j-1} Y_{il}X_{lj} = T_{ij}, \quad 1 \le i < j \le k, \tag{9}$$

has solutions. The system solutions are $X_{ij} = C_{ij}$, $Y_{ij} = B_{ij}$, $i < j$, $i, j = 1, \dots, k$. The solving of the system is reduced to the solving of the linear Sylvester matrix equations in the form

$$B_{ii}X_{ij} + Y_{ij}C_{jj} = T_{ij}, \quad 1 \le i < j \le k. \tag{10}$$

From (8) it follows that $(\det B_{ii}, \det C_{jj}) = 1$, $1 \le i < j \le k$. Then every linear matrix equation (10) has a solution [12]. Therefore, the system of the matrix equations (9) has a solution. Consequently, the matrix T has the factorization of the form (4) corresponding to the factorizations (5) of its diagonal blocks.

For the matrix T there exist such invertible matrices U and V over R that $TU = F$, $BV = H^B$, $V^{-1}CU = D$ are upper triangular matrices. The matrix H^B has the Hermite normal form [11]. It follows from (4) that $F = H^B D$:

$$\begin{bmatrix} F_{11} & F_{12} & \dots & F_{1k} \\ 0 & F_{22} & \dots & F_{2k} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & F_{kk} \end{bmatrix} = \begin{bmatrix} H^{B_{11}} & G_{12} & \dots & G_{1k} \\ 0 & H^{B_{22}} & \dots & G_{2k} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & H^{B_{kk}} \end{bmatrix} \begin{bmatrix} D_{11} & D_{12} & \dots & D_{1k} \\ 0 & D_{22} & \dots & D_{2k} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & D_{kk} \end{bmatrix} \tag{11}$$

where $H^{B_{pp}} = B_{pp}V_{pp} = [h_{ij}^{(p)}]_1^{n_p}$ is the Hermite normal form of the block B_{pp}. Each element of the $i-$th row of the matrix $G_{pq} = [g_{ij}^{(pq)}]_1^{n_p, n_q}$ lies in a prescribed complete set of residues modulo the diagonal element $h_{ii}^{(p)}$ of the matrix $H^{B_{pp}}$, that is $g_{ij}^{(pq)} \in R_{h_{ii}^{(p)}}$, $i = 1, \dots, n_p$, $j = 1, \dots, n_q$, $1 \le p < q \le k$.

It follows from the factorization (11) that the matrices $X_{pq} = D_{pq}$, $Y_{pq} = G_{pq}$, $1 \le p < q \le k$, are the solutions of the system of the linear matrix equations

$$H^{B_{pp}}X_{pq} + Y_{pq}D_{qq} + \sum_{l=p+1}^{q-1} Y_{pl}X_{lg} = F_{pq}, \quad 1 \le p < q \le k. \tag{12}$$

The solving of this system of the matrix equations is reduced to the solving of the linear Sylvester matrix equations in the form

$$H^{B_{pp}}X_{pq} + Y_{pq}D_{qq} = F_{pq}, \quad 1 \le p < q \le k. \tag{13}$$

It follows from [2] that the solution $X_{pq} = D_{pq}$, $Y_{pq} = G_{pq} = [g_{ij}^{(pq)}]_1^{n_p, n_q}$ of the equation (13), where $g_{ij}^{(pq)} \in R_{h_{ii}^{(p)}}$, $i = 1, \dots, n_p$, $j = 1, \dots, n_q$, $1 \le p < q \le k$, is unique if and only if $(\det H^{B_{ii}}, \det D_{jj}) = 1$, $i, j = 1, \dots, k$, $i < j$. These conditions hold if the conditions (8) are true. The factorizations (11) and (4) of the matrix T are associate. $\square$

Corollary 3. *Let the determinants* $\det T_{ii}$ *of the diagonal blocks* T_{ii}, $i = 1, \dots, k$, *of the matrix* $T \in BT(n_1, \dots, n_k, R)$ *have the factorizations*

$$\det T_{ii} = \varphi_i \psi_i, \ i = 1, \dots, k, \quad \textit{and} \quad \prod_{i=1}^{k} \varphi_i = \varphi, \ \prod_{i=1}^{k} \psi_i = \psi. \tag{14}$$

Let at least one of the following conditions hold:

