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Some estimates for angular derivative at the boundary

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Abstract. In this paper, we establish lower estimates for the modulus of the values of $f(z)$ on boundary of unit disc. For the function $f(z) = 1 + c_1 z + c_2 z^2 + ...$ defined in the unit disc such that $f(z) \in \mathcal{N}(\beta)$ assuming the existence of angular limit at the boundary point b, the estimations below of the modulus of angular derivative have been obtained at the boundary point b with $f(b) = \beta$. Moreover, Schwarz lemma for class $\mathcal{N}(\beta)$ is given. The sharpness of these inequalities has been proved.

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1 Introduction

Let f be a holomorphic function in the unit disc $D = \{z : |z| < 1\}$, $f(0) = 0$ and $|f(z)| < 1$ for $|z| < 1$. In accordance with the classical Schwarz lemma, for any point z in the disc D, we have $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. Equality in these inequalities (in the first one, for $z \neq 0$) occurs only if $f(z) = ze^{i\theta}$, where θ is a real number ([6], p.329). For historical background about the Schwarz lemma and its applications on the boundary of the unit disc, we refer to [2, 19].

The basic tool in proving our results is the following lemma due to Jack.

Lemma 1 (Jack's lemma)**.** *Let $f(z)$ be holomorphic function in the unit disc D with $f(0) = 0$. Then if $|f(z)|$ attains its maximum value on the circle $|z| = r$ at a point $z_0 \in D$, then there exists a real number $k \geq 1$ such that*

$$\frac{z_0 f'(z_0)}{f(z_0)} = k.$$

Let $\mathcal{A}$ denote the class of functions

$$f(z) = 1 + c_1 z + c_2 z^2 + ...$$

that are holomorphic in the unit disc D. Also, $\mathcal{N}(\beta)$ be the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which satisfy

$$\Re\left(f(z) - \frac{zf'(z)}{f(z)}\right) > \frac{\beta(3-2\beta)}{2(1-\beta)}, \tag{1.1}$$

where $\beta < 0$.

Let $f(z) \in \mathcal{N}(\beta)$ and define $\varphi(z)$ in D by

$$\varphi(z) = \frac{f(z) - 1}{f(z) - (2\beta - 1)}.$$

Obviously, $\varphi(z)$ is holomorphic function in the unit disc D and $\varphi(0) = 0$. We want to prove $|\varphi(z)| < 1$ for $|z| < 1$.

If there exists a point $z_0 \in D$ such that

$$\max_{|z| \leq |z_0|} |\varphi(z)| = |\varphi(z_0)| = 1,$$

then Jack's lemma gives us that $\varphi(z_0) = e^{i\theta}$ and $z_0\varphi'(z_0) = k\varphi(z_0)$ $(k \geq 1)$.

Thus we have

$$\frac{z_0 f'(z_0)}{f(z_0)} = \frac{\frac{2(1-\beta)z_0\varphi'(z_0)}{(1-\varphi(z_0))^2}}{(1-\beta)\frac{1+\varphi(z_0)}{1-\varphi(z_0)} + \beta} = \frac{\frac{2(1-\beta)ke^{i\theta}}{\left(1-e^{i\theta}\right)^2}}{(1-\beta)\frac{1+e^{i\theta}}{1-e^{i\theta}} + \beta}.$$

Since

$$\frac{e^{i\theta}}{\left(1-e^{i\theta}\right)^2} = \frac{e^{i\theta}}{1 - 2e^{i\theta} + e^{2i\theta}} = \frac{1}{e^{-i\theta} - 2 + e^{i\theta}} = \frac{1}{2\left(\cos\theta - 1\right)}$$

and

$$\begin{aligned} \frac{1+e^{i\theta}}{1-e^{i\theta}} &= \frac{1+\cos\theta + i\sin\theta}{1-\cos\theta - i\sin\theta} = \frac{\left(1+\cos\theta + i\sin\theta\right)\left(1-\cos\theta + i\sin\theta\right)}{\left(1-\cos\theta\right)^2 + \sin^2\theta} \\ &= \frac{i\sin\theta}{1-\cos\theta}, \end{aligned}$$

we obtain

$$\begin{aligned} \frac{z_0 f'(z_0)}{f(z_0)} &= -\frac{\frac{(1-\beta)k}{1-\cos\theta}}{(1-\beta)\frac{i\sin\theta}{1-\cos\theta} + \beta} = \frac{-\left(1-\beta\right)k}{\left(1-\beta\right)i\sin\theta + \beta\left(1-\cos\theta\right)} \\ &= \frac{-\left(1-\beta\right)k\left[-\left(1-\beta\right)i\sin\theta + \beta\left(1-\cos\theta\right)\right]}{\beta^2\left(1-\cos\theta\right)^2 + \left(1-\beta\right)^2\sin^2\theta} \end{aligned}$$

and

$$\Re\left(\frac{z_0 f'(z_0)}{f(z_0)}\right) = \frac{-\beta\left(1-\beta\right)k\left(1-\cos\theta\right)}{\beta^2\left(1-\cos\theta\right)^2 + \left(1-\beta\right)^2\sin^2\theta}.$$

