BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 2(84), 2017, Pages 71–76
ISSN 1024–7696

Unrefinable chains when taking the infimum in the lattice of ring topologies for a nilpotent ring

V. I. Arnautov, G. N. Ermakova

Abstract. A nilpotent ring $\widehat{R}$ and two ring topologies $\widehat{\tau}''$ and $\widehat{\tau}*$ on $\widehat{R}$ are constructed such that $\widehat{\tau}*$ is a coatom (i.e. between the discrete topology τ_d and $\widehat{\tau}*$ there no exists ring topologies) and such that between $\inf\{\widehat{\tau}'', \widehat{\tau}_d\}$ and $\inf\{\widehat{\tau}'', \widehat{\tau}*\}$ there exists an infinite chain of ring topologies in the lattice of all ring topologies of the ring $\widehat{R}$.

Mathematics subject classification: 22A05, 06B30, 22A30.
Keywords and phrases: Nilpotent ring, ring topology, lattice of ring topologies, unrefinable chains, coatoms, infimum of ring topologies.

1 Introduction

As is known, in any modular lattice, the lengths of any finite unrefinable chains with the same ends are equal and the lengths of finite unrefinable chains do not become greater if we take the infimum or the supremum in these lattices.

The lattice of all ring topologies for a nilpotent ring need not be modular [1]. However, as is shown in [2], in the lattice of all ring topologies on a nilpotent ring, the lengths of any finite unrefinable chains which have the same ends are equal.

Given the above, it was natural to expect that the lengths of any finite unrefinable chains do not become greater if for a nilpotent ring we take the infimum or the supremum in the lattice of all ring topologies. However, as shown in this article, it is not the case if we take the infimum.

An example of a nilpotent ring R and such ring topologies $\widehat{\tau}''$ and $\widehat{\tau}*$ that $\widehat{\tau}*$ is a coatom in the lattice of all ring topologies of the ring R (i.e. between the discrete topology τ_d and $\widehat{\tau}*$ there exist no ring topologies) is constructed, and an infinite chain of ring topologies, which are less than $\widehat{\tau}'' = \inf\{\tau'', \tau_d\}$ and more than $\inf\{\widehat{\tau}'', \widehat{\tau}*\}$, exists.

To present the further results we need the following known result (see [3], page 39 and page 51):

Theorem 1. *Let $\mathcal{B}$ be a collection of subsets of a ring R such that the following conditions are satisfied:*

1) $\{0\} = \bigcap\limits_{V \in \mathcal{B}} V$ *;*

2) for any $V_1, V_2 \in \mathcal{B}$ there exists $V_3 \in \mathcal{B}$ such that $V_3 \subseteq V_1 \bigcap V_2$;

3) for any $V_1 \in \mathcal{B}$ there exists $V_2 \in \mathcal{B}$ such that $V_2 + V_2 \subseteq V_1$;

4) for any $V_1 \in \mathcal{B}$ there exists $V_2 \in \mathcal{B}$ such that $-V_2 \subseteq V_1$;

5) for any $V_1 \in \mathcal{B}$ there exists $V_2 \in \mathcal{B}$ such that $V_2 \cdot V_2 \subseteq V_1$;

6) for any $V_1 \in \mathcal{B}$ and any element $r \in R$ there exists $V_2 \in \mathcal{B}$ such that $r \cdot V_2 \subseteq V_1$ and $V_2 \cdot r \subseteq V_1$.

Then there exists a unique ring topology τ on the ring R for which (R, τ) is a Hausdorff space and the collection $\mathcal{B}$ is a basis of neighborhoods of zero [1].

2 Basic results

To state basic results we need the following notations:

Notations 2.

2.1. $\mathbb{N}$ is the set of all natural numbers, $\mathbb{Z}$ is the set of all integers and $\mathbb{R}(+,\cdot)$ is the field of real numbers;

2.2. R is the set of all matrices of the dimension 3×3 over the field $\mathbb{R}$ of real numbers of the form $\begin{pmatrix} 0 & a_{1,2} & a_{1,3} \\ 0 & 0 & a_{2,3} \\ 0 & 0 & 0 \end{pmatrix}$

$$R' = \left\{ \begin{pmatrix} 0 & a_{1,2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Big| a_{1,2} \in \mathbb{R} \right\};$$

$$R'' = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a_{2,3} \\ 0 & 0 & 0 \end{pmatrix} \Big| a_{2,3} \in \mathbb{R} \right\};$$