(i) $(\prod_{i=1}^{s} \varphi_i, \psi_{s+1}) = 1, \quad s = 1, \dots, k-1, \quad$ *and* $\quad ((\varphi, \psi), d_{n-1}^{T}) = 1,$

(ii) $(\det T_{ii}, (\varphi, \psi)) = 1, \ i = 1, \dots, k-1.$

Then there exist the factorizations

$$T_{ii} = B_{ii} C_{ii}, \qquad \det B_{ii} = \varphi_i, \quad \det C_{ii} = \psi_i, \quad p = 1, \dots, k \tag{15}$$

of the diagonal blocks T_{ii} *and the factorization of the matrix* T

$$T = BC, \quad \det B = \varphi, \quad \det C = \psi, \tag{16}$$

in the ring $BT(n_1, \dots, n_k, R)$. *This factorization of the matrix* T *is unique up to the association.*

Theorem 2. *Let* $T = \operatorname{triang}(T_{11}, \dots, T_{kk})$ *be a nonsingular matrix and the determinants of the diagonal blocks* T_{ii}, $i = 1, \dots, k$, *have the factorizations in the form (14). If at least one of the following conditions holds:*

(i) $(\prod_{i=1}^{s} \varphi_i, \psi_{s+1}) = 1,$ *and* $\quad ((\varphi, \psi), d_{n-1}^{T}) = 1, \quad s = 1, \dots, k-1,$

(ii) $(\det T_{ii}, (\varphi, \psi)) = 1, \ i = 1, \dots, k-1,$

then there exists the parallel factorization of the matrix T *in the ring* $BT(n_1, \dots, n_k, R)$: $T = BC$, $B, C \in BT(n_1, \dots, n_k, R)$, *that is* $B = \operatorname{triang}(B_{11}, \dots, B_{kk})$, $C = \operatorname{triang}(C_{11}, \dots, C_{kk})$, $B_{ii}, C_{ii} \in M(n_i, R)$ *and* $\det B_i = \varphi_i$, $\det C_i = \psi_i$, $i{=}1, \dots, k$. *Each parallel factorization* $T = BC$, $B, C \in M(n, R)$, $\det B = \varphi$, $\det C = \psi$ *of the matrix* T *in the ring* $M(n, R)$ *is associate to the parallel factorization* $T = \widetilde{B}\widetilde{C}$, *where* $\widetilde{B} = \operatorname{triang}(\widetilde{B}_{11}, \dots, \widetilde{B}_{kk})$, $\widetilde{C} = \operatorname{triang}(\widetilde{C}_{11}, \dots, \widetilde{C}_{kk})$ *and* $\det \widetilde{B}_{ii} = \varphi_i$, $\det \widetilde{C}_{ii} = \psi_i$, $i = 1, \dots, k$, *in the ring* $BT(n_1, \dots, n_k, R)$.

Proof. Let $k = 2$, that is $T = \operatorname{triang}(T_{11}, T_{22})$. It follows from the conditions (7) that there exists such a factorization $T = BC$ of the matrix T that $\det B = \varphi$, $\det C = \psi$. We write it in an appropriate block form

$$\operatorname{triang}(T_{11}, T_{22}) = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} C, \tag{17}$$

$B_{ij} \in M(n_i, n_j, R)$, $\quad C \in M(n, R)$, $i, j = 1, 2$. It follows from the conditions *(i)* of the theorem that $(\det B, \det T_{22}) = \varphi_2$.

According to Lemma 1 $d_{n_2}^{B_2} = \varphi_2$, where $B_2 = \begin{bmatrix} B_{21} & B_{22} \end{bmatrix}$, there exists such a matrix $V \in GL(n, R)$ that $B_2 V = \begin{bmatrix} 0 & \widetilde{B}_{22} \end{bmatrix}$, where $\widetilde{B}_{22} \in M(n_2, R)$ and $\det \widetilde{B}_{22} = \varphi_2$. So, from (17) we get

$$\operatorname{triang}(T_{11}, T_{22}) = \operatorname{triang}(\widetilde{B}_{11}, \widetilde{B}_{22}) \operatorname{triang}(\widetilde{C}_{11}, \widetilde{C}_{22}) = \widetilde{B}\widetilde{C},$$

where $\widetilde{B} = BV$, $\widetilde{C} = V^{-1}C$, $V \in GL(n, R)$, $B_{ij}, C_{ij} \in M(n_i, n_j, R)$ and $\det \widetilde{B}_{ii} = \varphi_i$, $\det \widetilde{C}_{ii} = \psi_i$, $i = 1, 2$.