If we write $1 - \cos\theta = s$ and

$$h(s) = \frac{s}{\beta^2 s^2 + \left(1-\beta\right)^2\left(2s - s^2\right)},$$

then we have

$$\Re\left(\frac{z_0 f'(z_0)}{f(z_0)}\right) = -\beta\left(1-\beta\right)kh(s).$$

Since $h(s)$ takes its minimum value for $s=0$, we have that

$$\Re\left(\frac{z_0 f'(z_0)}{f(z_0)}\right) = -\beta(1-\beta)k\frac{1}{2(1-\beta)^2} \geq \frac{-\beta}{2(1-\beta)}.$$

Thus, we obtain

$$\Re\left(f(z_0) - \frac{z_0 f'(z_0)}{f(z_0)}\right) \leq \beta + \frac{\beta}{2(1-\beta)} = \frac{\beta(3-2\beta)}{2(1-\beta)}.$$

This contradict (1.1). So, there is no point $z_0 \in D$ such that $\varphi(z_0) = 1$. This means that $|\varphi(z)| < 1$ for $|z| < 1$. Thus, from the Schwarz lemma, we obtain

$$\left|f'(0)\right| \leq 2(1-\beta).$$

Moreover, the equality $|f'(0)| = 2(1-\beta)$ occurs for the function

$$f(z) = \frac{1-(2\beta-1)z}{1-z}.$$

That proves

Lemma 2. *If $f(z) \in \mathcal{N}(\beta)$, then we have*

$$\left|f'(0)\right| \leq 2(1-\beta) \tag{1.3}$$

The equality in (1.3) *occurs for the function*

$$f(z) = \frac{1-(2\beta-1)z}{1-z}.$$

This lemma yields a "$\mathcal{N}(\beta)$ *version*" of the classical Schwarz lemma for holomorphic function of one complex variable.

The following boundary version of the Schwarz lemma was proved in 1938 by Unkelbach in [20] and then rediscovered and partially improved by Osserman in 2000 [16].

Lemma 3. *Let $f(z)$ be a holomorphic function self-mapping of D, that is $|f(z)| < 1$ for all $z \in D$. Assume that there is a $b \in \partial D$ so that f extends continuously to b, $|f(b)| = 1$ and $f'(b)$ exists. Then*

$$\left|f'(b)\right| \geq \frac{2}{1+|f'(0)|}. \tag{1.4}$$

The equality in (1.4) *holds if and only if f is of the form*

$$f(z) = -z\frac{d-z}{1-dz}, \quad \forall z \in D,$$

for some constant $d \in (-1, 0]$.

Corollary 1. *Under the hypotheses lemma, we have*

$$\left|f'(b)\right| \geq 1, \tag{1.5}$$

with equality only if f is of the form

$$f(z) = ze^{i\theta},$$

where θ is a real number.

The following lemma, known as the Julia-Wolff lemma, is needed in the sequel [18].

Lemma 4 (Julia-Wolff lemma)**.** *Let f be a holomorphic function in E, $f(0) = 0$ and $f(D) \subset D$. If, in addition, the function f has an angular limit $f(b)$ at $b \in \partial D$, $|f(b)| = 1$, then the angular derivative $f'(b)$ exists and $1 \leq |f'(b)| \leq \infty$.*

Inequality (1.4) and its generalizations have important applications in geometric theory of functions (see, e.g., [6, 18]). Therefore, the interest to such type results is not vanished recently (see, e.g., [1, 2, 4, 5, 10, 11, 16, 17, 19] and references therein).

Vladimir N. Dubinin has continued this line and has made a refinement on the boundary Schwarz lemma under the assumption that $f(z) = c_p z^p + c_{p+1} z^{p+1} + ...$, with a zero set $\{z_k\}$ (see [4]).

S. G. Krantz and D. M. Burns [9] and D. Chelst [3] studied the uniqueness part of the Schwarz lemma. In M. Mateljević's papers, for more general results and related estimates, see also ([12–15]).

Also, M. Jeong [8] showed some inequalities at a boundary point for different form of holomorphic functions and found the condition for equality and in [7] a holomorphic self map defined on the closed unit disc with fixed points only on the boundary of the unit disc.

2 Main Results

In this section, for holomorphic function $f(z) = 1 + c_1 z + c_2 z^2 + ...$ belonging to the class of $\mathcal{N}(\beta)$, the modulus of the angular derivative of the function at the boundary point of the unit disc has been estimated.

Theorem 1. *Let $f(z) \in \mathcal{N}(\beta)$. Assume that, for some $b \in \partial D$, f has angular limit $f(b)$ at b and $f(b) = \beta$. Then we have the inequality*

$$\left|f'(b)\right| \geq \frac{1-\beta}{2}. \tag{2.1}$$

The equality in (2.1) occurs for the function

$$f(z) = \frac{1-(2\beta-1)z}{1-z}.$$

Proof. Consider the function

$$\varphi(z)=\frac{f(z)-1}{f(z)-(2\beta-1)}.$$

$\varphi(z)$ is a holomorphic function in the unit disc D and $\varphi(0)=0$. From the Jack's lemma and since $f(z)\in\mathcal{N}(\beta)$, we obtain $|\varphi(z)|<1$ for $|z|<1$. Also, we have $|\varphi(b)|=1$ for $b\in\partial D$.

