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On LCA groups whose ring of continuous endomorphisms satisfies DCC on closed ideals

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Abstract. We determine the structure of LCA (locally compact abelian) groups X with the property that the ring $E(X)$ of continuous endomorphisms of X, taken with the compact-open topology, satisfies DCC (descending chain condition) on different types of closed ideals.

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Introduction

A well known theorem of L. Fuchs [7, Theorem 111.3] asserts that the endomorphism ring of an (abstract) abelian group X is right (respectively, left) artinian if and only if X is the direct sum of a finite group and finitely many copies of the additive group of rational numbers. F. Szász observed [15] that the same conclusion about the structure of X remains true under weaker hypothesis that the endomorphism ring of X satisfies DCC on principal right (respectively, left) ideals.

The purpose of the present paper is to extend these results to the more general setting obtained by considering LCA groups and their rings of continuous endomorphisms. To be precise, let $\mathcal{L}$ be the class of all LCA groups. For $X \in \mathcal{L}$, let $E(X)$ denote the ring of continuous endomorphisms of X, endowed with the compact-open topology. We shall determine here the explicite structure of groups $X \in \mathcal{L}$ with the property that the ring $E(X)$ satisfies DCC on closed right (respectively, left) ideals, and we shall show that the corresponding class of groups coincides with the class of those groups $X \in \mathcal{L}$ whose ring $E(X)$ satisfies DCC on topologically principal right (respectively, left) ideals. We shall also determine the groups $X \in \mathcal{L}$ for which $E(X)$ is right (respectively, left) artinian.

1 Notation

Throughout the following, $\mathbb{N}$ is the set of natural numbers (including zero), $\mathbb{N}_0 = \mathbb{N} \setminus \{0\}$, and $\mathbb{P}$ is the set of prime numbers.

The groups in $\mathcal{L}$ which we shall mention frequently are the reals $\mathbb{R}$, the p-adic numbers $\mathbb{Q}_p$, the p-adic integers $\mathbb{Z}_p$ (all with their usual topologies), the rationals

$\mathbb{Q}$, the quasi-cyclic groups $\mathbb{Z}(p^\infty)$ and the cyclic groups $\mathbb{Z}(p^n)$ of order p^n (all with the discrete topology), where $p \in \mathbb{P}$ and $n \in \mathbb{N}$.

For $X \in \mathcal{L}$, we let 1_X, $c(X)$, $d(X)$, $k(X)$, $m(X)$, $t(X)$, and X^* denote respectively the identity map on X, the connected component of zero in X, the maximal divisible subgroup of X, the subgroup of compact elements of X, the smallest closed subgroup K of X such that the quotient group X/K is torsion-free, the torsion subgroup of X, and the character group of X.

We denote by $E(X)$ the ring of continuous endomorphisms of X and by $H(X,Y)$, where Y is another group in $\mathcal{L}$, the group of continuous homomorphisms from X to Y, both endowed with the compact-open topology.

For $n \in \mathbb{N}$ and $p \in \mathbb{P}$, we let $nX = \{nx \mid x \in X\}$, $X[n] = \{x \in X \mid nx = 0\}$, $X_p = \{x \in X \mid \lim_{k\to\infty} p^k x = 0\}$, and $S(X) = \{q \in \mathbb{P} \mid (k(X)/c(X))_q \neq 0\}$.

For $a \in X$ and $S \subset X$, $\langle a \rangle$ is the subgroup of X generated by a, $\overline{S}$ is the closure of S in X, and $A(X^*, S) = \{\gamma \in X^* \mid \gamma(x) = 0 \text{ for all } x \in S\}$.

Also, we write $X = A \oplus B$ (respectively, $X = A \dotplus B$) in case X is a topological (respectively, an algebraic) direct sum of its subgroups A and B.

If $(X_i)_{i\in I}$ is a family of groups in $\mathcal{L}$, we write $\prod_{i\in I} X_i$ for the topological direct product of the groups X_i and $\prod_{i\in I}(X_i; U_i)$ for the topological local direct product of the groups X_i relative to the compact open subgroups $U_i \subset X_i$. We recall that $\prod_{i\in I}(X_i; U_i)$ consists of all $(x_i)_{i\in I} \in \prod_{i\in I} X_i$ with $x_i \in U_i$ for all but finitely many i, topologized by declaring all neighbourhoods of zero in the topological group $\prod_{i\in I} U_i$ to be a fundamental system of neighbourhoods of zero in $\prod_{i\in I}(X_i; U_i)$.

If F is a field, $\mathbb{M}_n(F)$ stands for the ring of all $n \times n$ matrices with entries in F. The symbol $\cong$ denotes topological group (ring) isomorphism.

2 Topological Morita context rings

In our study of groups $X \in \mathcal{L}$ with the property that $E(X)$ satisfies DCC on different types of closed ideals, we will frequently make use of topological Morita context rings. Here we recall this construction and derive several facts about its closed ideals.

Let $\mathcal{M} = \big(R, S, {}_RP_S, {}_SQ_R, [\cdot,\cdot]_R, [\cdot,\cdot]_S\big)$ be a topological Morita context, that is R and S are topological rings with identity, ${}_RP_S$ is a unital topological (R,S)-bimodule, ${}_SQ_R$ is a unital topological (S,R)-bimodule, $[\cdot,\cdot]_R : {}_RP_S \times {}_SQ_R \to {}_RR_R$ is a continuous (R,R)-bilinear S-balanced mapping, and $[\cdot,\cdot]_S : {}_SQ_R \times {}_RP_S \to {}_SS_S$ is a continuous (S,S)-bilinear R-balanced mapping such that

$$[p,q]_R p' = p[q,p']_S \quad \text{and} \quad [q,p]_S q' = q[p,q']_R$$

for all $r \in R, s \in S, p, p' \in P$ and $q, q' \in Q$. By analogy with the case of abstract Morita contexts, we can associate to $\mathcal{M}$ a topological ring, called the topological Morita context ring of $\mathcal{M}$. Specifically, we endow the set

$$M = \left\{ \begin{pmatrix} r & p \\ q & s \end{pmatrix} \mid r \in R, p \in P, q \in Q, s \in S \right\}$$

with the product topology of $R \times P \times Q \times S$, and define addition and multiplication on M by setting:

$$\begin{pmatrix} r_1 & p_1 \\ q_1 & s_1 \end{pmatrix} + \begin{pmatrix} r_2 & p_2 \\ q_2 & s_2 \end{pmatrix} = \begin{pmatrix} r_1 + r_2 & p_1 + p_2 \\ q_1 + q_2 & s_1 + s_2 \end{pmatrix}$$

and

$$\begin{pmatrix} r_1 & p_1 \\ q_1 & s_1 \end{pmatrix} \begin{pmatrix} r_2 & p_2 \\ q_2 & s_2 \end{pmatrix} = \begin{pmatrix} r_1 r_2 + [p_1, q_2]_R & r_1 p_2 + p_1 s_2 \\ q_1 r_2 + s_1 q_2 & [q_1, p_2]_S + s_1 s_2 \end{pmatrix}$$

for all $r_1, r_2 \in R$, $p_1, p_2 \in P$, $q_1, q_2 \in Q$, and $s_1, s_2 \in S$. As is well known, the algebraic properties of operations of R, S, P and Q, and of mappings $[\cdot,\cdot]_R$ and $[\cdot,\cdot]_S$ ensure that, with respect to the above addition and multiplication, M is a ring with identity. It turns out that, in the considered topological situation, these operations on M are also compatible with the topology of M. To see this, it suffices in view of [3, Ch. I, §4, Proposition 1] to observe that composing the mentioned operations on M with the canonical projections on the components of M we get continuous mappings, because of the continuity of operations on R, S, P and Q, and of mappings $[\cdot,\cdot]_R$ and $[\cdot,\cdot]_S$. Thus M becomes a topological ring with identity, which we will denote by $\begin{pmatrix} R & P \\ Q & S \end{pmatrix}$. We will use frequently the special cases $\begin{pmatrix} R & 0 \\ Q & S \end{pmatrix}$ and $\begin{pmatrix} R & P \\ 0 & S \end{pmatrix}$ corresponding respectively to $P = \{0\}$ or $Q = \{0\}$.

As we will be working with closed ideals of $\begin{pmatrix} R & P \\ Q & S \end{pmatrix}$, it is desirable to relate them to closed subobjects of the components $R, P, Q,$ and S. For this purpose, we need to introduce four mappings of $\mathcal{M}$. Recall that if A and B are topological rings and if $h : A \to B$ is a continuous ring homomorphism, then any topological right (respectively, left) B-module X can be viewed as a topological right (respectively, left) A-module via the scalar multiplication given by $xa = xh(a)$ (respectively, $ax = h(a)x$) for all $a \in A$ and $x \in X$. For example, if $h_R : R \times S \to R$ and $h_S : R \times S \to S$ are the canonical projections, then R, S, P, Q and hence their products can be considered as topological right (respectively, left) modules over the topological direct product ring $R \times S$. We will use the following continuous mappings:

$$\begin{aligned}
\varphi_{R,Q,P} &: {}_{R\times S}((R \times Q)_R \times {}_R P)_S \to {}_{R\times S}(P \times S)_S,\ ((r,q),p) \to (rp, [q,p]_S),\\
\varphi_{P,S,Q} &: {}_{R\times S}((P \times S)_S \times {}_S Q)_R \to {}_{R\times S}(R \times Q)_R,\ ((p,s),q) \to ([p,q]_R, sq),\\
\varphi_{P,Q,S} &: {}_R(P_S \times {}_S(Q \times S))_{R\times S} \to {}_R(R \times P)_{R\times S},\ (p,(q,s)) \to ([p,q]_R, ps),\\
\varphi_{Q,R,P} &: {}_S(Q_R \times {}_R(R \times P))_{R\times S} \to {}_S(Q \times S)_{R\times S},\ (q,(r,p)) \to (qr, [q,p]_S).
\end{aligned}$$

It is easy to see that $\varphi_{R,Q,P}$ is R-balanced and $(R \times S, S)$-bilinear, $\varphi_{P,S,Q}$ is S-balanced and $(R \times S, R)$-bilinear, $\varphi_{P,Q,S}$ is S-balanced and $(R, R \times S)$-bilinear, and $\varphi_{Q,R,P}$ is R-balanced and $(S, R \times S)$-bilinear.

We have:

Lemma 1. *Let* $\big(R, S, {}_RP_S, {}_SQ_R, [\cdot,\cdot]_R, [\cdot,\cdot]_S\big)$ *be a topological Morita context.*

(i) The closed right ideals of $\begin{pmatrix} R & P \\ Q & S \end{pmatrix}$ *are of the form*

$$\begin{pmatrix} A & B \end{pmatrix} = \left\{ \begin{pmatrix} r & p \\ q & s \end{pmatrix} \mid (r,q) \in A, (p,s) \in B \right\},$$

where A *is a closed submodule of* $(R \times Q)_R$ *and* B *is a closed submodule of* $(P \times S)_S$ *such that* $\varphi_{P,S,Q}(B \times Q) \subset A$ *and* $\varphi_{R,Q,P}(A \times P) \subset B$.