Similarly, the theorem can be proved under the condition *(ii)*.

For an arbitrary k, we prove the theorem by induction. □

Corollary 4. *Let the determinants* $\det T_{ii}$ *of the diagonal blocks* T_{ii}, $i = 1, \ldots, k$, *of the matrix* $T \in BT(n_1, \ldots, n_k, R)$ *have the factorizations in the form (14). If at least one of the following conditions holds:*

(i) $(\prod_{i=1}^{s} \varphi_i, \psi_{s+1}) = 1, \quad s = 1, \ldots, k-1, \quad$ *and* $\quad ((\varphi, \psi), d_{n-1}^{T}) = 1,$

(ii) $(\det T_{ii}, (\varphi, \psi)) = 1, \ i = 1, \ldots, k-1,$

then there exist factorizations (15) of the diagonal blocks T_{ii} *and the factorization of matrix* T *(16) in the ring* $BT(n_1, \ldots, n_k, R)$. *This factorization of the matrix* T *is unique up to the association.*

Theorem 3. *Let the determinants of the diagonal blocks* T_{ii}, $i = 1, \ldots, k$, *of the matrix* $T \in BT(n_1, \ldots, n_k, R)$ *have the factorizations in the form (14). Then there exists the factorization of the matrix* T *parallel to the factorization (14) of the determinants of the diagonal blocks if and only if the following conditions hold:*

(i) $((\varphi_i, \psi_i), d_{n_i-1}^{T_{ii}}) = 1, \ i = 1, \ldots, k,$

(ii) $(\varphi_s, \psi_{s+t}) = 1$ *for all* $s = 1, \ldots, k-1, \quad t = 1, \ldots, k-s.$

This factorization of the matrix T *is unique up to the association.*

Proof. It follows from the factorizations (14) of the determinants $\det T_{ii}$ of the diagonal blocks T_{ii} of the matrix T that there exist the parallel factorizations of the diagonal blocks T_{ii}. When the condition *(i)* holds, these factorizations of the blocks T_{ii} are parallel to the factorizations of their determinants up to the association and they are unique. From Theorem 1 we conclude that there exists the factorization of the matrix T corresponding to the factorizations (5) of its diagonal blocks T_{ii} and parallel to the factorizations (14) of the determinants of the diagonal blocks T_{ii}, $i = 1, \ldots, k$, and it is unique up to the association. □

It should be highlighted that there does not exist the parallel factorization in the ring $BT(n_1, \ldots, n_k, R)$ for every factorization of the determinants of the diagonal blocks T_{ii}, $i = 1, \ldots, k$, of the matrix T.

We establish the matrices having such a property in the following corollary.

Corollary 5. *Let the determinants of the diagonal blocks* $T_{ii},\ i=1,\dots,k$, *of the matrix* $T \in BT(n_1,\dots,n_k,R)$ *be pairwise relatively prime, that is* $(\det T_{ii}, \det T_{jj}) = 1$. *Then for each factorization (14) of the determinants* $\det T_{ii}$ *of the diagonal blocks* $T_{ii},\ i=1,\dots,k$, *there exists the parallel factorization of the matrix* T, *that is the matrix* T *has the maximum number of the parallel factorizations.*

The block diagonal matrices $D = \mathrm{diag}(D_{11},\dots,D_{kk}),\ D_{ii} \in M(n_i,R),\ i = 1,\dots,k$, form the subring $BD(n_1,\dots,n_k,R)$ of the ring of the block triangular matrices. We consider the factorizations of the matrices in the ring $BD(n_1,\dots,n_k,R)$.

Definition 2. Let the determinants of the diagonal blocks $D_{ii} \in M(n_i,R),\ i = 1,\dots,k$, of the matrix $D = \mathrm{diag}(D_{11},\dots,D_{kk})$ have the factorizations

$$\det D_{ii} = \varphi_i\psi_i,\ i=1,\dots,k. \tag{18}$$

The factorization $D = BC$, $B = \mathrm{diag}(B_{11},\dots,B_{kk})$, $C = \mathrm{diag}(C_{11},\dots,C_{kk})$, of the matrix D is such that $\det B_{ii} = \varphi_i,\ \det C_{ii} = \psi_i,\ i=1,\dots,k$, and is called the parallel factorization to the factorizations (18) of the determinants of the diagonal blocks $D_{ii},\ i=1,\dots,k$, or briefly, the parallel factorization of the matrix D in the ring $BD(n_1,\dots,n_k,R)$ of the block diagonal matrices.