From (1.5), we obtain

$$1\leq\left|\varphi'(b)\right|=\frac{2(1-\beta)\left|f'(b)\right|}{\left|f(b)-(2\beta-1)\right|^2}=\frac{2(1-\beta)\left|f'(b)\right|}{(\beta-(2\beta-1))^2}=\frac{2(1-\beta)\left|f'(b)\right|}{(1-\beta)^2}$$

and

$$1\leq\frac{2\left|f'(b)\right|}{1-\beta}.$$

So, we take the inequality (2.1).

Now, we shall show that the inequality (2.1) is sharp. Let

$$f(z)=\frac{1-(2\beta-1)z}{1-z}.$$

Then, we have

$$f'(z)=2\frac{1-\beta}{(z-1)^2},$$

and

$$\left|f'(-1)\right|=\frac{1-\beta}{2}.$$

□

Theorem 2. *Under the same assumptions as in Theorem 1, we have*

$$\left|f'(b)\right|\geq\frac{(1-\beta)^2}{1-\beta+\left|f'(0)\right|}. \tag{2.2}$$

The inequality (2.2) *is sharp with equality for the function*

$$f(z)=\frac{1+az+(2\beta-1)\left(z^2+az\right)}{1+2az+z^2},$$

where $a=\frac{|f'(0)|}{2(1-\beta)}$ *is an arbitrary number from* $[0,1]$ (*see* (1.3)).

Proof. Let $\varphi(z)$ be as in the proof of Theorem 1. Using the inequality (1.4) for the function $\varphi(z)$, we obtain

$$\frac{2}{1+\left|\varphi'(0)\right|}\leq\left|\varphi'(b)\right|=\frac{2\left|f'(b)\right|}{1-\beta}.$$

Since

$$\varphi'(z) = \frac{2(1-\beta) f'(z)}{(f(z) - (2\beta - 1))^2}$$

and

$$|\varphi'(0)| = \frac{2(1-\beta)|f'(0)|}{(f(0) - (2\beta - 1))^2} = \frac{2(1-\beta)|f'(0)|}{(1 - (2\beta - 1))^2} = \frac{|f'(0)|}{2(1-\beta)},$$

we have

$$\frac{2}{1 + \frac{|f'(0)|}{2(1-\beta)}} \leq \frac{2|f'(b)|}{1-\beta}$$

and

$$|f'(b)| \geq \frac{2(1-\beta)^2}{2(1-\beta) + |f'(0)|}.$$

To show that the inequality (2.2) is sharp, take the holomorphic function

$$f(z) = \frac{1 + az + (2\beta - 1)\left(z^2 + az\right)}{1 + 2az + z^2}.$$

Then

$$f'(1) = -\frac{1-\beta}{1+a}$$

and

$$|f'(1)| = \frac{1-\beta}{1+a}.$$

Since $a = \frac{|f'(0)|}{2(1-\beta)}$, we have

$$|f'(1)| = \frac{1-\beta}{1 + \frac{|f'(0)|}{2(1-\beta)}} = \frac{2(1-\beta)^2}{2(1-\beta) + |f'(0)|}.$$

□

The inequality (2.2) can be strengthened as below by taking into account c_2 which is second coefficient in the expansion of the function $f(z)$. That is, taking into account two consecutive coeffients, the inequality (2.2) has been strengthened. This is given by the following Theorem.

Theorem 3. *Let $f(z) \in \mathcal{N}(\beta)$. Assume that, for some $b \in \partial D$, f has angular limit $f(b)$ at b and $f(b) = \beta$. Then we have the inequality*

$$|f'(b)| \geq \frac{1-\beta}{2}\left(1 + \frac{2\left(2(1-\beta) - |c_1|\right)^2}{4(1-\beta)^2 - |c_1|^2 + \left|2(1-\beta)c_2 - c_1^2\right|}\right). \qquad (2.3)$$

The inequality (2.3) *is sharp with extremal function*

$$f(z) = \frac{1 - (2\beta - 1)z}{1 - z}.$$

Proof. Let $\varphi(z)$ be as in the proof of Theorem 1. By the maximum principle for each $z \in D$, we have $|\varphi(z)| \leq |z|$. So,

$$\Theta(z) = \frac{\varphi(z)}{z}$$

is a holomorphic function in D and $|\Theta(z)| < 1$ for $|z| < 1$.

From equality of $\Theta(z)$, we have

$$\begin{aligned} \Theta(z) &= \frac{\varphi(z)}{z} = \frac{1}{z}\frac{f(z)-1}{f(z)-(2\beta-1)} \\ &= \frac{1+c_1z+c_2z^2+c_3z^3+...-1}{z\left(1+c_1z+c_2z^2+c_3z^3+...-(2\beta-1)\right)} \\ &= \frac{c_1+c_2z+c_3z^2+...}{2\left(1-\beta\right)+c_1z+c_2z^2+c_3z^3+...}. \end{aligned}$$

Thus, we take

$$|\Theta(0)| = \frac{|c_1|}{2\left(1-\beta\right)} \leq 1 \tag{2.4}$$

and

$$\left|\Theta'(0)\right| = \frac{\left|2\left(1-\beta\right)c_2 - c_1^2\right|}{4\left(1-\beta\right)^2}.$$

Moreover, it can be seen that

$$\frac{b\varphi'(b)}{\varphi(b)} = \left|\varphi'(b)\right| \geq \left|(b)'\right| = \frac{b\,(b)'}{b}.$$

The function

$$\Phi(z) = \frac{\Theta(z)-\Theta(0)}{1-\overline{\Theta(0)}\Theta(z)}$$

is a holomorphic in the unit disc D, $|\Phi(z)| < 1$ for $|z| < 1$, $\Phi(0) = 0$ and $|\Phi(b)| = 1$ for $b \in \partial D$.