(ii) The closed left ideals of $\begin{pmatrix} R & P \\ Q & S \end{pmatrix}$ *are of the form*

$$\begin{pmatrix} C \\ D \end{pmatrix} = \left\{ \begin{pmatrix} r & p \\ q & s \end{pmatrix} \mid (r,p) \in C, (q,s) \in D \right\},$$

where C *is a closed submodule of* ${}_R(R \times P)$ *and* D *is a closed submodule of* ${}_S(Q \times S)$ *such that* $\varphi_{P,Q,S}(P \times D) \subset C$ *and* $\varphi_{Q,R,P}(Q \times C) \subset D$.

(iii) The closed ideals of $\begin{pmatrix} R & P \\ Q & S \end{pmatrix}$ *are of the form*

$$\begin{pmatrix} I & U \\ V & J \end{pmatrix} = \left\{ \begin{pmatrix} r & p \\ q & s \end{pmatrix} \mid r \in I, p \in U, q \in V, s \in J \right\},$$

where I *is a closed ideal of* R, J *is a closed ideal of* S, U *is a closed subbimodule of* ${}_RP_S$, V *is a closed subbimodule of* ${}_SQ_R$, *and the following conditions hold:* $[U,Q]_R \subset I$, $[P,V]_R \subset I$, $[Q,U]_S \subset J$, $[V,P]_S \subset J$, $IP \subset U$, $PJ \subset U$, $QI \subset V$, $JQ \subset V$.

Proof. (i) Let A and B be as stated in (i). Clearly, the additive group of $\begin{pmatrix} A & B \end{pmatrix}$ is a closed subgroup of the additive group of $\begin{pmatrix} R & P \\ Q & S \end{pmatrix}$. Given any $\begin{pmatrix} r_0 & p_0 \\ q_0 & s_0 \end{pmatrix} \in \begin{pmatrix} A & B \end{pmatrix}$ and $\begin{pmatrix} r & p \\ q & s \end{pmatrix} \in \begin{pmatrix} R & P \\ Q & S \end{pmatrix}$, we also have

$$(r_0 r, q_0 r) \in A, \quad ([p_0, q]_R, s_0 q) = \varphi_{P,S,Q}((p_0, s_0), q) \in A,$$

$$(p_0 s, s_0 s) \in B \quad \text{and} \quad (r_0 p, [q_0, p]_S) = \varphi_{R,Q,P}((r_0, q_0), p) \in B,$$

so

$$\begin{pmatrix} r_0 & p_0 \\ q_0 & s_0 \end{pmatrix} \begin{pmatrix} r & p \\ q & s \end{pmatrix} = \begin{pmatrix} r_0 r + [p_0, q]_R & r_0 p + p_0 s \\ q_0 r + s_0 q & [q_0, p]_S + s_0 s \end{pmatrix} \in \begin{pmatrix} A & B \end{pmatrix},$$

and hence $\begin{pmatrix} A & B \end{pmatrix}$ is a closed right ideal of $\begin{pmatrix} R & P \\ Q & S \end{pmatrix}$.

To show the converse, we first make the following observations. Since, clearly, $r \mapsto \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}$ is a continuous ring homomorphism from R into $\begin{pmatrix} R & P \\ Q & S \end{pmatrix}$, $\begin{pmatrix} R & P \\ Q & S \end{pmatrix}$ can be regarded as a topological right R-module. Then $\begin{pmatrix} R & 0 \\ Q & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & P \\ 0 & S \end{pmatrix}$ become topological submodules of $\begin{pmatrix} R & P \\ Q & S \end{pmatrix}_R$, and $\begin{pmatrix} R & P \\ Q & S \end{pmatrix}_R$ can be written in the form

$$\begin{pmatrix} R & P \\ Q & S \end{pmatrix}_R = \begin{pmatrix} R & 0 \\ Q & 0 \end{pmatrix}_R \oplus \begin{pmatrix} 0 & P \\ 0 & S \end{pmatrix}_R.$$

In particular, the mapping

$$\pi_{R\times Q} : \begin{pmatrix} R & P \\ Q & S \end{pmatrix}_R \to (R \times Q)_R, \ \begin{pmatrix} r & p \\ q & s \end{pmatrix} \mapsto (r, q),$$

is a continuous morphism of R-modules whose restriction to $\begin{pmatrix} R & 0 \\ Q & 0 \end{pmatrix}_R$ is an isomorphism of topological R-modules. Similarly, by using the ring homomorfism $s \mapsto \begin{pmatrix} 0 & 0 \\ 0 & s \end{pmatrix}$ from S into $\begin{pmatrix} R & P \\ Q & S \end{pmatrix}$, $\begin{pmatrix} R & P \\ Q & S \end{pmatrix}$ can be given the structure of topological right S-module. Then $\begin{pmatrix} R & 0 \\ Q & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & P \\ 0 & S \end{pmatrix}$ become topological submodules of $\begin{pmatrix} R & P \\ Q & S \end{pmatrix}_S$, and $\begin{pmatrix} R & P \\ Q & S \end{pmatrix}_S$ can be written in the form

$$\begin{pmatrix} R & P \\ Q & S \end{pmatrix}_S = \begin{pmatrix} R & 0 \\ Q & 0 \end{pmatrix}_S \oplus \begin{pmatrix} 0 & P \\ 0 & S \end{pmatrix}_S.$$

In particular, the mapping

$$\pi_{P\times S} : \begin{pmatrix} R & P \\ Q & S \end{pmatrix}_S \to (P \times S)_S, \ \begin{pmatrix} r & p \\ q & s \end{pmatrix} \mapsto (p, s),$$

is a continuous morphism of S-modules whose restriction to $\begin{pmatrix} 0 & P \\ 0 & S \end{pmatrix}_S$ is an isomorphism of topological S-modules.

Now, let Y be an arbitrary closed right ideal of $\begin{pmatrix} R & P \\ Q & S \end{pmatrix}$. It is clear that $Y_R \subset \begin{pmatrix} R & P \\ Q & S \end{pmatrix}_R$ and $Y_S \subset \begin{pmatrix} R & P \\ Q & S \end{pmatrix}_S$. Given any $\begin{pmatrix} r & p \\ q & s \end{pmatrix} \in Y$, we have

$$\begin{pmatrix} r & 0 \\ q & 0 \end{pmatrix} = \begin{pmatrix} r & p \\ q & s \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in Y$$

and

$$\begin{pmatrix} 0 & p \\ 0 & s \end{pmatrix} = \begin{pmatrix} r & p \\ q & s \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in Y.$$

It follows that

$$Y_R = (Y \cap \begin{pmatrix} R & 0 \\ Q & 0 \end{pmatrix})_R \oplus (Y \cap \begin{pmatrix} 0 & P \\ 0 & S \end{pmatrix})_R.$$

and

$$Y_S = (Y \cap \begin{pmatrix} R & 0 \\ Q & 0 \end{pmatrix})_S \oplus (Y \cap \begin{pmatrix} 0 & P \\ 0 & S \end{pmatrix})_S.$$

In particular, $A = \pi_{R\times Q}(Y) = \pi_{R\times Q}(Y \cap \begin{pmatrix} R & 0 \\ Q & 0 \end{pmatrix})$ is a closed submodule of $(R\times Q)_R$ and $B = \pi_{P\times S}(Y) = \pi_{P\times S}(Y \cap \begin{pmatrix} 0 & P \\ 0 & S \end{pmatrix})$ is a closed submodule of $(P \times S)_S$.

It only remains for us to show that $\varphi_{P,S,Q}(B\times Q) \subset A$ and $\varphi_{R,Q,P}(A\times P) \subset B$. Pick arbitrary $(p, s) \in B$ and $q' \in Q$. Then $\begin{pmatrix} 0 & p \\ 0 & s \end{pmatrix} \in Y$, so

$$\begin{pmatrix} [p, q']_R & 0 \\ sq' & 0 \end{pmatrix} = \begin{pmatrix} 0 & p \\ 0 & s \end{pmatrix} \begin{pmatrix} 0 & 0 \\ q' & 0 \end{pmatrix} \in Y,$$

and hence $([p, q']_R, sq') \in A$. Since $(p, s) \in B$ and $q' \in Q$ were arbitrary, we conclude that $\varphi_{P,S,Q}(B\times Q) \subset A$. Next pick arbitrary $(r, q) \in A$ and $p' \in P$. Then $\begin{pmatrix} r & 0 \\ q & 0 \end{pmatrix} \in Y$, so

$$\begin{pmatrix} 0 & rp' \\ 0 & [q, p']_S \end{pmatrix} = \begin{pmatrix} r & p \\ q & s \end{pmatrix} \begin{pmatrix} 0 & p' \\ 0 & 0 \end{pmatrix} \in Y,$$

and hence $(rp', [q, p']_S) \in B$. It follows that $\varphi_{R,Q,P}(A \times P) \subset B$.

(ii) The proof of (ii) is similar to that of (i).

(iii) The fact that $\begin{pmatrix} I & U \\ V & J \end{pmatrix}$ is a closed ideal of $\begin{pmatrix} R & P \\ Q & S \end{pmatrix}$ is clear. For the converse, pick an arbitrary closed ideal Y of $\begin{pmatrix} R & P \\ Q & S \end{pmatrix}$. Given any $\begin{pmatrix} r & p \\ q & s \end{pmatrix} \in Y$, we have

$$\begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} r & p \\ q & s \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in Y$$

$$\begin{pmatrix} 0 & p \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} r & p \\ q & s \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in Y$$

$$\begin{pmatrix} 0 & 0 \\ q & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r & p \\ q & s \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in Y$$

and

$$\begin{pmatrix} 0 & 0 \\ 0 & s \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r & p \\ q & s \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in Y.$$

Set $I' = Y \cap \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix}$, $U' = Y \cap \begin{pmatrix} 0 & P \\ 0 & 0 \end{pmatrix}$, $V' = Y \cap \begin{pmatrix} 0 & 0 \\ Q & 0 \end{pmatrix}$ and $J' = Y \cap \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix}$. It follows that the additive group of Y is a topological direct sum of the additive groups of I', U', V' and J', proving the closeness of $I = \pi_R(I')$, $U = \pi_P(U')$, $V = \pi_Q(V')$, and $J = \pi_S(J')$, where π_R, π_P, π_Q, and π_S are the canonical projections of $\begin{pmatrix} R & P \\ Q & S \end{pmatrix}$ onto R, P, Q, and S respectively. It is also clear that I is an ideal of R, J is an ideal of S, U is a subbimodule of P, and V is a subbimodule of Q. Finally, the inclusions in (iii) follow from the inclusions in (i) and (ii). $\square$

Specializing to $\begin{pmatrix} R & P \\ 0 & S \end{pmatrix}$, we obtain the following corollary.

Corollary 1. *Let R and S be topological rings with identity, and let P be a unital topological (R, S)-bimodule.*

(i) The closed right ideals of $\begin{pmatrix} R & P \\ 0 & S \end{pmatrix}$ are of the form

$$\left\{ \begin{pmatrix} r & p \\ 0 & s \end{pmatrix} \mid r \in I, (p, s) \in B \right\},$$

where I is a closed right ideal of R and B is a closed submodule of $(P \times S)_S$ such that $IP \times \{0\} \subset B$.