$R(A) = \left\{ \begin{pmatrix} 0 & 0 & a_{1,3} \\ 0 & 0 & a_{2,3} \\ 0 & 0 & 0 \end{pmatrix} \Big| a_{1,3} \in A,\ a_{2,3} \in \mathbb{R} \right\}$ for any subgroup $A(+)$ of the group $\mathbb{R}(+)$ of the field $\mathbb{R}(+,\cdot)$;

2.3. $R_i = R$, $R'_i = R'$ and $R''_i = R''$ for every natural number i;

2.4. $R_i(A) = R(A)$ for every natural number i and any subgroup $A(+)$ of the group $\mathbb{R}(+)$ of the field $\mathbb{R}(+,\cdot)$;

2.5. $\widehat{R} = \sum\limits_{i=1}^{\infty} R_i$, $\widehat{R}' = \sum\limits_{i=1}^{\infty} R'_i$ and $\widehat{R}'' = \sum\limits_{i=1}^{\infty} R''_i$; $\widehat{R}(A) = \sum\limits_{i=1}^{\infty} R_i(A)$;

2.6. $\widehat{V}_n = \{\widehat{g} \in \widehat{R} \mid pr_i(\widehat{g}) = 0 \text{ if } i \leq n\}$ for any $n \in \mathbb{N}$;

2.7. $\widehat{R}_k(A) = \{\widehat{g} \in \widehat{R} | pr_k(\widehat{g}) \in R_k(A)$ and $pr_j(\widehat{g}) = \{0\}$ if $j \neq k\}$, where $k \in \mathbb{N}$ and $A(+)$ is a subgroup of the group $\mathbb{R}(+)$.

Remark 3. It is easy to see that R with the usual operation of matrix is a ring and $R^3 = 0$ and $(R')^2 = (R'')^2 = (R(A))^2 = 0$.

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[1] As usual, the set V is called a neighborhood of an element a in the topological space (X, τ) if $a \in U \subseteq V$ for some $U \in \tau$.

In addition, since

$$\begin{pmatrix} 0 & a_{1,2} & a_{1,3} \\ 0 & 0 & a_{2,3} \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & b_{1,2} & b_{1,3} \\ 0 & 0 & b_{2,3} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a_{1,2} \cdot b_{2,3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

then it is obvious that $R^3 = 0$ and $(R')^2 = (R'')^2 = (R(A))^2 = 0$.

Proposition 4. *For the ring $\widehat{R}(+,\cdot)$ the following statements are true:*

1. *The collection $\mathcal{B}' = \{\widehat{V}_i \bigcap \widehat{R}' | i \in \mathbb{N}\}$ satisfies the conditions of Theorem 1, and hence, it is a basis of neighborhoods of zero for a ring topology $\widehat{\tau}'$ on the ring $\widehat{R}(+,\cdot)$;*

2. *The collection $\mathcal{B}'' = \{\widehat{V}_i \bigcap \widehat{R}'' | i \in \mathbb{N}\}$ satisfies the conditions of Theorem 1, and hence, is a basis of neighborhoods of zero for a ring topology $\widehat{\tau}''$ on the ring $\widehat{R}(+,\cdot)$;*

3. *If A is a subgroup of the group $\mathbb{R}(+)$ of the field $\mathbb{R}(+,\cdot)$, then the collection $\mathcal{B}(A) = \{\widehat{R}(A) \bigcap \widehat{V}_n | n \in \mathbb{N}\}$ satisfies all the conditions of Theorem 1, and hence, it is a basis of neighborhoods of zero for a ring topology $\widehat{\tau}(A)$ on the ring $\widehat{R}(+,\cdot)$.*

Proof. In addition, taking into consideration the definitions of sets $\widehat{V}_n$, $\widehat{R}'$, $\widehat{R}''$, and $\widehat{R}(A)$ we obtain that any set from the collection $\mathcal{B}' \bigcup \mathcal{B}'' \bigcup \mathcal{B}(A, \mathcal{F})$ is a subring of the ring $\widehat{R}(+,\cdot)$, and hence, any collection $\mathcal{B}'$, $\mathcal{B}''$, and $\mathcal{B}(A, \mathcal{F})$ satisfies conditions 1, 2, 3, 4 and 5 of Theorem 1.

To complete the proof of the theorem it remains to verify that for any of the mentioned collections the condition 6 of Theorem 1 are also satisfied.

Let now $\widehat{g} \in \widehat{R}$, then there exists a natural number n such that $pr_i(\widehat{g}) = 0$ for $i > m$.