From (1.4), we obtain

$$\begin{aligned} \frac{2}{1+|\Phi'(0)|} &\leq \left|\Phi'(b)\right| = \frac{1-|\Theta(0)|^2}{\left|1-\overline{\Theta(0)}\Theta(b)\right|^2}\left|\Theta'(b)\right| \leq \frac{1+|\Theta(0)|}{1-|\Theta(0)|}\left|\Theta'(b)\right| \\ &= \frac{1+|\Theta(0)|}{1-|\Theta(0)|}\left\{\left|\varphi'(b)\right|-1\right\}. \end{aligned}$$

Since

$$\Phi'(z) = \frac{1-|\Theta(0)|^2}{\left(1-\overline{\Theta(0)}\Theta(z)\right)^2}\Theta'(z),$$

$$\left|\Phi'(0)\right| = \frac{|\Theta'(0)|}{1-|\Theta(0)|^2} = \frac{\frac{\left|2(1-\beta)c_2-c_1^2\right|}{4(1-\beta)^2}}{1-\left(\frac{|c_1|}{2(1-\beta)}\right)^2} = \frac{\left|2\left(1-\beta\right)c_2-c_1^2\right|}{4\left(1-\beta\right)^2-|c_1|^2},$$

we take

$$\begin{aligned}\frac{2}{1+\frac{\left|2(1-\beta)c_2-c_1^2\right|}{4(1-\beta)^2-|c_1|^2}} &\leq \frac{1+\frac{|c_1|}{2(1-\beta)}}{1-\frac{|c_1|}{2(1-\beta)}}\left\{\frac{2\left|f'(b)\right|}{1-\beta}-1\right\} \\ &= \frac{2\left(1-\beta\right)+|c_1|}{2\left(1-\beta\right)-|c_1|}\left\{\frac{2\left|f'(b)\right|}{1-\beta}-1\right\}.\end{aligned}$$

Therefore, we obtain

$$1+\frac{2\left(4\left(1-\beta\right)^2-|c_1|^2\right)}{4\left(1-\beta\right)^2-|c_1|^2+\left|2\left(1-\beta\right)c_2-c_1^2\right|}\frac{2\left(1-\beta\right)-|c_1|}{2\left(1-\beta\right)+|c_1|}\leq\frac{2\left|f'(b)\right|}{1-\beta}$$

and

$$\left|f'(b)\right|\geq\frac{1-\beta}{2}\left(1+\frac{2\left(2\left(1-\beta\right)-|c_1|\right)^2}{4\left(1-\beta\right)^2-|c_1|^2+\left|2\left(1-\beta\right)c_2-c_1^2\right|}\right).$$

So, we obtain the inequality (2.3).

To show that the inequality (2.3) is sharp, take the holomorphic function

$$f(z)=\frac{1-(2\beta-1)\,z}{1-z}.$$

Then

$$\left|f'(-1)\right|=\frac{1-\beta}{2}.$$

Since $|c_1|=2\left(1-\beta\right)$, (2.3) is satisfied with equality. □

If $f(z)-1$ has no zeros different from $z=0$ in Theorem 3, the inequality (2.3) can be further strengthened. It has been investigated in the case of having only one point b in the unit disc D of the function $f(z)$. That inequality is stronger than the inequalities which have been expressed above. This is given by the following Theorem.

Theorem 4. *Let $f(z)\in\mathcal{N}(\beta)$ and $f(z)-1$ has no zeros in D except $z=0$ and $c_1>0$. Assume that, for some $b\in\partial D$, f has angular limit $f(b)$ at b and $f(b)=\beta$. Then we have the inequality*

$$\left|f'(b)\right|\geq\frac{1-\beta}{2}\left(1-\frac{2\left(1-\beta\right)|c_1|\ln^2\left(\frac{|c_1|}{2(1-\beta)}\right)}{2\left(1-\beta\right)|c_1|\ln\left(\frac{|c_1|}{2(1-\beta)}\right)-\left|2\left(1-\beta\right)c_2-c_1^2\right|}\right). \tag{2.5}$$

In addition, the equality in (2.5) *occurs for the function*

$$f(z)=\frac{1-(2\beta-1)\,z}{1-z}.$$

Proof. Let $c_1 > 0$ in the expression of the function $f(z)$. Having in mind the inequality (2.4) and the function $f(z) - 1$ has no zeros in D except $D - \{0\}$, we denote by $\ln \Theta(z)$ the holomorphic branch of the logarithm normed by the condition

$$\ln \Theta(0) = \ln \left(\frac{|c_1|}{2(1-\beta)} \right) < 0.$$

The auxiliary function

$$\Gamma(z) = \frac{\ln \Theta(z) - \ln \Theta(0)}{\ln \Theta(z) + \ln \Theta(0)}$$

is holomorphic in the unit disc D, $|\Gamma(z)| < 1$, $\Gamma(0) = 0$ and $|\Gamma(b)| = 1$ for $b \in \partial D$.