(ii) The closed left ideals of $\begin{pmatrix} R & P \\ 0 & S \end{pmatrix}$ are of the form

$$\left\{ \begin{pmatrix} r & p \\ 0 & s \end{pmatrix} \mid s \in J, (r, p) \in C \right\},$$

where J is a closed left ideal of S and C is a closed submodule of ${}_R(R \times P)$ such that $\{0\} \times PJ \subset C$.

(iii) The closed ideals of $\begin{pmatrix} R & P \\ 0 & S \end{pmatrix}$ are of the form

$$\left\{ \begin{pmatrix} r & p \\ 0 & s \end{pmatrix} \mid r \in I, s \in J, p \in U \right\},$$

where I is a closed ideal of R, J is a closed ideal of S, and U is a closed subbimodule of ${}_R P_S$ such that $IP + PJ \subset U$.

Next we consider chain conditions in $\begin{pmatrix} R & P \\ 0 & S \end{pmatrix}$. In accordance with [10, (1.22)], we have:

Lemma 2. *Let R and S be topological rings with identity, and let P be a unital topological (R, S)-bimodule. The ring $\begin{pmatrix} R & P \\ 0 & S \end{pmatrix}$ satisfies DCC on closed right (respectively, left) ideals if and only if so does R (respectively, S), and the right S-module $(P \times S)_S$ (respectively, left R-module ${}_R(R \times P)$) satisfies DCC on closed submodules.*

The same statement is true if we replace throughout DCC by ACC.

Proof. Assume $\begin{pmatrix} R & P \\ 0 & S \end{pmatrix}$ satisfies DCC on closed right ideals, and let $(I_n)_n \subset R_R$ and $(B_n)_n \subset (P \times S)_S$ be descending chains of closed submodules. Passing to the chain $\big((I_n \times \{0\} \quad B_n)\big)_n$ of $\begin{pmatrix} R & P \\ 0 & S \end{pmatrix}$, we see that $(I_n)_n$ and $(B_n)_n$ must stabilise.

For the converse, let $(Y_n)_n$ be a descending chain of closed right ideals of $\begin{pmatrix} R & P \\ 0 & S \end{pmatrix}$. For each n, we can write $Y_n = \big(I_n \times \{0\} \quad B_n\big)$, where $I_n \subset R_R$ and $B_n \subset (P \times S)_S$ are closed submodules such that $I_n \supset I_{n+1}$ and $B_n \supset B_{n+1}$. As $(I_n)_n$ and $(B_n)_n$ are stationary, $(Y_n)_n$ must be stationary as well. □

We close this section by pointing out the specific topological Morita context rings, which we will be working with. Let $X \in \mathcal{L}$. To any two closed subgroups A and B of X such that $X = A \oplus B$, we associate the topological Morita context

$$\mathcal{M}(A,B) = \big(E(A), E(B), {}_{E(A)}H(B,A)_{E(B)}, {}_{E(B)}H(A,B)_{E(A)}, [\cdot,\cdot]_{E(A)}, [\cdot,\cdot]_{E(B)}\big),$$

where $[f,g]_{E(A)} = f \circ g$ and $[g,f]_{E(B)} = g \circ f$ for all $f \in H(B,A)$ and $g \in H(A,B)$. We write $\begin{pmatrix} E(A) & H(B,A) \\ H(A,B) & E(B) \end{pmatrix}$ for the topological Morita context ring of $\mathcal{M}(A,B)$.

Lemma 3. *Let X be a group in $\mathcal{L}$ which can be written in the form $X = A \oplus B$ for some closed subgroups A and B of X. Then*

$$E(X) \cong \begin{pmatrix} E(A) & H(B,A) \\ H(A,B) & E(B) \end{pmatrix}.$$

If A is topologically fully invariant in X, then

$$E(X) \cong \begin{pmatrix} E(A) & H(B,A) \\ 0 & E(B) \end{pmatrix}.$$

If A and B are both topologically fully invariant in X, then

$$E(X) \cong E(A) \times E(B).$$

Proof. Let $\eta_A : A \to X$, $\eta_B : B \to X$ and $\pi_A : X \to A$, $\pi_B : X \to B$ denote respectively the canonical injections and the canonical projections corresponding to the above decomposition of X. Define

$$\xi : E(X) \to \begin{pmatrix} E(A) & H(B,A) \\ H(A,B) & E(B) \end{pmatrix}$$

by setting

$$\xi(u) = \begin{pmatrix} \pi_A \circ u \circ \eta_A & \pi_A \circ u \circ \eta_B \\ \pi_B \circ u \circ \eta_A & \pi_B \circ u \circ \eta_B \end{pmatrix}$$

for all $u \in E(X)$. It is easy to see that ξ establishes a topological ring isomorphism between $E(X)$ and $\begin{pmatrix} E(A) & H(B,A) \\ H(A,B) & E(B) \end{pmatrix}$.

If A is topologically fully invariant, then $\pi_B \circ u \circ \eta_A = 0$ for all $u \in E(X)$, so $\operatorname{im}(\xi) = \begin{pmatrix} E(A) & H(B,A) \\ 0 & E(B) \end{pmatrix}$. If B is topologically fully invariant as well, then $\operatorname{im}(\xi) = \begin{pmatrix} E(A) & 0 \\ 0 & E(B) \end{pmatrix}$. □

3 Reduction to topological p-primary groups

In this section, we establish some necessary conditions in order for the ring $E(X)$ of a group $X \in \mathcal{L}$ satisfy DCC on topologically principal ideals, i.e. on ideals of the form $\overline{(f)}$ with $f \in E(X)$.

We begin by recalling that for any group $X \in \mathcal{L}$, $E(X)$ and $E(X^*)$ are topologically anti-isomorphic [11, (2.1)]. Recall also that the group X is called residual if $\overline{d(X)} \subset k(X)$ and $c(X) \subset m(X)$, and that X is called topologically torsion in case $\lim_{n\in\mathbb{N}}(n!)x = 0$ for all $x \in X$.

Theorem 1. *Let X be a residual group in $\mathcal{L}$ such that the collection*

$$\mathcal{E} = \{\overline{nE(X)} \mid n \in \mathbb{N}_0\}$$

has a minimal element with respect to set inclusion. Then X is a topological torsion group, and there exists a finite subset S of $S(X)$ such that the following conditions hold:

(i) For each $p \in S(X) \setminus S$, X_p is densely divisible and torsionfree;

(ii) For each $p \in S$, there exists an $n(p) \in \mathbb{N}$ such that $m(X_p) = X_p[p^{n(p)}]$ and $\overline{d(X_p)} = \overline{p^{n(p)}X_p}$.

Proof. Let $\overline{n_0E(X)}$, where $n_0 \in \mathbb{N}_0$, be a minimal element of $\mathcal{E}$. Then

$$\overline{n_0E(X)} = \overline{pn_0E(X)} \tag{1}$$

for all $p \in \mathbb{P}$. Our first objective is to show that $\overline{n_0X}$ and $\overline{n_0X^*}$ are densely divisible. Fix any $q \in \mathbb{P}$. We show first that

$$\overline{n_0X} = \overline{q\overline{n_0X}} \qquad \text{and} \qquad \overline{n_0X^*} = \overline{q\overline{n_0X^*}}.$$

To this end, pick any $x \in X$ and define $\delta_x : E(X) \to X$ by setting $\delta_x(u) = u(x)$ for all $u \in E(X)$. In view of the equality (1), we can find a net $(u_i^{(q)})_{i\in I_q}$ of elements in $E(X)$ such that $n_0 1_X = \lim_{i\in I_q} qn_0u_i^{(q)}$. Since δ_x is a continuous [5, Ch. X, §3, Theorem 3, Corollary 1] group homomorphism, it follows that

$$n_0x = \delta_x(n_0 1_X) = \lim_{i\in I_q} \delta_x(qn_0u_i^{(q)}) = \lim_{i\in I_q} qn_0u_i^{(q)}(x),$$

and so $n_0x \in \overline{qn_0X}$. As x was arbitrarily chosen in X, this gives $n_0X \subset \overline{qn_0X}$, so $\overline{n_0X} \subset \overline{qn_0X}$. It follows that $\overline{n_0X} = \overline{qn_0X}$ because the reverse inclusion is obvious.

On the other hand, the multiplication by q being continuous, we have $q\overline{n_0X} \subset \overline{qn_0X}$ [3, Ch. I, §2, Theorem 1], whence $\overline{q\overline{n_0X}} \subset \overline{qn_0X}$. As the opposite inclusion is obvious, it follows that $\overline{q\overline{n_0X}} = \overline{qn_0X} = \overline{n_0X}$. Further, since $E(X)$ and $E(X^*)$ are topologically anti-isomorphic, the equality (1) also gives $\overline{n_0E(X^*)} = \overline{pn_0E(X^*)}$ for all $p \in \mathbb{P}$. Applying the preceding argument to X^*, we conclude that $\overline{n_0X^*} = \overline{q\overline{n_0X^*}}$.

Now we show that $\overline{n_0X}$ and $\overline{n_0X^*}$ are densely divisible. By [8, (24.22) and (22.17)], we have

$$(\overline{n_0X})^*[q] = A((\overline{n_0X})^*, \overline{q\overline{n_0X}}) = A((\overline{n_0X})^*, \overline{n_0X}) = \{0\}.$$

Analogously, $(\overline{n_0X^*})^*[q] = \{0\}$. Since $q \in \mathbb{P}$ was arbitrary, it follows that $(\overline{n_0X})^*$ and $(\overline{n_0X^*})^*$ are torsion-free, so $\overline{n_0X}$ and $\overline{n_0X^*}$ are densely divisible by [13, (5.2)]. In particular, $\overline{d(X)} \supset \overline{n_0X}$ and $\overline{d(X^*)} \supset \overline{n_0X^*}$, whence $\overline{d(X)} = \overline{n_0X}$ and $\overline{d(X^*)} = \overline{n_0X^*}$ because the opposite inclusions are obvious. By taking annihilators, we also obtain

$$m(X) = A(X, \overline{d(X)}) = A(X, \overline{n_0X}) = X[n_0]$$

and $m(X^*) = X^*[n_0]$. Finally, since X and X^* are residual groups, we must have

$$c(X) \subset m(X) = X[n_0] \quad \text{and} \quad c(X^*) \subset m(X^*) = X^*[n_0],$$

so $c(X) = \{0\} = c(X^*)$ because $X[n_0]$ and $X^*[n_0]$ are totally disconnected [8, (24.21)]. This implies that X is a topological torsion group [1, (3.5)], and hence $X \cong \prod_{p\in S(X)}(X_p; U_p)$, where, for each $p \in S(X)$, U_p is a compact open subgroup of X_p [1, (3.13)]. Let

$$n_0 = p_1^{n_1} \cdots p_t^{n_t} \quad \text{and} \quad S = \{p_1, \dots, p_t\},$$

where $p_1, \dots, p_t$ are the distinct prime divisors of n_0 and $t, n_1, \dots, n_t \in \mathbb{N}_0$. We can write

$$X = X_{p_1} \oplus \cdots \oplus X_{p_t} \oplus G \quad \text{and} \quad X^* = X^*_{p_1} \oplus \cdots \oplus X^*_{p_t} \oplus H,$$

where $G = \overline{\sum_{p\nmid n_0} X_p} \cong \prod_{p\nmid n_0}(X_p; U_p)$ and $H = \overline{\sum_{p\nmid n_0} X^*_p} \cong \prod_{p\nmid n_0}(X^*_p; A(X^*_p, U_p))$. It is clear that G and $H \cong G^*$ are torsion-free, so (i) holds [13, (5.2)]. For each $i = 1, \dots, t$, we also have $m(X_{p_i}) = X_{p_i}[p_i^{n_i}]$ and $m(X^*_{p_i}) = X^*_{p_i}[p_i^{n_i}]$, so (ii) holds as well. □

In order to deal with general groups $X \in \mathcal{L}$, we need the following lemma which is inspired by [7, p. 236, (b)] and [9, Lemma 64.1].