If $\widehat{V}_k \bigcap \widehat{R}' \in \mathcal{B}'$ and $m = max\{k, n\}$, then $\widehat{g} \cdot \widehat{a} = 0$ and $\widehat{a} \cdot \widehat{g} = 0$ for any $\widehat{a} \in \widehat{V}_m \bigcap \widehat{R}'$, and hence, $\widehat{g} \cdot (\widehat{V}_m \bigcap \widehat{R}') \subseteq \widehat{V}_k \bigcap \widehat{R}'$ and $(\widehat{V}_m \bigcap \widehat{R}') \cdot \widehat{g} \subseteq \widehat{V}_k \bigcap \widehat{R}'$, i.e. the condition 6 of Theorem 1 holds for the collection $\mathcal{B}'$.

Analogously, if $\widehat{V}_k \bigcap \widehat{R}'' \in \mathcal{B}''$ and $m = max\{k, n\}$, then $\widehat{g} \cdot \widehat{a} = 0 \in \widehat{V}_k \bigcap \widehat{R}''$ for any $\widehat{a} \in \widehat{V}_m \bigcap \widehat{R}''$, and $\widehat{a} \cdot \widehat{g} = 0 \in \widehat{V}_k \bigcap \widehat{R}''$ for any $\widehat{a} \in \widehat{V}_m \bigcap \widehat{R}''$. Then $\widehat{g} \cdot (\widehat{V}_m \bigcap \widehat{R}'') \subseteq \widehat{V}_k \bigcap \widehat{R}''$ and $\widehat{g} \cdot (\widehat{V}_m \bigcap \widehat{R}'') \subseteq \widehat{V}_k \bigcap \widehat{R}''$, i.e. the condition 6 of Theorem 1 holds for the collection $\mathcal{B}''$.

If $\widehat{V}(A) \bigcap \widehat{V}_k \in \mathcal{B}(A)$ and $m = max\{n, k\}$, then $\widehat{V}(A) \bigcap V_m \subseteq \widehat{V}(A) \bigcap \widehat{V}_k$ and $\widehat{a} \cdot \widehat{g} = 0$ for any $\widehat{a} \in \widehat{V}(A) \bigcap \widehat{V}_m$, and $\widehat{V}(A, F) \bigcap \widehat{V}_k \in \mathcal{B}(A)$ and $m = max\{n, k\}$. Then $\widehat{V}(A) \bigcap \widehat{V}_m \subseteq \widehat{V}(A) \bigcap \widehat{V}_k$ and $\widehat{a} \cdot \widehat{g} = 0$ for any $\widehat{a} \in \widehat{V}(A) \bigcap \widehat{V}_m$.

Hence, $\widehat{g} \cdot (\widehat{V}(A) \bigcap \widehat{V}_m) = \{0\} \subseteq \widehat{V}(A) \bigcap \widehat{V}_k$ and $(\widehat{V}(A) \bigcap \widehat{V}_m) \cdot \widehat{g} = \{0\} \subseteq \widehat{V}(A) \bigcap \widehat{V}_k$, i.e. the condition 6 of Theorem 1 holds for the collection $\mathcal{B}(A)$.

By this, the proposition is completely proved. □

Proposition 5. *Let $\widehat{\tau}'$ and $\widehat{\tau}''$ be ring topologies on the ring $\widehat{R}$, defined in Proposition 5, and $n \in \mathbb{N}$. If τ is a non-discrete ring topology on the ring $\widehat{R}$ such that*

$\tau \geq \widehat{\tau}'$, then for any neighborhood W of zero in the topological ring $(\widehat{R}, \inf\{\tau, \widehat{\tau}''\})$ there exists a natural number $k \geq n$ such that $\widehat{R}_k(\mathbb{R}) \subseteq W$. (see 2.7)

Proof. Let W be a neighborhood of zero in the topological ring $(\widehat{R}, \inf\{\tau, \widehat{\tau}''\})$, and let W_1 be a neighborhood of zero in the topological ring $(\widehat{R}, \inf\{\tau, \widehat{\tau}''\})$ such that $W_1 \cdot W_1 + W_1 \subseteq W$. Then W_1 is a neighborhood of zero in each of the topological ring $(\widehat{R}, \tau)$ and $(\widehat{R}, \widehat{\tau}'')$, and hence, there exists a natural number $n_0 \in \mathbb{N}$ such that $n_0 \geq n$ and $\widehat{V}_{n_0} \bigcap \widehat{R}'' \subseteq W_1$. Since $\tau \geq \widehat{\tau}'$, then $\widehat{R}' \bigcap \widehat{V}_{n_0}$ is a neighborhood of zero in the topological ring $(\widehat{R}, \tau)$. Hence $\widehat{R}' \bigcap \widehat{V}_{n_0} \bigcap W_1$ is a neighborhood of zero in the topological ring $(\widehat{R}, \tau)$.