From (1.4), we obtain

$$\begin{aligned} \frac{2}{1 + |\Gamma'(0)|} &\leq \left|\Gamma'(b)\right| = \frac{|2 \ln \Theta(0)|}{|\ln \Theta(b) + \ln \Theta(0)|^2} \left| \frac{\Theta'(b)}{\Theta(b)} \right| \\ &= \frac{-2 \ln \Theta(0)}{\ln^2 \Theta(0) + \arg^2 \Theta(b)} \left\{ \left|\varphi'(b)\right| - 1 \right\}. \end{aligned}$$

Since

$$\begin{aligned} \left|\Gamma'(0)\right| &= \frac{-1}{\ln\left(\frac{|c_1|}{2(1-\beta)}\right)} \frac{\frac{\left|2(1-\beta)c_2 - c_1^2\right|}{4(1-\beta)^2}}{\frac{|c_1|}{2(1-\beta)}} \\ &= \frac{-1}{\ln\left(\frac{|c_1|}{2(1-\beta)}\right)} \frac{\left|2(1-\beta)c_2 - c_1^2\right|}{2(1-\beta)|c_1|} \end{aligned}$$

and replacing $\arg^2 \Theta(b)$ by zero, then we have

$$\frac{1}{1 - \frac{1}{\ln\left(\frac{|c_1|}{2(1-\beta)}\right)} \frac{\left|2(1-\beta)c_2 - c_1^2\right|}{2(1-\beta)|c_1|}} \leq \frac{-1}{\ln\left(\frac{|c_1|}{2(1-\beta)}\right)} \left\{ \frac{2|f'(b)|}{1-\beta} - 1 \right\}$$

and

$$1 - \frac{2(1-\beta)|c_1| \ln^2\left(\frac{|c_1|}{2(1-\beta)}\right)}{2(1-\beta)|c_1| \ln\left(\frac{|c_1|}{2(1-\beta)}\right) - \left|2(1-\beta)c_2 - c_1^2\right|} \leq \frac{2|f'(b)|}{1-\beta}.$$

Thus, we obtain the inequality (2.5) with an obvious equality case. □

In the following Theorem, we shall give an estimate below $|f'(b)|$ according to the first nonzero Taylor coeffcient of about two zeros, namely $z = 0$ and $z_1 \neq 0$.

Theorem 5. *Let $f(z) \in \mathcal{N}(\beta)$ and $f(z_1) = 1$ for $0 < |z_1| < 1$. Suppose that, for some $b \in \partial D$, f has an angular limit $f(b)$ at b, $f(b) = \beta$. Then we have the inequality*

$$|f'(b)| \geq \frac{1-\beta}{2}\left(1+\frac{1-|z_1|^2}{|b-z_1|^2}+\frac{2(1-\beta)|z_1|-|f'(0)|}{2(1-\beta)|z_1|+|f'(0)|}\right. \tag{2.6}$$
$$\left.\times\left[1+\frac{4(1-\beta)^2|z_1|^2+|f'(z_1)|\left(1-|z_1|^2\right)|f'(0)|-2(1-\beta)|f'(z_1)|\left(1-|z_1|^2\right)-2(1-\beta)|f'(0)|}{4(1-\beta)^2|z_1|^2+|f'(z_1)|\left(1-|z_1|^2\right)|f'(0)|+2(1-\beta)|f'(z_1)|\left(1-|z_1|^2\right)+2(1-\beta)|f'(0)|}\frac{1-|z_1|^2}{|b-z_1|^2}\right]\right).$$

The inequality (2.6) *is sharp, with equality for each possible values* $|f'(0)| = 2e\,(1-\beta)$ and $|f'(z_1)| = 2f\,(1-\beta)\left(0 \leq e \leq 2\,(1-\beta)\,|z_1|\,,\ 0 \leq f \leq 2\,(1-\beta)\,\frac{|z_1|}{1-|z_1|^2}\right)$.

Proof. Let

$$q(z) = \frac{z-z_1}{1-\overline{z_1}z}.$$

Also, let $h : D \to D$ be a holomorphic function and a point $z_1 \in D$ in order to satisfy

$$\left|\frac{h(z)-h(z_1)}{1-\overline{h(z_1)}h(z)}\right| \leq \left|\frac{z-z_1}{1-\overline{z_1}z}\right| = |q(z)|$$

and

$$|h(z)| \leq \frac{|h(z_1)|+|q(z)|}{1+|h(z_1)|\,|q(z)|} \tag{2.7}$$

by Schwarz-Pick lemma [8]. If $p : D \to D$ is holomorphic function and $0 < |z_1| < 1$, letting