Lemma 4. *Let X be a group in $\mathcal{L}$ for which there exist two sequences $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$ of non-zero closed subgroups such that*

$$X = A_0 \oplus \cdots \oplus A_n \oplus B_n \quad \textit{and} \quad B_n = A_{n+1} \oplus B_{n+1}$$

for all $n \in \mathbb{N}$. Then $E(X)$ fails to satisfy DCC on topologically principal right (respectively, left) ideals.

Proof. For $n \in \mathbb{N}$, let $\varepsilon_n \in E(X)$ denote the canonical projection of X onto B_n. As in the proof of [7, p. 236, (b)] or [9, Lemma 64.1], one can see that $\big(\varepsilon_n E(X)\big)_{n\in\mathbb{N}}$ and $\big(E(X)\varepsilon_n\big)_{n\in\mathbb{N}}$ are strictly descending chains of right, respectively, left ideals. It remains to observe that, for every $n \in \mathbb{N}$, $\varepsilon_n E(X)$ and $E(X)\varepsilon_n$ are closed in $E(X)$ because ε_n is idempotent. $\square$

For general groups in $\mathcal{L}$, we have:

Theorem 2. *Let X be a group in $\mathcal{L}$ such that $E(X)$ satisfies DCC on topologically principal ideals. Then $X = U \oplus V \oplus W \oplus Y$, where $U \cong \mathbb{R}^d$ for some $d \in \mathbb{N}$, $V \cong \mathbb{Q}^{(\mu)}$ and $W \cong (\mathbb{Q}^*)^\nu$ for some cardinal numbers μ and ν, and Y is a topological torsion group in $\mathcal{L}$ satisfying the following conditions:*

(i) $S(Y) = S(X)$ is finite;

(ii) for each $p \in S(Y)$, there exists $n(p) \in \mathbb{N}$ such that $m(Y_p) = Y[p^{n(p)}]$ and $\overline{d(Y_p)} = \overline{p^{n(p)}Y_p}$.

Proof. By [1, (9.3)], we can write $X = U \oplus V \oplus W \oplus Y$, where $U \cong \mathbb{R}^d$ for some $d \in \mathbb{N}$, $V \cong \mathbb{Q}^{(\mu)}$ and $W \cong (\mathbb{Q}^*)^\nu$ for some cardinal numbers μ and ν, and Y is residual. In particular, $k(X) = W \oplus k(Y)$ and $c(X) \cap k(X) = W \oplus \big(c(Y) \cap k(Y)\big)$, so $k(X)/\big(c(X) \cap k(X)\big) \cong k(Y)/\big(c(Y) \cap k(Y)\big)$, and hence $S(Y) = S(X)$. Our first aim is to show that the collection $\mathcal{E} = \{\overline{nE(Y)} \mid n \in \mathbb{N}_0\}$ has a minimal element with respect to inclusion. Let $Z = U \oplus V \oplus W$, so

$$E(X) \cong \begin{pmatrix} E(Z) & H(Y,Z) \\ H(Z,Y) & E(Y) \end{pmatrix},$$

as it follows from Lemma 3. For $n \in \mathbb{N}_0$, let $\mathcal{I}_n$ be the closed ideal of $\begin{pmatrix} E(Z) & H(Y,Z) \\ H(Z,Y) & E(Y) \end{pmatrix}$ generated by $\begin{pmatrix} 0 & 0 \\ 0 & n1_Y \end{pmatrix}$. We assert that

$$\mathcal{I}_n = \begin{pmatrix} \overline{\big(H(Y,Z)H(Z,Y)\big)} & H(Y,Z) \\ H(Z,Y) & \overline{nE(Y)} \end{pmatrix},$$

where $\overline{\big(H(Y,Z)H(Z,Y)\big)} \subset E(Z)$. To see that

$$\mathcal{I}_n \subset \begin{pmatrix} \overline{\big(H(Y,Z)H(Z,Y)\big)} & H(Y,Z) \\ H(Z,Y) & \overline{nE(Y)} \end{pmatrix},$$

it suffices to show that

$$\begin{pmatrix} \overline{\big(H(Y,Z)H(Z,Y)\big)} & H(Y,Z) \\ H(Z,Y) & \overline{nE(Y)} \end{pmatrix}$$

is a closed ideal of $\begin{pmatrix} E(Z) & H(Y,Z) \\ H(Z,Y) & E(Y) \end{pmatrix}$. We will show the later by applying Lemma 1(iii). Clearly, we have

$$\overline{\big(H(Y,Z)H(Z,Y)\big)}H(Y,Z) \subset H(Y,Z),$$

$$H(Y,Z)\overline{nE(Y)} \subset H(Y,Z),$$
$$H(Z,Y)\overline{\big(H(Y,Z)H(Z,Y)\big)} \subset H(Z,Y),$$
$$\overline{nE(Y)}H(Z,Y) \subset H(Z,Y),$$

and

$$\big[H(Y,Z), H(Z,Y)\big]_{E(Z)} \subset \overline{\big(H(Y,Z)H(Z,Y)\big)}.$$

Further, since $\frac{1}{n}1_Z$ is a continuous endomorphism of Z, every $f \in H(Y,Z)$ and $g \in H(Z,Y)$ can be written in the form $f = n(\frac{1}{n}f)$ and $g = n(\frac{1}{n}g)$. Consequently, we also have

$$\big[H(Z,Y), H(Y,Z)\big]_{E(Y)} \subset \overline{nE(Y)}.$$

It follows that Lemma 1(iii) is applicable, so

$$\begin{pmatrix} \overline{\big(H(Y,Z)H(Z,Y)\big)} & H(Y,Z) \\ H(Z,Y) & \overline{nE(Y)} \end{pmatrix}$$

is a closed ideal of $\begin{pmatrix} E(Z) & H(Y,Z) \\ H(Z,Y) & E(Y) \end{pmatrix}$, and hence

$$\mathcal{I}_n \subset \begin{pmatrix} \overline{\big(H(Y,Z)H(Z,Y)\big)} & H(Y,Z) \\ H(Z,Y) & \overline{nE(Y)} \end{pmatrix}.$$

On the other hand, given any $f \in H(Y,Z)$ and $g \in H(Z,Y)$, we have

$$\begin{pmatrix} 0 & f \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{n}f \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & n1_Y \end{pmatrix} \in \mathcal{I}_n,$$

$$\begin{pmatrix} 0 & 0 \\ g & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & n1_Y \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \frac{1}{n}g & 0 \end{pmatrix} \in \mathcal{I}_n,$$

and

$$\begin{pmatrix} fg & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & f \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ g & 0 \end{pmatrix} \in \mathcal{I}_n,$$

so

$$\mathcal{I}_n \supset \begin{pmatrix} \overline{\big(H(Y,Z)H(Z,Y)\big)} & H(Y,Z) \\ H(Z,Y) & \overline{nE(Y)} \end{pmatrix},$$

and hence

$$\mathcal{I}_n = \begin{pmatrix} \overline{\big(H(Y,Z)H(Z,Y)\big)} & \overline{H(Y,Z)} \\ H(Z,Y) & \overline{nE(Y)} \end{pmatrix}.$$

Now, since $\begin{pmatrix} E(Z) & H(Y,Z) \\ H(Z,Y) & E(Y) \end{pmatrix}$ satisfies DCC on topologically principal ideals, we conclude that the collection $\{\mathcal{I}_n \mid n \in \mathbb{N}_0\}$ has a minimal element, which implies that the collection

$$\mathcal{E} = \{\overline{nE(Y)} \mid n \in \mathbb{N}_0\}$$

has a minimal element as well. It follows that Theorem 1 is applicable to Y. In particular, Y is a topological torsion group, so

$$Y \cong \prod_{p \in S(Y)} (Y_p; O_p),$$

where, for each $p \in S(Y)$, O_p is a compact open subgroup of Y_p [1, (3.13)]. It remains to observe that if $S(Y)$ were infinite, say $S(Y) = \{p_0, p_1, \ldots\}$, then we could construct, by setting $A_n = Y_{p_n}$ and $B_n = \overline{\sum_{i>n} Y_{p_i}}$, two sequences $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$ of closed subgroups of Y as in Lemma 4, a contradiction. □

4 The necessary condition in case of topological p-primary groups

As we saw in the preceding section, the problem of determining the groups $X \in \mathcal{L}$ for which the ring $E(X)$ satisfies DCC on topologically principal right (respectively, left) ideals reduces to the case of topological p-primary groups. In the present section, we deal with this last type of groups.

We begin by extending and sharpening a result of L. Robertson, which asserts that $\mathbb{Q}_p$ is splitting in the class of torsion-free groups in $\mathcal{L}$ (see [1, Proposition 6.23]).

Theorem 3. *Let $X \in \mathcal{L}$ and let D be a closed subgroup of X such that $D \cong \mathbb{Q}_p$ for some $p \in \mathbb{P}$. The following conditions are equivalent:*

(i) D splits topologically from X.

(ii) $D \not\subset \big(c(X) \cap k(X)\big) + m(X)$.

Proof. Assume (i). Then we can write $X = D \oplus G$ for some closed subgroup G of X. Since $X/G \cong D$ is torsion-free, we have $m(X) \subset G$. Also, since X/G is totally disconnected, we have $c(X) \subset G$. Consequently, $c(X) + m(X) \subset G$ and hence (ii) holds.

Assume (ii). By [1, (9.3)], we can write $X = U \oplus V \oplus W \oplus Y$, where $U \cong \mathbb{R}^d$ for some $d \in \mathbb{N}$, $V \cong \mathbb{Q}^{(\mu)}$ and $W \cong (\mathbb{Q}^*)^\nu$ for some cardinal numbers μ and ν, and Y is residual. Since $D = k(D)$ and $k(X) = W \oplus Y$, we have $D \subset W \oplus Y$. Consequently, it suffices to show that D splits topologically from $W \oplus Y$.