Since τ is a non-discrete topology, then $\widehat{R}' \bigcap \widehat{V}_{n_0} \bigcap W_1 \neq \{0\}$.

If $0 \neq \widehat{g}_0 \in \widehat{R}' \bigcap \widehat{V}_{n_0} \bigcap W_1 \neq \{0\}$, then there exists a natural number $k \geq n_0 \geq n$ such that $pr_k(\widehat{g}_0) \neq 0$.

Since $\widehat{g}_0 \in \widehat{R}'$, then $pr_k(\widehat{g}_0) = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and $a \neq 0$. Now if $\widehat{g}_1 \in \widehat{R}_k(\mathbb{R})$ then $pr_k(\widehat{g}_1) = \begin{pmatrix} 0 & 0 & r \\ 0 & 0 & a_{2,3} \\ 0 & 0 & 0 \end{pmatrix}$ and $pr_i(\widehat{g}_1) = 0$ for $i \neq k$.

If $\widehat{g}_2 \in \widehat{R}''$ and $\widehat{g}_3 \in \widehat{R}''$ are such elements that $pr_k(\widehat{g}_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a^{-1} \cdot r \\ 0 & 0 & 0 \end{pmatrix}$, $pr_k(\widehat{g}_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a_{2,3} \\ 0 & 0 & 0 \end{pmatrix}$, and $pr_i(\widehat{g}_2) = pr_i(\widehat{g}_3) = 0$ for $i \neq k$, then $\widehat{g}_2 \in \widehat{R}''_k \bigcap \widehat{V}_{n_0} \subseteq W_1$. Then $\widehat{g}_0 \cdot \widehat{g}_2 + \widehat{g}_3 \in W_1 \cdot W_1 + W_1 \subseteq W$. As

$$pr_k(\widehat{g}_1) = \begin{pmatrix} 0 & 0 & r \\ 0 & 0 & a_{2,3} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a^{-1} \cdot r \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & r \\ 0 & 0 & a_{2,3} \\ 0 & 0 & 0 \end{pmatrix} =$$

$pr_k(\widehat{g}_0) \cdot pr_k(\widehat{g}_2) + pr_k(\widehat{g}_3)$ and $pr_i(\widehat{g}_1) = 0 = pr_i(\widehat{g}_0) \cdot pr_i(\widehat{g}_2) + pr_i(\widehat{g}_2)$ for $i \neq k$ then $\widehat{g}_1 = \widehat{g}_0 \cdot \widehat{g}_2 + \widehat{g}_3 \in W$. From the arbitrariness of the element $\widehat{g}_1$ it follows then that $\widehat{R}_k(\mathbb{R}) \subseteq W$.

By this, the proposition is completely proved. □

Theorem 6. *Let $\widehat{\tau}'$ and $\widehat{\tau}''$ be ring topologies on the ring $\widehat{R}$, defined in Proposition 5. Then the following statements are true:*

1. *If τ is a ring topology on the ring $\widehat{R}$ such that $\tau \geq \widehat{\tau}'$, then*

$$\sup\{\widehat{\tau}(A), \inf\{\widehat{\tau}'', \tau\}\} > \sup\{\widehat{\tau}(B), \inf\{\widehat{\tau}'', \tau\}\}.$$

for any subgroups $A \subset B$ of the group $\mathbb{R}(+)$.

2. *If $\widehat{\tau}_d$ is the discrete topology on the ring $\widehat{R}$, and $\widehat{\tau}*$ is a coatom in the lattice of all ring topologies on the ring $\widehat{R}$ such that $\widehat{\tau}* \geq \widehat{\tau}'$, then between the topologies $\inf\{\widehat{\tau}_d, \widehat{\tau}''\}$ and $\inf\{\widehat{\tau}*, \widehat{\tau}''\}$, there exists a chain of ring topologies on the ring $\widehat{R}$ which is infinitely decreasing and infinitely increasing.*

Proof. Proof of Statement 7.1. Since $A \subset B$, then (see the notation at the beginning of this article) $\widehat{V}_n(A) \subseteq \widehat{V}_n(B)$ for any a natural number n. Then (see Proposition 5) $\widehat{\tau}(A) \geq \widehat{\tau}(B)$, and hence,

$$\sup\{\widehat{\tau}(A), \inf\{\widehat{\tau}'', \tau\}\} \geq \sup\{\widehat{\tau}(B), \inf\{\widehat{\tau}'', \tau\}\}.$$