$$h(z) = \frac{p(z)-p(0)}{z\left(1-\overline{p(0)}p(z)\right)}$$

in (2.7), we obtain

$$\left|\frac{p(z)-p(0)}{z\left(1-\overline{p(0)}p(z)\right)}\right| \leq \frac{\left|\frac{p(z_1)-p(0)}{z_1\left(1-\overline{p(0)}p(z_1)\right)}\right|+|q(z)|}{1+\left|\frac{p(z_1)-p(0)}{z_1\left(1-\overline{p(0)}p(z_1)\right)}\right||q(z)|}$$

and

$$|p(z)| \leq \frac{|p(0)|+|z|\,\frac{|C|+|q(z)|}{1+|C||q(z)|}}{1+|p(0)|\,|z|\,\frac{|C|+|q(z)|}{1+|C||q(z)|}}, \tag{2.8}$$

where

$$C = \frac{p(z_1)-p(0)}{z_1\left(1-\overline{p(0)}p(z_1)\right)}.$$

Without loss of generality, we will assume that $b = 1$. If we take

$$p(z) = \frac{\varphi(z)}{z\frac{z-z_1}{1-\overline{z_1}z}},$$

then

$$p(z_1) = \frac{\varphi'(z_1)\left(1-|z_1|^2\right)}{z_1}, \quad p(0) = \frac{\varphi'(0)}{-z_1}$$

and

$$C=\frac{\frac{\varphi'(z_1)\left(1-|z_1|^2\right)}{z_1}+\frac{\varphi'(0)}{z_1}}{z_1\left(1+\frac{\varphi'(z_1)\left(1-|z_1|^2\right)}{z_1}\frac{\varphi'(0)}{z_1}\right)},$$

where $|C|\leq 1$. Let $|p(0)|=\alpha$ and

$$\mathrm{T}=\frac{\left|\frac{\varphi'(z_1)\left(1-|z_1|^2\right)}{z_1}\right|+\left|\frac{\varphi'(0)}{z_1}\right|}{|z_1|\left(1+\left|\frac{\varphi'(z_1)\left(1-|z_1|^2\right)}{z_1}\right|\left|\frac{\varphi'(0)}{z_1}\right|\right)}.$$

From (2.8), we get

$$|\varphi(z)|\leq |z|\,|q(z)|\,\frac{\alpha+|z|\,\frac{\mathrm{T}+|q(z)|}{1+\mathrm{T}|q(z)|}}{1+\alpha\,|z|\,\frac{\mathrm{T}+|q(z)|}{1+\mathrm{T}|q(z)|}}$$

and

$$\frac{1-|\varphi(z)|}{1-|z|}\geq\frac{1+\alpha\,|z|\,\frac{\mathrm{T}+|q(z)|}{1+\mathrm{T}|q(z)|}-\alpha\,|z|\,|q(z)|-|q(z)|\,|z|^2\,\frac{\mathrm{T}+|q(z)|}{1+\mathrm{T}|q(z)|}}{(1-|z|)\left(1+\alpha\,|z|\,\frac{\mathrm{T}+|q(z)|}{1+\mathrm{T}|q(z)|}\right)}=s(z). \tag{2.9}$$

Let $\kappa(z)=1+\alpha\,|z|\,\frac{\mathrm{T}+|q(z)|}{1+\mathrm{T}|q(z)|}$ and $\tau(z)=1+\mathrm{T}\,|q(z)|$. Then

$$s(z)=\frac{1-|z|^2\,|q(z)|^2}{(1-|z|)\,\kappa(z)\tau(z)}+\mathrm{T}\,|q(z)|\,\frac{1-|z|^2}{(1-|z|)\,\kappa(z)\tau(z)}+|z|\,\mathrm{T}\alpha\frac{1-|q(z)|^2}{(1-|z|)\,\kappa(z)\tau(z)}.$$

Since

$$\lim_{z\to 1}\kappa(z)=\lim_{z\to 1}1+\alpha\,|z|\,\frac{\mathrm{T}+|q(z)|}{1+\mathrm{T}\,|q(z)|}=1+\alpha,$$
$$\lim_{z\to 1}\tau(z)=\lim_{z\to 1}1+\mathrm{T}\,|q(z)|=1+\mathrm{T}$$

and

$$1-|q(z)|^2=1-\left|\frac{z-z_1}{1-\overline{z_1}z}\right|^2=\frac{\left(1-|z_1|^2\right)\left(1-|z|^2\right)}{|1-\overline{z_1}z|^2},$$

passing to the angular limit in (2.9) gives

$$\begin{aligned}\left|\varphi'(z)\right| &\geq \frac{2}{(1+\alpha)\,(1+\mathrm{T})}\left(1+\frac{1-|z_1|^2}{|1-z_1|^2}+\mathrm{T}+\alpha\mathrm{T}\frac{1-|z_1|^2}{|1-z_1|^2}\right)\\ &= 1+\frac{1-|z_1|^2}{|1-z_1|^2}+\frac{1-\alpha}{1+\alpha}\left(1+\frac{1-\mathrm{T}}{1+\mathrm{T}}\frac{1-|z_1|^2}{|1-z_1|^2}\right).\end{aligned}$$