Now, since Y is residual, we have $c(Y) \subset m(Y) = m(X)$, which implies

$$(c(X) \cap k(X)) + m(X) = W \oplus m(Y).$$

Our assumption then gives $D \not\subset W \oplus m(Y)$, and hence $W \oplus Y \setminus W \oplus m(Y)$ must contain elements of D. Denote by $\varphi : W \oplus Y \to (W \oplus Y)/(W \oplus m(Y))$ the canonical projection, and let f be the restriction of φ to D. By [8, (5.27)], we have $D/\ker(f) \cong f(D)$. Since $(W \oplus Y)/(W \oplus m(Y)) \cong Y/m(Y)$ is torsion-free and since every quotient of $\mathbb{Q}_p$ by a proper closed subgroup is torsion, we conclude that

$$D \cap (W \oplus m(Y)) = \ker f = \{0\}.$$

In particular, $f(D) \cong \mathbb{Q}_p$, and hence $f(D)$ splits topologically from $(W \oplus Y)/(W \oplus m(Y))$ [1, (6.23)]. Write $(W \oplus Y)/(W \oplus m(Y)) = f(D) \oplus G$ for some closed subgroup G of $(W \oplus Y)/(W \oplus m(Y))$, and set $G_0 = \varphi^{-1}(G)$. We assert that $W \oplus Y = D \oplus G_0$. Indeed, it is clear that G_0 is a closed subgroup of $W \oplus Y$. If $a \in D \cap G_0$, then $\varphi(a) \in \varphi(D) \cap \varphi(G_0) = f(D) \cap G = \{0\}$, so $a \in D \cap \big(W \oplus m(Y)\big) = \{0\}$. Further, given any $z \in W \oplus Y$, we have $\varphi(z) = \varphi(a) + \varphi(b)$ for some $a \in D$ and $b \in G_0$. Consequently, $z - a - b = t$ for some $t \in W \oplus m(Y)$, and hence $z = a + b + t$. Since $b + t \in G_0$, we conclude that $W \oplus Y = D \dotplus G_0$. Since $\mathbb{Q}_p$ is σ-compact, it then follows from [1, (6.5)] that $W \oplus Y = D \oplus G_0$. □

Corollary 2. *Let X be a group in $\mathcal{L}$ such that $t(X)$ is reduced and closed in X. If D is a closed subgroup of X satisfying $D \cong \mathbb{Q}_p$, then D splits topologically from X.*

Proof. As in the proof of Lemma 3, write $X = U \oplus V \oplus W \oplus Y$, where $U \cong \mathbb{R}^d$ for some $d \in \mathbb{N}$, $V \cong \mathbb{Q}^{(\mu)}$ and $W \cong (\mathbb{Q}^*)^\nu$ for some cardinal numbers μ and ν, and Y is residual. Since $t(X)$ is closed in X, we have $m(X) = t(X) = t(Y)$, so $(c(X) \cap k(X)) + m(X) = W \oplus t(Y)$. It is also clear that $D \subset k(X) = W \oplus Y$. In order to apply Theorem 3, we have to show that $D \not\subset W \oplus t(Y)$. Assume this is not so, and let $\varepsilon \in E(X)$ denote the canonical projection of X onto Y. It follows that $\varepsilon(D)$ is a subgroup of $t(Y)$. Since $\varepsilon(D)$ is divisible and $t(Y)$ is reduced, we get $\varepsilon(D) = \{0\}$, so $D \subset W$, which is a contradiction because W is compact and D is not. □

We continue with the following

Lemma 5. *Let $p \in \mathbb{P}$, and let X be a non-reduced topological p-primary group in $\mathcal{L}$ such that $t(X) = X[p^{n_0}]$ for some $n_0 \in \mathbb{N}$. For any non-zero $a \in d(X)$, let D_a be the smallest divisible subgroup of X containing a. Then $\overline{D_a} \cong \mathbb{Q}_p$ and $X = \overline{D_a} \oplus G$ for some closed subgroup G of X.*

Proof. Since $t(X) = X[p^{n_0}]$, $d(X)$ cannot contain copies of $\mathbb{Z}(p^\infty)$, so D_a is algebraically isomorphic to $\mathbb{Q}$. It follows from [2, Theorem 1] that $\overline{D_a}$ is divisible. Since X is a topological p-primary group, there exists a topological group isomorphism f from $\mathbb{Z}_p$ onto $\overline{\langle a \rangle}$. Let $\eta : \overline{\langle a \rangle} \to \overline{D_a}$ denote the canonical injection, and set $h = \eta \circ f$. Since $\mathbb{Z}_p$ is open in $\mathbb{Q}_p$, h extends to a continuous group homomorphism

$h_0 : \mathbb{Q}_p \to \overline{D_a}$ [8, (A.7)]. Now, since $\mathbb{Q}_p$ is the minimal divisible extension of $\mathbb{Z}_p$, $\mathbb{Z}_p$ is essential in $\mathbb{Q}_p$ [6, Lemma 24.3], and hence $\ker(h_0) = \{0\}$ [6, Lemma 24.2]. We deduce that h_0 is a topological isomorphism from $\mathbb{Q}_p$ onto a closed subgroup of $\overline{D_a}$ [1, (4.21)]. Now, since $h_0(\mathbb{Q}_p)$ is divisible and $a \in h_0(\mathbb{Q}_p)$, we must have $h_0(\mathbb{Q}_p) = \overline{D_a}$, so $\overline{D_a} \cong \mathbb{Q}_p$. It remains to apply Corollary 2. □

Now we can concretize the structure of topological p-primary groups in $\mathcal{L}$ with the property in question.

Theorem 4. *Let $p \in \mathbb{P}$, and let X be a topological p-primary group in $\mathcal{L}$ such that $E(X)$ satisfies DCC on topologically principal right* (*respectively, left*) *ideals. Then*

$$X \cong \prod_{i=0}^{k(p)} \mathbb{Z}(p^{r_i(p)}) \times \mathbb{Q}_p^{l(p)}$$

for some $k(p), r_0(p), \ldots, r_{k(p)}(p), l(p) \in \mathbb{N}$.

Proof. By Theorem 1, there exists an $n(p) \in \mathbb{N}$ such that $m(X) = X[p^{n(p)}]$ and $\overline{d(X)} = \overline{p^{n(p)}X}$. We will distinguish two cases: $d(X) = \{0\}$ and $d(X) \neq \{0\}$.

First assume $d(X) = \{0\}$, so $X = X[p^{n(p)}]$. To decompose X, pick an element of maximal order $x_0 \in X$, and set $A_0 = \langle x_0 \rangle$. Clearly, $A_0 \cong \mathbb{Z}(p^{r_0(p)})$ for some $r_0(p) \in \mathbb{N}$. By [12, Lemma 2], we can write $X = A_0 \oplus B_0$ for some closed subgroup B_0 of X. If $B_0 \neq \{0\}$, choose an element of maximal order $x_1 \in B_0$ and write $X = A_0 \oplus A_1 \oplus B_1$, where $A_1 \cong \mathbb{Z}(p^{r_1(p)})$ for some $r_1(p) \in \mathbb{N}$ and B_1 is a closed subgroup of B_0. As Lemma 4 shows, if we continue in this way, we must arrive at a step $k(p)$ with $B_{k(p)} = \{0\}$.

Next assume $d(X) \neq \{0\}$. Picking any non-zero $y_0 \in d(X)$, let D_0 be the closure of the smallest divisible subgroup of X containing y_0. By Lemma 5, $D_0 \cong \mathbb{Q}_p$ and $X = D_0 \oplus G_0$ for some closed subgroup G_0 of X. If $d(G_0) \neq 0$, pick any non-zero $y_1 \in d(G_0)$ and let D_1 be the closure of the smallest divisible subgroup of D_0 containing y_1. As above, we have $D_1 \cong \mathbb{Q}_p$ and $X = D_0 \oplus D_1 \oplus G_1$ for some closed subgroup G_1 of G_0. By Lemma 4 again, this procedure must stop after a finite number, say $l(p)$, of steps, and so

$$X = D_0 \oplus \cdots \oplus D_{l(p)-1} \oplus G_{l(p)},$$

where $G_{l(p)}$ is reduced. This shows that

$$d(X) = D_0 \oplus \cdots \oplus D_{l(p)-1} = \overline{d(X)} \quad \text{and} \quad X[p^{n(p)}] \subset G_{l(p)}.$$

Therefore

$$\begin{aligned} p^{n(p)}G_{l(p)} &\subset \overline{p^{n(p)}X} \cap G_{l(p)} = \overline{d(X)} \cap G_{l(p)} \\ &= (D_0 \oplus \cdots \oplus D_{l(p)-1}) \cap G_{l(p)} = \{0\}, \end{aligned}$$

so $G_{l(p)} = X[p^{n(p)}]$, and hence

$$X = D_0 \oplus \cdots \oplus D_{l(p)-1} \oplus X[p^{n(p)}].$$

Since $D_0 \oplus \cdots \oplus D_{l(p)-1}$ and $X[p^{n(p)}]$ are fully invariant in X, we deduce from Lemma 3 that

$$E(X) \cong E(D_0 \oplus \cdots \oplus D_{l(p)-1}) \times E(X[p^{n(p)}]),$$

and hence $E(X[p^{n(p)}])$ satisfies DCC on topologically principal ideals. It follows that the first case applies to $X[p^{n(p)}]$, completing the proof. □

5 Characterizations

In this last section, we establish our results. We begin with two lemmas, which are needed in the proof of the main result. For the former, recall that every divisible torsion-free abelian group D can be considered as a vector space over the field of rational numbers, $\mathbb{Q}$, and this $\mathbb{Q}$-vector space structure is the only one existing on D. Moreover, every group homomorphism between such groups is in fact a homomorphism of $\mathbb{Q}$-vector spaces.

We have:

Lemma 6. *Let $d, n, l_1, \ldots, l_n \in \mathbb{N}$ and $p_1, \ldots, p_n \in \mathbb{P}$. The $\mathbb{Q}$-vector spaces $\mathbb{R}^d \times \prod_{i=1}^n \mathbb{Q}_{p_i}^{l_i}$ and $(\mathbb{Q}^*)^d$ satisfy both ACC and DCC on closed $\mathbb{Q}$-subspaces.*

Proof. It is clear that in either of $\mathbb{Q}$-vector spaces $\mathbb{R}^d$ and $\mathbb{Q}_p^l$, where $d, l \in \mathbb{N}$ and $p \in \mathbb{P}$, the closed $\mathbb{Q}$-subspaces are in fact $\mathbb{R}$-subspaces and respectively $\mathbb{Q}_p$-subspaces. As $\dim_{\mathbb{R}}(\mathbb{R}^d) = d$ and $\dim_{\mathbb{Q}_p}(\mathbb{Q}_p^l) = l$, we conclude that $\mathbb{R}^d$ and $\mathbb{Q}_p^l$ satisfy ACC and DCC on closed $\mathbb{Q}$-subspaces. Now, write the $\mathbb{Q}$-vector space $G = \mathbb{R}^d \times \prod_{i=1}^n \mathbb{Q}_{p_i}^{l_i}$ in the form

$$G = G_0 \oplus G_1 \oplus \cdots \oplus G_n,$$

where $G_0 \cong \mathbb{R}^d$, $G_1 \cong \mathbb{Q}_{p_1}^{l_1}, \ldots, G_n \cong \mathbb{Q}_{p_n}^{l_n}$. Given a closed $\mathbb{Q}$-subspace H of G, it is clear that $c(H) \subset c(G) = G_0$. It is also clear that, for any $x \in G_0 \cap H$, the $\mathbb{Q}$-subspace $\mathbb{Q}x \subset G_0 \cap H$, so $\mathbb{R}x = \overline{\mathbb{Q}x} \subset G_0 \cap H$, and hence $G_0 \cap H$ is connected [3, Ch. 1, §11, Proposition 2]. It follows that $c(H) = G_0 \cap H$. Further, since H is torsion-free, we can write $H = H_0 \oplus K$ (a topological direct sum of topological groups), where $H_0 = c(H)$ [1, (6.13)]. Moreover, since $H_0 \subset G_0$, we have $K \subset G_1 \oplus \cdots \oplus G_n$, so $K = H_1 \oplus \cdots \oplus H_n$, where $H_i \subset G_i$ for all $i = 1, \ldots, n$ [1, (3.13)]. Thus we obtained a decomposition of H as a topological direct sum $H = H_0 \oplus H_1 \oplus \cdots \oplus H_n$ of $\mathbb{Q}$-vector spaces. Since the $\mathbb{Q}$-vector spaces $G_0, G_1, \ldots, G_n$ satisfy ACC and DCC on closed $\mathbb{Q}$-subspaces, we conclude that so does G.