We will show that

$$\sup\{\widehat{\tau}(A), \inf\{\widehat{\tau}'', \tau\}\} > \sup\{\widehat{\tau}(B), \inf\{\widehat{\tau}'', \tau\}\}.$$

Assume the contrary, i.e. that

$$\sup\{\widehat{\tau}(A), \inf\{\widehat{\tau}'', \tau\}\} = \sup\{\widehat{\tau}(B), \inf\{\widehat{\tau}'', \tau\}\}.$$

Then $\widehat{R}(A)$ is a neighborhood of zero in the topological ring $(\widehat{R}, \widehat{\tau}(A))$, and hence, $\widehat{R}(A)$ is a neighborhood of zero in the topological ring $(\widehat{R}, \sup\{\widehat{\tau}(B), \inf\{\widehat{\tau}'', \tau\}\})$. Then there exists a neighborhood W of zero in the topological ring $(\widehat{R}, \inf\{\widehat{\tau}'', \tau\})$ and a natural number $n \in \mathbb{N}$ such that $W \bigcap (\widehat{V}(B) \bigcap \widehat{V}_n) \subseteq \widehat{R}(A)$.

By Proposition 5, there exists a natural number $k \geq n$ such that $\widehat{R}_k(\mathbb{R}) \subseteq W$, and hence, $\widehat{R}_k(B) \subseteq \widehat{R}_k(\mathbb{R}) \subseteq W$. As $k \geq n$ then $\widehat{R}_k(B) \subseteq \widehat{V}_n$.

Since $k > m$, then (see 3.7)

$$R_k(B) = pr_k(\widehat{R}_k(B)) \subseteq pr_k(\widehat{R}(A)) = R_k(A),$$

but this contradicts $B \not\subseteq A$.

By this, Statement 1 is proved.

Proof of Statement 2. There exists a chain $\{A_i \mid i \in \mathbb{Z}\}$ of subgroups A_i of the group $\mathbb{R}(+)$ such that $A_i \subseteq A_{i+1}$ for any $i \in \mathbb{Z}$, i.e. this chain of subgroups is infinitely decreasing and infinitely increasing.

For any subgroup A_i let us consider the ring topology $\widehat{\tau}(A_i)$ on the ring $\widehat{R}$. Since $\widehat{\tau}* \geq \widehat{\tau}'$, then by statement 1, of this theorem

$$\sup\{\widehat{\tau}(A_i), \inf\{\widehat{\tau}'', \widehat{\tau}*\}\} > \sup\{\widehat{\tau}(A_{i+1}), \inf\{\widehat{\tau}'', \widehat{\tau}*\}\},$$

and hence, the chain of ring topologies $\sup\{\widehat{\tau}(A_i), \inf\{\widehat{\tau}'', \widehat{\tau}*\}\}$ is infinitely decreasing and infinitely increasing.

To complete the proof of the theorem it remains to verify that

$$\inf\{\widehat{\tau}*, \widehat{\tau}''\} \leq \sup\{\widehat{\tau}(A_i), \inf\{\widehat{\tau}*, \widehat{\tau}''\}\} \leq \inf\{\widehat{\tau}_d, \widehat{\tau}''\}$$

for any subgroup $A_i(+)$ of the group $\mathbb{R}(+)$, where $i \in \mathbb{Z}$.

In fact, from the definition of the sets $R(A)$ and R'' (see 3.2) it follows that $R(\{0\}) = R''$, and hence, $\widehat{\tau}(\{0\}) = \widehat{\tau}'' = \inf\{\widehat{\tau}_d, \widehat{\tau}''\}$. Then

$$\inf\{\widehat{\tau*}, \widehat{\tau''}\} \leq \sup\{\widehat{\tau}(\mathbb{R}), \inf\{\widehat{\tau}*, \widehat{\tau}''\}\} \leq \sup\{\widehat{\tau}(A_i), \inf\{\widehat{\tau}*, \widehat{\tau}''\}\} \leq$$

$$\sup\{\widehat{\tau}(\{0\}), \inf\{\widehat{\tau}_d, \widehat{\tau}''\}\} = \inf\{\widehat{\tau}'', \widehat{\tau}_d, \widehat{\tau}''\} = \inf\{\widehat{\tau}_d, \widehat{\tau}''\}$$

By this, the theorem is proved. □

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V. I. Arnautov
Institute of Mathematics and Computer Science
Academy of Sciences of Moldova
5 Academiei str., MD-2028, Chisinau
Moldova
E-mail: *arnautov@math.md*

Received February 02, 2017

G. N. Ermacova
Transnistrian State University
25 October str., 128, Tiraspol, 278000
Moldova
E-mail: *galla0808@yandex.ru*