Moreover, since

$$\frac{1-\alpha}{1+\alpha}=\frac{1-|p(0)|}{1+|p(0)|}=\frac{1-\left|\frac{\varphi'(0)}{z_1}\right|}{1+\left|\frac{\varphi'(0)}{z_1}\right|}=\frac{|z_1|-|\varphi'(0)|}{|z_1|+|\varphi'(0)|}$$

$$= \frac{2(1-\beta)|z_1| - |f'(0)|}{2(1-\beta)|z_1| + |f'(0)|},$$

$$\frac{1-\mathrm{T}}{1+\mathrm{T}} = \frac{1 - \dfrac{\left|\frac{\varphi'(z_1)\left(1-|z_1|^2\right)}{z_1}\right| + \left|\frac{\varphi'(0)}{z_1}\right|}{|z_1|\left(1+\left|\frac{\varphi'(z_1)\left(1-|z_1|^2\right)}{z_1}\right|\left|\frac{\varphi'(0)}{z_1}\right|\right)}}{1 + \dfrac{\left|\frac{\varphi'(z_1)\left(1-|z_1|^2\right)}{z_1}\right| + \left|\frac{\varphi'(0)}{z_1}\right|}{|z_1|\left(1+\left|\frac{\varphi'(z_1)\left(1-|z_1|^2\right)}{z_1}\right|\left|\frac{\varphi'(0)}{z_1}\right|\right)}}$$

$$= \frac{1 - \dfrac{\left|\frac{\frac{f'(z_1)}{2(1-\beta)}\left(1-|z_1|^2\right)}{z_1}\right| + \left|\frac{\frac{f'(0)}{2(1-\beta)}}{z_1}\right|}{|z_1|\left(1+\left|\frac{\frac{f'(z_1)}{2(1-\beta)}\left(1-|z_1|^2\right)}{z_1}\right|\left|\frac{\frac{f'(0)}{2(1-\beta)}}{z_1}\right|\right)}}{1 + \dfrac{\left|\frac{\frac{f'(z_1)}{2(1-\beta)}\left(1-|z_1|^2\right)}{z_1}\right| + \left|\frac{\frac{f'(0)}{2(1-\beta)}}{z_1}\right|}{|z_1|\left(1+\left|\frac{\frac{f'(z_1)}{2(1-\beta)}\left(1-|z_1|^2\right)}{z_1}\right|\left|\frac{\frac{f'(0)}{2(1-\beta)}}{z_1}\right|\right)}}$$

and

$$\frac{1-\mathrm{T}}{1+\mathrm{T}} = \frac{|z_1|\left(1+\left|\frac{\frac{f'(z_1)}{2(1-\beta)}\left(1-|z_1|^2\right)}{z_1}\right|\left|\frac{\frac{f'(0)}{2(1-\beta)}}{z_1}\right|\right) - \left|\frac{\frac{f'(z_1)}{2(1-\beta)}\left(1-|z_1|^2\right)}{z_1}\right| - \left|\frac{\frac{f'(0)}{2(1-\beta)}}{z_1}\right|}{|z_1|\left(1+\left|\frac{\frac{f'(z_1)}{2(1-\beta)}\left(1-|z_1|^2\right)}{z_1}\right|\left|\frac{\frac{f'(0)}{2(1-\beta)}}{z_1}\right|\right) + \left|\frac{\frac{f'(z_1)}{2(1-\beta)}\left(1-|z_1|^2\right)}{z_1}\right| + \left|\frac{\frac{f'(0)}{2(1-\beta)}}{z_1}\right|}$$

$$= \frac{4(1-\beta)^2|z_1|^2 + |f'(z_1)|\left(1-|z_1|^2\right)|f'(0)| - 2(1-\beta)|f'(z_1)|\left(1-|z_1|^2\right) - 2(1-\beta)|f'(0)|}{4(1-\beta)^2|z_1|^2 + |f'(z_1)|\left(1-|z_1|^2\right)|f'(0)| + 2(1-\beta)|f'(z_1)|\left(1-|z_1|^2\right) + 2(1-\beta)|f'(0)|},$$

we obtain

$$|\varphi'(1)| \geq 1 + \frac{1-|z_1|^2}{|1-z_1|^2} + \frac{2(1-\beta)|z_1| - |f'(0)|}{2(1-\beta)|z_1| + |f'(0)|}$$
$$\times \left[1 + \frac{4(1-\beta)^2|z_1|^2 + |f'(z_1)|\left(1-|z_1|^2\right)|f'(0)| - 2(1-\beta)|f'(z_1)|\left(1-|z_1|^2\right) - 2(1-\beta)|f'(0)|}{4(1-\beta)^2|z_1|^2 + |f'(z_1)|\left(1-|z_1|^2\right)|f'(0)| + 2(1-\beta)|f'(z_1)|\left(1-|z_1|^2\right) + 2(1-\beta)|f'(0)|} \frac{1-|z_1|^2}{|1-z_1|^2}\right].$$

From definition of $\varphi(z)$, we have

$$\varphi'(z) = \frac{2(1-\beta) f'(z)}{(f(z) - (2\beta - 1))^2}$$

and

$$|\varphi'(1)| = \left|\frac{2(1-\beta) f'(1)}{(f(1) - (2\beta-1))^2}\right| = \frac{2|f'(1)|}{1-\beta}.$$

Thus, we obtain the inequality (2.6).

Now, we shall show that the inequality (2.6) is sharp.