Now let us consider the case of $(\mathbb{Q}^*)^d$. It suffices to observe that a closed subgroup C of $(\mathbb{Q}^*)^d$ is a $\mathbb{Q}$-subspace if and only if its annihilator $A(\mathbb{Q}^d, C)$ is a $\mathbb{Q}$-subspace of $\mathbb{Q}^d$. Indeed, if C is a $\mathbb{Q}$-subspace of $(\mathbb{Q}^*)^d$ and $x \in A(\mathbb{Q}^d, C)$, then $\gamma(\frac{p}{q}x) = \frac{p}{q}\gamma(x) = 0$ for all $\gamma \in C$ and $\frac{p}{q} \in \mathbb{Q}$. Consequently, $\frac{p}{q}x \in A(\mathbb{Q}^d, C)$ for all $\frac{p}{q} \in \mathbb{Q}$, so $A(\mathbb{Q}^d, C)$

is a $\mathbb{Q}$-subspace of $\mathbb{Q}^d$. In a similar way, if $A(\mathbb{Q}^d, C)$ is a $\mathbb{Q}$-subspace of $\mathbb{Q}^d$, then C is a closed $\mathbb{Q}$-subspace of $(\mathbb{Q}^*)^d$. Since $\mathbb{Q}^d$ is of finite dimension, the proof is complete. □

Lemma 7. *Let R be a topological ring, M a topological (right or left) R-module, and C a closed submodule of M.*

(i) If M satisfies DCC on closed submodules, then so do C and M/C.

(ii) If C is either compact or open in M and if C and M/C satisfy DCC on closed submodules, then so does M.

Proof. The proof follows the same pattern as in the abstract case (see, for examle,[9, Proposition 27.1]). The requirement in (ii) that C is either compact or open in M assures that the image through the canonical projection of any closed submodule of M is closed in M/C. □

We are now prepared to prove our main result.

Theorem 5. *For a group $X \in \mathcal{L}$, the following statements are equivalent:*

(i) $E(X)$ satisfies both ACC and DCC on closed right ideals.

(ii) $E(X)$ satisfies DCC on closed right ideals.

(iii) $E(X)$ satisfies DCC on topologically principal right ideals, i.e. ideals of the form $\overline{fE(X)}$ with $f \in E(X)$.

(iv) $X \cong \mathbb{R}^d \times \mathbb{Q}^n \times (\mathbb{Q}^*)^m \times \prod_{p\in S_1} \mathbb{Q}_p^{l(p)} \times \prod_{p\in S_2} \prod_{i=0}^{k(p)} \mathbb{Z}(p^{r_i(p)})$, where S_1, S_2 are finite subsets of $\mathbb{P}$, and d, n, m, the $k(p)$'s, the $r_i(p)$'s and the $l(p)$'s are natural numbers.

Proof. Clearly, (i) implies (ii) and (ii) implies (iii). The fact that (iii) implies (iv) follows from Theorem 2 and Theorem 4.

Now assume (iv). We can write $X = D \oplus T$, where

$$D \cong \mathbb{R}^d \times \mathbb{Q}^n \times (\mathbb{Q}^*)^m \times \prod_{p\in S_1} \mathbb{Q}_p^{l(p)} \quad \text{and} \quad T \cong \prod_{p\in S_2} \prod_{i=0}^{k(p)} \mathbb{Z}(p^{r_i(p)}).$$

It is clear that $D = d(X)$ and $T = t(X)$, so D and T are topologically fully invariant subgroups of X. It follows from Lemma 3 that $E(X) \cong E(D) \times E(T)$. Since $E(T)$ is finite and since every right ideal $\mathcal{J}$ of $E(D) \times E(T)$ is of the form $\mathcal{J} = \mathcal{J}_d \times \mathcal{J}_t$, where $\mathcal{J}_d$ is a right ideal of $E(D)$ and $\mathcal{J}_t$ is a right ideal of $E(T)$, it suffices to show that $E(D)$ satisfies ACC and DCC on closed right ideals. In order to do this, write $D = M \oplus W$, where

$$M \cong \mathbb{Q}^n \quad \text{and} \quad W \cong \mathbb{R}^d \times (\mathbb{Q}^*)^m \times \prod_{p\in S_1} \mathbb{Q}_p^{l(p)}.$$

We have $W = c(D) + k(D)$, so W is topologically fully invariant in D, and hence

$$E(D) \cong \begin{pmatrix} E(W) & H(M,W) \\ 0 & E(M) \end{pmatrix}$$

by Lemma 3 again. It follows from Lemma 2 that we will achieve our goal if we show that $E(W)_{E(W)}$ and $\big(H(M,W)\times E(M)\big)_{E(M)}$ satisfy ACC and DCC on closed submodules.

First we consider the case of $\big(H(M,W) \times E(M)\big)_{E(M)}$. Since $E(M) \cong \mathbb{M}_n(\mathbb{Q})$, we deduce that $E(M)$ is discrete and satisfies ACC and DCC on right ideals. As then $H(M,W) \times \{0\}$ is open in $H(M,W) \times E(M)$, it suffices by Lemma 7 to show that $H(M,W)$ satisfies ACC and DCC on closed $E(M)$-submodules. To this end, we write

$$W = V \oplus K \oplus L, \tag{2}$$

where $V \cong \mathbb{R}^d$, $K \cong (\mathbb{Q}^*)^m$, and $L = \oplus_{p\in S_1} L_p$ with $L_p \cong \mathbb{Q}_p^{l(p)}$ for all $p \in S_1$. We know from [8, (23.34)(d)] that

$$H(M,W) \cong H(M,V) \times H(M,K) \times \prod_{p\in S_1} H(M,L_p) \tag{3}$$

as topological groups, and hence as topological $E(M)$-modules because the corresponding canonical isomorphism in (3) is easily seen to be an isomorphism of $E(M)$-modules . Now, since M is discrete and K is compact, it follows by the Ascoli theorem that $H(M,K)$ is compact. Therefore to see that $H(M,W)$ satisfies ACC and DCC on closed $E(M)$-submodules, it suffices by Lemma 7 to show that so do $H(M,K)$ and $H(M,V) \times \prod_{p\in S_1} H(M,L_p)$. For this purpose, we will consider $H(M,K)$ and $H(M,V) \times \prod_{p\in S_1} H(M,L_p)$ as vector spaces over $\mathbb{Q}$, by using the inclusion $\lambda \mapsto \lambda I_n$ of $\mathbb{Q}$ into $\mathbb{M}_n(\mathbb{Q}) \cong E(M)$. It is then clear that the closed $E(M)$-submodules of $H(M,K)$ and those of $H(M,V) \times \prod_{p\in S_1} H(M,L_p)$ are closed $\mathbb{Q}$-subspaces, so it will suffice to show that $H(M,K)$ and $H(M,V)\times\prod_{p\in S_1} H(M,L_p)$ satisfy both ACC and DCC on closed $\mathbb{Q}$-subspaces. Now, since $H(\mathbb{Q},\mathbb{Q}^*) \cong \mathbb{Q}^*$, $H(\mathbb{Q},\mathbb{R}) \cong \mathbb{R}$, and $H(\mathbb{Q},\mathbb{Q}_p) \cong \mathbb{Q}_p$ for all $p \in \mathbb{P}$, we deduce from [8, (23.34)(c, d)] that

$$H(M,K) \cong (\mathbb{Q}^*)^{nm} \ \text{ and } \ H(M,V) \times \prod_{p\in S_1} H(M,L_p) \cong \mathbb{R}^{nd} \times \prod_{p\in S_1} \mathbb{Q}_p^{nl(p)}$$

as topological groups, and hence as topological vector spaces over $\mathbb{Q}$. It follows from Lemma 6 that both $H(M,K)$ and $H(M,V) \times \prod_{p\in S_1} H(M,L_p)$ satisfy ACC and DCC on closed $\mathbb{Q}$-subspaces. This proves that $H(M,W)\times E(M)$ satisfies ACC and DCC on closed $E(M)$-submodules.

Further, we consider the case of $E(W)$. Since $K \oplus L = k(W)$ is topologically fully invariant in W, we deduce from (2) and Lemma 3 that

$$E(W) \cong \begin{pmatrix} E(K\oplus L) & H(V, K\oplus L) \\ 0 & E(V) \end{pmatrix}.$$

By Lemma 2, we have to show that the modules $E(K \oplus L)_{E(K \oplus L)}$ and $\big(H(V, K \oplus L) \times E(V)\big)_{E(V)}$ satisfy ACC and DCC on closed submodules.

First we consider the case of $\big(H(V, K \oplus L) \times E(V)\big)_{E(V)}$. By use of the inclusion $\lambda \mapsto \lambda I_d \in \mathbb{M}_d(\mathbb{R}) \cong E(V)$, the group $H(V, K \oplus L) \times E(V)$ can be given a topological vector space structure over the field of reals, $\mathbb{R}$. It is clear that every $E(V)$-submodules of $H(V, K \oplus L) \times E(V)$ becomes an $\mathbb{R}$-subspace. So to achieve our goal, it suffices to show that $H(V, K \oplus L) \times E(V)$ is of finite dimension. This is clear for $E(V)$. On the other hand, $H(V, K \oplus L) = H(V, K)$ because $V = c(V)$ and $c(L) = \{0\}$. Since, by [8, (23.34)(c,d)], $H(V, K) \cong \mathbb{R}^{md}$ as topological groups and hence as topological $\mathbb{R}$-spaces, $H(V, K)$ has finite dimension as well.