Theorem 4. *Let* $D \in BD(n_1,\dots,n_k,R)$, *that is* $D = \mathrm{diag}(D_{11},\dots,D_{kk}),\ D_{ii} \in M(n_i,R),\ i=1,\dots,k$, *and the determinants of its diagonal blocks* D_{ii} *have the factorizations:*

$$\det D_{ii} = \varphi_i\psi_i,\ \prod_{i=1}^{k}\varphi_i = \varphi,\ \prod_{i=1}^{k}\psi_i = \psi,\ i=1,\dots,k. \tag{19}$$

If $((\det D_{ii}, \det D_{jj}),(\varphi,\psi)) = 1$, $i,j = 1,\dots,k$, $i \neq j$, *then for the matrix* D *there exists the factorization* $D = BC$, $B, C \in M(n,R)$, $\det B = \varphi$, $\det C = \psi$, *in the ring* $M(n,R)$ *and each of such factorizations is associate to the parallel factorization of the matrix* D *in the ring* $BD(n_1,\dots,n_k,R)$, *that is* $D = \widetilde{B}\widetilde{C}$, *where* $\widetilde{B} = BV = \mathrm{diag}(\widetilde{B}_{11},\dots,\widetilde{B}_{kk})$, $\widetilde{C} = V^{-1}C = \mathrm{diag}(\widetilde{C}_{11},\dots,\widetilde{C}_{kk})$, $V \in GL(n,R)$, $\widetilde{B}_{ii}, \widetilde{C}_{ii} \in M(n_i,R)$, $\det \widetilde{B}_{ii} = \varphi_i$, $\det \widetilde{C}_{ii} = \psi_i$, $i=1,\dots,k$.

Proof. Let $k=2$. It follows from (19) that there exists such a factorization $D = BC$ of the matrix D that $\det B = \varphi$, $\det C = \psi$. We write it in the block form

$$\begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} C, \tag{20}$$

where $B_{ii} \in M(n_i,R)$, $C \in M(n,R)$, $i=1,2$. Then, from (19) we have that $(\det B, \det D_i) = \varphi_i$, $i=1,2$.

Based on Lemma 2, $\det B = d_{n_1}^{B_1} d_{n_2}^{B_2}$, where $B_i = \begin{bmatrix} B_{i1} & B_{i2} \end{bmatrix}$, $i=1,2$. Since $d_{n_i}^{B_i} | \varphi_i$, $i=1,2$, and $\det B = \varphi_1\varphi_2$, it follows that $d_{n_i}^{B_i} = \varphi_i$, $i=1,2$. For the matrix

B_2 there exists such a matrix $U \in GL(n,R)$ that $B_2U = \begin{bmatrix} 0 & \widetilde{B}_{22} \end{bmatrix}$, $\widetilde{B}_{22} \in M(n_2,R)$, $\det \widetilde{B}_{22} = \varphi_2$.

Then from the equality (20) we obtain:

$$\begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} = \begin{bmatrix} \widetilde{B}_{11} & \widetilde{B}_{12} \\ 0 & \widetilde{B}_{22} \end{bmatrix} \widetilde{C}, \text{ where } \begin{bmatrix} \widetilde{B}_{11} & \widetilde{B}_{12} \\ 0 & \widetilde{B}_{22} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} U, \quad \widetilde{C} = U^{-1}C.$$

According to Corollary 2 there exists such a matrix $Q = \begin{bmatrix} I_{n_1} & Q_{12} \\ 0 & I_{n_2} \end{bmatrix}$ that $\begin{bmatrix} \widetilde{B}_{11} & \widetilde{B}_{12} \end{bmatrix} Q = \begin{bmatrix} \widetilde{B}_{11} & 0 \end{bmatrix}$.