Since

$$p(z) = \frac{\varphi(z)}{z\frac{z-z_1}{1-\overline{z_1}z}}$$

is holomorphic function in the unit disc and $|p(z)| \leq 1$ for $z \in D$, we obtain

$$\left|\varphi'(0)\right| \leq |z_1|$$

and

$$\left|\varphi'(z_1)\right| \leq \frac{|z_1|}{1-|z_1|^2}.$$

We take $z_1 \in (-1, 0)$ and arbitrary two numbers e and f, such that $0 \leq e \leq 2\,(1-\beta)\,|z_1|$, $0 \leq f \leq 2\,(1-\beta)\,\frac{|z_1|}{1-|z_1|^2}$.

Let

$$\mathtt{K} = \frac{\frac{f\left(1-|z_1|^2\right)}{z_1} + \frac{e}{z_1}}{z_1\left(1+ef\frac{1-|z_1|^2}{z_1^2}\right)} = \frac{1}{z_1^2}\frac{f\left(1-|z_1|^2\right)+e}{1+ef\frac{1-|z_1|^2}{z_1^2}}.$$

The auxiliary function

$$t(z) = z\frac{z-z_1}{1-\overline{z_1}z}\frac{\frac{-e}{z_1} + z\frac{\mathtt{K}+\frac{z-z_1}{1-\overline{z_1}z}}{1+\mathtt{K}\frac{z-z_1}{1-\overline{z_1}z}}}{1-\frac{e}{z_1}z\frac{\mathtt{K}+\frac{z-z_1}{1-\overline{z_1}z}}{1+\mathtt{K}\frac{z-z_1}{1-\overline{z_1}z}}}$$

is holomorphic in D and $|t(z)| < 1$ for $z \in D$. Let

$$\frac{f(z)-1}{f(z)-(2\beta-1)} = z\frac{z-z_1}{1-\overline{z_1}z}\frac{\frac{-e}{z_1} + z\frac{\mathtt{K}+\frac{z-z_1}{1-\overline{z_1}z}}{1+\mathtt{K}\frac{z-z_1}{1-\overline{z_1}z}}}{1-\frac{e}{z_1}z\frac{\mathtt{K}+\frac{z-z_1}{1-\overline{z_1}z}}{1+\mathtt{K}\frac{z-z_1}{1-\overline{z_1}z}}}. \tag{2.10}$$

So, we have

$$f(z) = \frac{1-(2\beta-1)\,z\frac{z-z_1}{1-\overline{z_1}z}\frac{\frac{-e}{z_1}+z\frac{\mathtt{K}+\frac{z-z_1}{1-\overline{z_1}z}}{1+\mathtt{K}\frac{z-z_1}{1-\overline{z_1}z}}}{1-\frac{e}{z_1}z\frac{\mathtt{K}+\frac{z-z_1}{1-\overline{z_1}z}}{1+\mathtt{K}\frac{z-z_1}{1-\overline{z_1}z}}}}{1-z\frac{z-z_1}{1-\overline{z_1}z}\frac{\frac{-e}{z_1}+z\frac{\mathtt{K}+\frac{z-z_1}{1-\overline{z_1}z}}{1+\mathtt{K}\frac{z-z_1}{1-\overline{z_1}z}}}{1-\frac{e}{z_1}z\frac{\mathtt{K}+\frac{z-z_1}{1-\overline{z_1}z}}{1+\mathtt{K}\frac{z-z_1}{1-\overline{z_1}z}}}}.$$

Therefore, we take $|f'(0)| = 2e\,(1-\beta)$ and $|f'(z_1)| = 2f\,(1-\beta)$.

From (2.10), with the simple calculations, we obtain

$$\frac{2(1-\beta)f'(1)}{(f(1)-(2\beta-1))^2} = 1+\frac{1-z_1^2}{(1-z_1)^2}+\frac{\left(1+\frac{1-z_1^2}{(1-z_1)^2}\frac{1-\mathtt{K}^2}{(1+\mathtt{K})^2}\right)\left(1-\frac{e}{z_1}\right)+\frac{e}{z_1}\left(1+\frac{1-z_1^2}{(1-z_1)^2}\frac{1-\mathtt{K}^2}{(1+\mathtt{K})^2}\right)\left(1-\frac{e}{z_1}\right)}{\left(1-\frac{e}{z_1}\right)^2}$$

$$= 1+\frac{1-z_1^2}{(1-z_1)^2}+\frac{e+z_1}{-e+z_1}\left(1+\frac{1-z_1^2}{(1-z_1)^2}\frac{z_1^2+ef\left(1-z_1^2\right)-f\left(1-z_1^2\right)-e}{z_1^2+ef\left(1-z_1^2\right)+f\left(1-z_1^2\right)+e}\right)$$

and

$$f'(1)=\frac{1-\beta}{2}\left(1+\frac{1-z_1^2}{(1-z_1)^2}+\frac{e+z_1}{-e+z_1}\left(1+\frac{1-z_1^2}{(1-z_1)^2}\frac{z_1^2+ef\left(1-z_1^2\right)-f\left(1-z_1^2\right)-e}{z_1^2+ef\left(1-z_1^2\right)+f\left(1-z_1^2\right)+e}\right)\right).$$

Since $z_1\in(-1,0)$, the last equality shows that (2.6) is sharp. □

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