Next consider the case of $E(K \oplus L) = E(K \oplus \oplus_{p \in S_1} L_p)$. We will proceed by induction on $n = card(S_1)$. If $S_1 = \varnothing$, then $E(K \oplus L) = E(K)$. Since $E(K)$ and $E(K^*)$ are topologically anti-isomrphic, and since $E(K^*) \cong \mathbb{M}_m(\mathbb{Q})^{opp}$, the fact that $E(K)$ satisfies ACC and DCC on closed right ideals is clear. Assume $S_1 = \{p\}$, so $L = L_p$. Since $K = c(K \oplus L_p)$ is topologically fully invariant in $K \oplus L_p$, it follows that

$$E(K \oplus L) = E(K \oplus L_p) \cong \begin{pmatrix} E(K) & H(L_p, K) \\ 0 & E(L_p) \end{pmatrix}.$$

To see that $E(K \oplus L_p)_{E(K \oplus L_p)}$ satisfies ACC and DCC on closed submodules, it suffices to show that so do $E(K)_{E(K)}$ and $\big(H(L_p, K) \times E(L_p)\big)_{E(L_p)}$. The case of $E(K)$ is clear. Further, by use of the inclusion $\lambda \mapsto \lambda I_{l(p)}$ of the field $\mathbb{Q}_p$ of p-adic numbers into $\mathbb{M}_{l(p)}(\mathbb{Q}_p) \cong E(L_p)$, the group $H(L_p, K) \times E(L_p)$ can be given a vector space structure over $\mathbb{Q}_p$. Since every $E(L_p)$-submodule of $\big(H(L_p, K) \times E(L_p)\big)_{E(L_p)}$ is a $\mathbb{Q}_p$-vector space, it suffices to show that $\big(H(L_p, K) \times E(L_p)\big)_{\mathbb{Q}_p}$ has finite dimension. This is clear for $E(L_p)_{\mathbb{Q}_p}$ because $E(L_p) \cong \mathbb{M}_{l(p)}(\mathbb{Q}_p)$. Also, since $H(L_p, K) \cong H(K^*, L_p^*) \cong H(\mathbb{Q}, \mathbb{Q}_p)^{ml(p)} \cong \mathbb{Q}_p^{ml(p)}$, we have $\dim_{\mathbb{Q}_p} H(L_p, K) = ml(p)$, proving the case $n = 1$. Assume $n \geq 2$ and that for every proper subset S' of S_1, the ring $E(K \oplus \oplus_{p \in S'} L_p)$ satisfies ACC and DCC on closed right ideals. Pick any $p \in S_1$. We have

$$E(K \oplus L) \cong \begin{pmatrix} E(K \oplus \oplus_{q \in S_1 \setminus \{p\}} L_q) & H(L_p, K \oplus \oplus_{q \in S_1 \setminus \{p\}} L_q) \\ 0 & E(L_p) \end{pmatrix}.$$

By the induction hypothesis, the ring $E(K \oplus \oplus_{q \in S_1 \setminus \{p\}} L_q)$ satisfies ACC and DCC on closed right ideals. Observing that

$$H(L_p, K \oplus \oplus_{q \in S_1 \setminus \{p\}} L_q) = H(L_p, K),$$

we conclude from the preceding case that $H(L_p, K \oplus \oplus_{q \in S_1 \setminus \{p\}} L_q)_{E(L_p)}$ satisfies ACC and DCC on closed submodules, Consequently, Lemma 2 is applicable, and the proof is complete. □

Corollary 3. *For a group $X \in \mathcal{L}$, the following statements are equivalent:*

(i) $E(X)$ satisfies both ACC and DCC on closed left ideals.

(ii) $E(X)$ *satisfies DCC on closed left ideals.*

(iii) $E(X)$ *satisfies DCC on topologically principal left ideals, i.e. ideals of the form* $\overline{E(X)f}$ *with* $f \in E(X)$.

(iv) $X \cong \mathbb{R}^d \times \mathbb{Q}^n \times (\mathbb{Q}^*)^m \times \prod_{p\in S_1} \mathbb{Q}_p^{l(p)} \times \prod_{p\in S_2} \prod_{i=0}^{k(p)} \mathbb{Z}(p^{r_i(p)})$, *where* S_1, S_2 *are finite subsets of* $\mathbb{P}$, *and* d, n, m, *the* $k(p)$*'s, the* $r_i(p)$*'s and the* $l(p)$*'s are natural numbers.*

In particular, $E(X)$ *satisfies DCC on closed left ideals if and only if it satisfies DCC on closed right ideals.*

Proof. The assertion follows from the fact that $E(X)$ and $E(X^*)$ are topologically anti-isomorphic. □

Specializing to the case of discrete groups, we see that the result of L. Fuchs and F. Szász, mentioned in Introduction, can be supplemented as follows.

Corollary 4. *For a discrete group* $X \in \mathcal{L}$, *the following statements are equivalent:*

(i) $E(X)$ *is right* (*respectively, left*) *artinian.*

(ii) $E(X)$ *satisfies DCC on principal right* (*respectively, left*) *ideals.*

(iii) $E(X)$ *satisfies DCC on closed right* (*respectively, left*) *ideals.*

(iv) $E(X)$ *satisfies DCC on topologically principal right* (*respectively, left*) *ideals.*

(v) $X \cong \mathbb{Q}^n \times \prod_{p\in S} \mathbb{Z}(p^{k(p)})$, *where* $n \in \mathbb{N}$, S *is a finite subset of* $\mathbb{P}$ *and* $k(p) \in \mathbb{N}$ *for all* $p \in S$.

Proof. Since (i) $\Rightarrow$ (ii) $\Rightarrow$ (iv) $\Rightarrow$ (v) and (i) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v), it remains to apply [7, Theorem 111.3]. □

In the following, we drop the assumption that the ideals are closed. First, we consider the problem of determining the groups $X \in \mathcal{L}$ for which the ring $E(X)$ is right (respectively, left) artinian. We need the following

Lemma 8. *Let* Y *be one of the groups* $\mathbb{R}^d$, $(\mathbb{Q}^*)^m$, *or* $\mathbb{Q}_p^{l(p)}$, *where* $d, m, l(p) \in \mathbb{N}_0$ *and* $p \in \mathbb{P}$. *For any* $n \in \mathbb{N}_0$, *the module* $H(\mathbb{Q}^n, Y)_{E(\mathbb{Q}^n)}$ *fails to be artinian.*

Proof. Let C be a $\mathbb{Q}$-basis of Y and $\{\gamma_k \mid k \in \mathbb{N}\}$ a countable subset of C. For $i \in \mathbb{N}$, let

$$H_i = \{h \in H(\mathbb{Q}^n, Y) \mid \operatorname{im}(h) \subset \langle \gamma_k \mid k \geq i\rangle_{\mathbb{Q}}\},$$

where $\langle \gamma_k \mid k \geq i\rangle_{\mathbb{Q}}$ is the $\mathbb{Q}$-subspace of Y generated by the γ_k with $k \geq i$. Then $(H_i)_{i\in\mathbb{N}}$ is a strictly decreasing sequence of $E(\mathbb{Q}^n)$-submodules of $H(\mathbb{Q}^n, Y)_{E(\mathbb{Q}^n)}$. □

We have:

Corollary 5. *For a group $X \in \mathcal{L}$, the following statements are equivalent:*

(i) *$E(X)$ is right artinian.*

(ii) *X is topologically isomorphic with one of the groups*
$\mathbb{R}^d \times (\mathbb{Q}^*)^n \times \prod_{p\in S_1} \mathbb{Q}_p^{l(p)} \times \prod_{p\in S_2} \prod_{i=0}^{k(p)} \mathbb{Z}(p^{r_i(p)})$,
or $\mathbb{Q}^n \times \prod_{p\in S_2} \prod_{i=0}^{k(p)} \mathbb{Z}(p^{r_i(p)})$, *where S_1, S_2 are finite subsets of $\mathbb{P}$ and $d, n, k(p), l(p), r_i(p) \in \mathbb{N}$ for all $i \in \{0, \ldots, k(p)\}$ and $p \in S_1 \cup S_2$.*

Proof. Assume (i). Then, clearly, $E(X)$ satisfies DCC on closed right ideals, so

$$X \cong \mathbb{R}^d \times \mathbb{Q}^n \times (\mathbb{Q}^*)^m \times \prod_{p\in S_1} \mathbb{Q}_p^{l(p)} \times \prod_{p\in S_2} \prod_{i=0}^{k(p)} \mathbb{Z}(p^{r_i(p)})$$

for some finite subsets S_1, S_2 of $\mathbb{P}$ and natural numbers $d, n, m, k(p), l(p)$, and $r_i(p)$ with $i \in \{0, \ldots, k(p)\}$ and $p \in S_1 \cup S_2$. Writing $X = D \oplus T$, where $D \cong \mathbb{R}^d \times \mathbb{Q}^n \times (\mathbb{Q}^*)^m \times \prod_{p\in S_1} \mathbb{Q}_p^{l(p)}$ and $T \cong \prod_{p\in S_2} \prod_{i=0}^{k(p)} \mathbb{Z}(p^{r_i(p)})$. we have $E(X) \cong E(D) \times E(T)$. It follows that $E(D)$ is right artinian. Now, write $D = M \oplus W$, where $M \cong \mathbb{Q}^n$ and $W \cong \mathbb{R}^d \times (\mathbb{Q}^*)^m \times \prod_{p\in S_1} \mathbb{Q}_p^{l(p)}$. Hence $E(D) \cong \begin{pmatrix} E(W) & H(M,W) \\ 0 & E(M) \end{pmatrix}$, where $H(M,W)_{E(\mathbb{Q}^n)}$ is topologically isomorphic with $H(\mathbb{Q}^n, \mathbb{R}^d)_{E(\mathbb{Q}^n)} \times H(\mathbb{Q}^n, (\mathbb{Q}^*)^m)_{E(\mathbb{Q}^n)} \times \prod_{p\in S_1} H(\mathbb{Q}^n, \mathbb{Q}_p^{l(p)})_{E(\mathbb{Q}^n)}$, as easily follows from [8, (23,34)(d)]. If M and W were both non-zero, it would follow from Lemma 8 and [10, (1,2)] that $E(D)$ is not right artinian. This contradiction proves (ii).

To see the converse, we have, by Corollary 4, to consider only the case of $X = \mathbb{R}^d \times (\mathbb{Q}^*)^n \times \prod_{p\in S_1} \mathbb{Q}_p^{l(p)} \times \prod_{p\in S_2} \prod_{i=0}^{k(p)} \mathbb{Z}(p^{r_i(p)})$. Then writing $X = C \oplus T$, where $C \cong \mathbb{R}^d \times (\mathbb{Q}^*)^m \times \prod_{p\in S_1} \mathbb{Q}_p^{l(p)}$ and $T \cong \prod_{p\in S_2} \prod_{i=0}^{k(p)} \mathbb{Z}(p^{r_i(p)})$. we have $E(X) \cong E(C) \times E(T)$. Consequently, it suffices to show that $E(C)$ is right artinian. Write $C = V \oplus K \oplus L$, where $V \cong \mathbb{R}^d$, $K \cong (\mathbb{Q}^*)^n$ and $L = \oplus_{p\in S_1} L_p$ with $L_p \cong \mathbb{Q}_p^{l(p)}$ for all $p \in S_1$. Then $E(C) \cong \begin{pmatrix} E(K) & H(V \oplus L, K) \\ 0 & E(V \oplus L) \end{pmatrix}$. Since $E(K) \cong \mathbb{M}_d(\mathbb{Q})^{opp}$ and $E(V \oplus L) \cong \mathbb{M}_d(\mathbb{R}) \times \prod_{p\in S_1} \mathbb{M}_{l(p)}(\mathbb{Q}_p)$, it suffices by [10, (1.2)] to show that $H(V \oplus L, K)_{E(V\oplus L)}$ is artinian. It is clear from [8, (23,34)(c)] that

$$H(V \oplus L, K)_{E(V\oplus L)} \cong H(V, K)_{E(V\oplus L)} \times \prod_{p\in S_1} H(L_p, K)_{E(V\oplus L)},$$

where the scalar multiplication of the modules $H(V,K)_{E(V\oplus L)}$ and respectively $H(L_p, K)_{E(V\oplus L)}$ with $p \in S_1$ is given by using the projection of $E(V \oplus L) \cong E(V) \times \prod_{q\in S_1} E(L_q)$ onto $E(V)$ respectively $E(L_p)$. Thus it suffices to show that $H(V,K)_{E(V)}$ and respectively $H(L_p, K)_{E(L_p)}$ with $p \in S_1$ are artinian. Now, since the field $\mathbb{R}$ embeds in $E(V)$ and the field $\mathbb{Q}_p$ embeds in $E(L_p)$, $H(V,K)$ can be considered as a vector space over $\mathbb{R}$ and $H(L_p, K)_{E(L_p)}$ as a vector space over $\mathbb{Q}_p$. The conclusion follows because these spaces are finite dimensional. $\square$

Corollary 6. *For a group $X \in \mathcal{L}$, the following statements are equivalent:*

(i) $E(X)$ is left artinian.