Thus, we get $D = \widetilde{B}\widetilde{C}$, $\widetilde{B} = BW = \operatorname{diag}(\widetilde{B}_{11}, \widetilde{B}_{22})$, $\widetilde{C} = W^{-1}C = \operatorname{diag}(\widetilde{C}_{11}, \widetilde{C}_{22})$, $W = UQ$, $\widetilde{B}_{ii}, \widetilde{C}_{ii} \in M(n_i,R)$, $\det \widetilde{B}_{ii} = \varphi_i$, $\det \widetilde{C}_{ii} = \psi_i$, $i = 1,2$.

For an arbitrary k, we prove the theorem by induction. □

Corollary 6. *If the determinants of the diagonal blocks D_{ii} of the matrix $D \in BD(n_1,\dots,n_k,R)$ are pairwise relatively prime, then each factorization $D = BC$, $B, C \in M(n,R)$ of the matrix D in the ring $M(n,R)$ is associate to a certain parallel factorization of the matrix D in the ring $BD(n_1,\dots,n_k,R)$.*

References

[1] Borevich Z. I. *On factorizations of matrices over a principal ideal ring.* Proceedings of the Third All-Union Symposium on the Theory of Rings, Algebras and Modules: Abstracts of talks. Tartu, University Press, 1976, 19 (in Russian)

[2] Dzhaliuk N. S., Petrychkovych V. M. *The matrix linear unilateral and bilateral equations with two variables over commutative rings.* International Scholarly Research Network, ISRN Algebra, Volume 2012, Article ID 205478, 14 pages, doi: 10.5402/2012/205478.

[3] Feinberg R. B. *Equivalence of partitioned matrices.* J. Res. Bur. Stand. Sect., 1976, **80**, No. 1, 89–97.

[4] Gohberg I., Lancaster P., Rodman L. *Matrix polynomials.* New York, Academic Press, 1982.

[5] Gustafson W. H. *Roth's theorem over commutative rings.* Linear Algebra Appl., 1979, **23**, 245–251.

[6] Gustafson W. H., Zelmanowitz J. M. *On matrix equivalence and matrix equations.* Linear Algebra Appl., 1979, **27**, 219–224.

[7] Kazimirs'kii P. S. *Factorization of matrix polynomials.* L'viv, Pidstryhach Institute for Applied Problems of Mechanics and Mathematics of the NAS of Ukraine, 2015 (in Ukrainian).

[8] Lancaster P. *Lambda-matrices and vibrating systems.* Reprint of the 1966 original [New York, Pergamon Press], Mineola, NY, Dover Publications, nc., 2002.

[9] Markus A. S. *Introduction to the spectral theory of polynomial operator pencils.* Transl. from the Russian by H. H. McFaden. Translations of Mathematical Monographs, 71. Providence, RI: American Mathematical Society (AMS), 1988.

[10] Martines F., Pereira E. *Block matrices and stability theory.* Tatra Mt. Math. Publ., 2007, **38**, 147–162.

[11] Newman M. *Integral matrices.* New York, Academic Press, 1972.

[12] Newman M. *The Smith normal form of a partitioned matrix.* J. Res. Bur. Stand. Sect., 1974, **78B**, No. 1, 3–6.

[13] Petrychkovych V. M. *Cell-triangular and cell-diagonal factorizations of cell-triangular and cell-diagonal polynomial matrices.* Math. Notes, 1985, **37**, No. 6, 431–435.

[14] Petrychkovych V. M. *Generalized Equivalence of Matrices and its Collections and Factorization of Matrices over Rings.* L'viv, Pidstryhach Inst. Appl. Probl. Mech. and Math. of the NAS of Ukraine, 2015 (in Ukrainian).

[15] Roth W. E. *The equations $AX - YB = C$ and $AX - XB = C$ in matrices.* Proc. Amer. Math. Soc., 1952, **3**, 392–396.

[16] Yang Y., Holtti H. *The factorization of block matrices with generalized geometric progression rows.* Linear Algebra Appl., 2004, **387**, 51–67.

Vasyl' Petrychkovych
Nataliia Dzhaliuk

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Pidstryhach Institute for Applied Problems of Mechanics
and Mathematics of the NAS
of Ukraine Department of Algebra
3b Naukova Str., 79060, L'viv, Ukraine

E-mail: *vas_petrych@yahoo.com*
E-mail: *nataliya.dzhalyuk@gmail.com*