(ii) X is topologically isomorphic with one of the groups

$\mathbb{R}^d \times \mathbb{Q}^n \times \prod_{p\in S_1} \mathbb{Q}_p^{l(p)} \times \prod_{p\in S_2} \prod_{i=0}^{k(p)} \mathbb{Z}(p^{r_i(p)})$,

or $(\mathbb{Q}^*)^n \times \prod_{p\in S_2} \prod_{i=0}^{k(p)} \mathbb{Z}(p^{r_i(p)})$, *where S_1, S_2 are finite subsets of $\mathbb{P}$ and $d, n, k(p), l(p), r_i(p) \in \mathbb{N}$ for all $i \in \{0, \dots, k(p)\}$ and $p \in S_1 \cup S_2$.*

Proof. Since $E(X)$ and $E(X^*)$ are topologically anti-isomorphic, the assertion follows from Corollary 5 and duality. $\square$

We close the paper by determining the groups $X \in \mathcal{L}$ with the property that $E(X)$ satisfies DCC on principal right (respectively, left) ideals. It turns out that this last condition on $E(X)$ is equivalent to those of Theorem 5. First we establish the following

Lemma 9. *Let $X = \mathbb{Q}^n$ and $Y = \mathbb{R}^d \times (\mathbb{Q}^*)^m \times \prod_{p\in S_1} \mathbb{Q}_p^{l(p)}$, where S is a subset of $\mathbb{P}$ and d, n, m, and $l(p)$ for $p \in S$ are natural numbers. If $u, v \in H(X, Y)$ satisfy $v = u \circ w$ for some $w \in E(X)$ and* $\dim_{\mathbb{Q}} \operatorname{im}(v) = \dim_{\mathbb{Q}} \operatorname{im}(u)$, *then $v = u \circ w'$ for some invertible $w' \in E(X)$.*

Proof. It is clear that the morphisms in $H(X, Y)$ are $\mathbb{Q}$-linear mappings. Since $\dim \operatorname{im}(v) = \dim \operatorname{im}(u)$, it follows by rank-nullity connection [14, Theorem 2.12] that $\ker(u)$ and $\ker(v)$ have the same dimension, say k. Let $e_1, \dots, e_n$ and $e'_1, \dots, e'_n$ be bases in X such that $e_1, \dots, e_k$ is a basis in $ker(u)$ and $e'_1, \dots, e'_k$ is a basis in $ker(v)$. Clearly, $v(e'_i) = u(w(e'_i))$ for all $i = 1, \dots, n$. We define $w' \in E(X)$ by setting

$$w'(e'_i) = \begin{cases} e_i, & \text{if} \quad i = 1, \dots, k; \\ w(e_i), & \text{if} \quad i = k+1, \dots, n. \end{cases}$$

Then w' is invertible and $(u \circ w')(e'_i) = v(e'_i)$ for all $i = 1, \dots, n$, so $v = u \circ w'$. $\square$

We have:

Corollary 7. *For a group $X \in \mathcal{L}$, the following statements are equivalent:*

(i) $E(X)$ satisfies DCC on principal right (respectively, left) ideals.

(ii) $X \cong \mathbb{R}^d \times \mathbb{Q}^n \times (\mathbb{Q}^*)^m \times \prod_{p\in S_1} \mathbb{Q}_p^{l(p)} \times \prod_{p\in S_2} \prod_{i=0}^{k(p)} \mathbb{Z}(p^{r_i(p)})$, where S_1, S_2 are disjoint finite subsets of $\mathbb{P}$, and d, n, m, the $k(p)$'s, the $r_i(p)$'s and the $l(p)$'s are natural numbers.

Proof. The fact that (i) implies (ii) follows from Theorem 5. Assume (ii) and write $X = D \oplus T$, where $D \cong \mathbb{R}^d \times \mathbb{Q}^n \times (\mathbb{Q}^*)^m \times \prod_{p \in S_1} \mathbb{Q}_p^{l(p)}$ and $T \cong \prod_{p \in S_2} \prod_{i=0}^{k(p)} \mathbb{Z}(p^{r_i(p)})$. Since $E(X) \cong E(D) \times E(T)$, it suffices to show that $E(D)$ satisfies DCC on principal right (respectively, left) ideals. We will first consider the case of principal right ideals. Write $D = M \oplus W$, where

$$M \cong \mathbb{Q}^n \quad \text{ and } \quad W \cong \mathbb{R}^d \times (\mathbb{Q}^*)^m \times \prod_{p \in S_1} \mathbb{Q}_p^{l(p)}.$$

Since W is topologically fully invariant in D, it follows that

$$E(D) \cong \begin{pmatrix} E(W) & H(M,W) \\ 0 & E(M) \end{pmatrix}.$$

Let

$$\begin{pmatrix} f_1 & g_1 \\ 0 & h_1 \end{pmatrix} \begin{pmatrix} E(W) & H(M,W) \\ 0 & E(M) \end{pmatrix} \supset \ldots \supset \begin{pmatrix} f_i & g_i \\ 0 & h_i \end{pmatrix} \begin{pmatrix} E(W) & H(M,W) \\ 0 & E(M) \end{pmatrix} \supset \ldots$$

be a descending chain of principal right ideals. For any $i \in \mathbb{N}_0$, we have

$$\begin{pmatrix} f_i & g_i \\ 0 & h_i \end{pmatrix} \begin{pmatrix} E(W) & H(M,W) \\ 0 & E(M) \end{pmatrix} = \begin{pmatrix} f_i E(W) & f_i H(M,W) + g_i E(M) \\ 0 & h_i E(M) \end{pmatrix},$$

so $\big(f_i E(W)\big)_i$, $\big(f_i H(M,W) + g_i E(M)\big)_i$, and respectively $\big(h_i E(M)\big)_i$ are descending chains of submodules in $E(W)_{E(W)}$, $H(M,W)_{E(M)}$, and respectively $E(M)_{E(M)}$. Moreover, the chain $\big(f_i H(M,W)\big)_i$ of submodules of $H(M,W)_{E(M)}$ decreases as well, because so does the chain $\big(f_i E(W)\big)_i$. Now, since $E(W)$ and $E(M)$ are artinian rings by Corollary 5, the chains $\big(f_i E(W)\big)_i$ and $\big(h_i E(M)\big)_i$ are stationary. It remains to show that the chain $\big(f_i H(M,W) + g_i E(M)\big)_i$ stabilises as well. Fix any $i_0 \in \mathbb{N}_0$ such that $f_i E(W) = f_{i_0} E(W)$ for all $i \geq i_0$. Using this representation, we get easily $f_i H(M,W) = f_{i_0} H(M,W)$ for all $i \geq i_0$. Observe also that, without loss of generality, we may consider $g_i E(M) \supset g_{i+1} E(M)$ for all $i \geq i_0$. Indeed, given any such i, we can write $g_{i+1} = f_i \circ u_i + g_i \circ v_i$ for some $u_i, v_i \in E(M)$. It follows easily that, for $g'_{i+1} = g_i \circ v_i$, we have

$$f_{i+1} H(M,W) + g_{i+1} E(M) = f_{i+1} H(M,W) + g'_{i+1} E(M).$$

Thus, replacing g_{i+1} with g'_{i+1}, we get our claim by induction. Now, we clearly have $\operatorname{im}(g_i) \supset \operatorname{im}(g_{i+1})$, so

$$\dim \operatorname{im}(g_{i_0}) \geq \dim \operatorname{im}(g_{i_0+1}) \geq \ldots,$$

and hence there is $j_0 \geq i_0$ such that $\dim \operatorname{im}(g_i) = \dim \operatorname{im}(g_{j_0})$ for all $i \geq j_0$. It follows from Lemma 9 that for every $i \geq j_0$ there is an invertible $w_i \in E(M)$ such that $g_i = g_{j_0} \circ w_i$, whence $g_{j_0} = g_i \circ w_i^{-1}$. Consequently, the chain $\big(f_i H(M,W) + g_i E(M)\big)_i$ stabilises.

Next we consider the case of left principal ideals. Because of the form of D, it is clear that the preceding argument can be applied to $E(D^*)$ to conclude that $E(D^*)$ satisfies DCC on principal right ideals. As $E(D)$ and $E(D^*)$ are topologically anti-isomorphic, it follows that $E(D)$ must satisfy DCC on principal left ideals. $\square$

References

[1] Armacost D. L. *The structure of locally compact abelian groups.* Pure and Applied Mathematics Series, Vol. 68 (Marcel Dekker, ed.), New York, 1981.

[2] Armacost D. L. and Armacost W. L. *On $\mathbb{Q}$-dense and densely divisible LCA groups,* Proc. Amer. Math. Sos., 1976, **36**, 301–305.

[3] Bourbaki N. *Topologie generale, Chapter 1-2, Éléments de mathematique,* Nauka, Moscow, 1968.

[4] Bourbaki N. *Topologie generale, Chapter 3-8, Éléments de mathematique,* Nauka, Moscow, 1969.

[5] Bourbaki N. *Topologie generale, Chapter 9-20, Éléments de mathematique.* Nauka, Moscow, 1975.

[6] Fuchs L. *Infinite abelian groups, Vol. 1.* Academic Press, New York and London, 1970.

[7] Fuchs L. *Infinite abelian groups, Vol. 2.* Academic Press, New York and London, 1973.

[8] Hewitt E., Ross K. *Abstract Harmonic Analysis, Vol. 1.* Academic Press, New York, 1963.

[9] Kertész A. *Lectures on artinian rings,* Académiai Kiadó, Budapest, 1987.

[10] Lam T. Y. *A first course in noncommutative rings,* Graduate texts in mathematics, Vol. 131, Springer-Verlag, Berlin-Heidelberg-New York, 1991.

[11] Popa V. *Units, idempotents and nilpotents of an endomorphism ring. I,* Bul. Acad. Ştiinţe Repub. Moldova, Mat., 1996, No. 3(22), 83–93.

[12] Popa V. *On LCA groups whose rings of continuous endomorphisms have at most two non-trivial closed ideals. I,* Bul. Acad. Ştiinţe Repub. Moldova, Mat., 2011, No. 3(67), (2011), 91–107.

[13] Robertson L. *Connectivity, divisibility and torsion,* Trans. Amer. Moath. Sos., 1967, **128**, 482–505.

[14] Rose H. *Linear algebra: a pure mathematical approach,* Birkhåuser Verlag, Basel-Boston-Berlin, 2002.

[15] Szász F. *Die abelschen Gruppen, deren volle Endomorphismenringe die Minimalbedingung für Hauptrechtsideale erfüllen,* Monatshefte Math., 1961, **65**, 150–153.

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