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Some examples of topological modules

Mihail Ursul, Adela Tripe

Abstract. In the paper examples of modules which do not admit topologies of different types are constructed.

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1 Introduction

In the monograph [2] (Chapter 5) the problem of topologization of rings and modules is discussed. The aim of this paper is to construct examples of modules which do not admit some types of topologies.

2 Notation and conventions

An elementary p-group A, where p is a prime number is an abelian group with identity $px = 0$. By [6], Theorem 17.2 (Prüfer, Baer) A is a direct sum of cyclic groups of order p. Rings are assumed to be associative with identity and modules unitary. Topological rings are assumed to be Hausdorff, but topological modules are not assumed to be Hausdorff.

Let R be a ring and M an (R,R)-bimodule. The product $R \times M$ is endowed with the multiplication

$$(r,m)(r',m') = (rr', rm' + mr').$$

If R is a topological ring and M a topological (R,R)-bimodule, then $R \times M$ endowed with the product topology becomes a topological ring. It is called the trivial extension of R by M and is denoted by $R \ltimes M$ (see [8]).

3 Preliminaries

The problem of topologization of a module is stated as follows: Let ${}_RM$ be a left R-module and $\mathcal{T}$ be a ring topology on R. Does there exist a group topology $\mathcal{U}$ such that ${}_{(R,\mathcal{T})}(M,\mathcal{U})$ is a topological module? This problem has a satisfactory solution in the case when $\mathcal{T}$ is the discrete topology.

It can be considered another problem: Let ${}_RM$ be a left R-module. Let $\mathcal{U}$ be a group topology on M. Does there exist a ring topology $\mathcal{T}$ such that ${}_{(R,\mathcal{T})}(M,\mathcal{U})$ is a topological module?

Recall that a left R-module M, where R is a topological ring and M a topological group is called a topological module if the mapping

$$R \times M \to M, (r, m) \mapsto rm$$

is continuous.

As a corollary we obtain that if $(R, \mathcal{T}_d)$ is a ring with discrete topology $\mathcal{T}_d$ and $\mathcal{U}$ is a group topoloogy on M, then ${}_{(R,\mathcal{T}_d)}(M,\mathcal{U})$ is a topological module if and only if the mapping $M \to M, m \mapsto rm$ is continuous for every $r \in R$.

It follows from these statements Theorem 5.1.2, [2]: Every infinite module ${}_RM$ admits a nondiscrete Hausdorff R-module topology if R is viewed as a topological ring with the discrete topology.

A short proof: Consider on M the maximal totally bounded group topology. It is well-known that every endomorphism of M is continuous [4], [5].

4 Examples

Example 1. A topological ring and an overring such that the topology of ring cannot be extended to the overring.

Let R be a second countable connected Boolean topological ring with identity. (The existence of such topological rings has been proved in [3]). Let M be a maximal ideal of R. Then M is a dense subspace of R. Indeed, otherwise M will be open and R/M will be a discrete connected space of cardinality 2, a contradiction.

Consider the simple R-module $N = R/M$ and the trivial extension $R \ltimes N$. Then $(R, 0)$ is a subring of index 2 of $R \ltimes N$ and we can identify it with R. We claim that the topology of $(R, 0)$ cannot be extended to a Hausdorff topology of $R \ltimes N$. Indeed, otherwise $(0, N) = (R, 0)(0, 1 + M)$ will be a nonzero connected discrete topological group, a contradiction.

Remark 1. An example of a ring having a subring whose topology cannot be extended has been constructed in [7].

Lemma 1 (folklore)**.** *If A is a dense subgroup of a connected abelian group G, then A is generated by each of its neighborhoods V of zero.*

Example 2. A countable topological ring R and a countable R-module ${}_RM$ such that every module topology is the antidiscrete topology.

Let S be a connected second countable Boolean topological ring with identity and R a dense countable subring containing identity. By Lemma 1 the additive group of R is generated by each of its neighborhoods of zero. Let M be a maximal ideal of R and $N = R/M$. Then the unique module topology on N will be antidiscrete.

Indeed, if $\mathcal{T}$ is a module topology on N and L the intersection of all neighborhoods of zero, then L is a submodule. If $L = 0$, then $(N, \mathcal{T})$ is Hausdorff, hence N is a nonzero discrete group generated by each its neighborhood of zero, a contradiction. Therefore, $L = N$, hence $\mathcal{T}$ is the antidiscrete topology.

Now let $L = \oplus_{i \in \mathbb{N}} N_i$, where $N_i = N (i \in \mathbb{N})$. We claim that every module topology on L is the antidiscrete topology.

Indeed, assume that $\mathcal{T}$ is a module topology and let P be the intersection of all neighborhoods of zero of $(L, \mathcal{T})$. Then $P \supseteq N_i$ for every $i \in \mathbb{N}$. Since P is a submodule, $P = L$, hence $\mathcal{T}$ is the antidiscrete topology.

Another example of this kind has been constructed in [1].

Next example is related to the example 3.4 from [1].

Example 3. Let p be a prime number, A a countable elementary p-group, $\mathcal{T}_d$ be the discrete topology on $\operatorname{End} A$, and $\mathcal{T}_{Bohr}$ the Bohr topology on A , i.e., the finest totally bounded group topology on A (see [5]). We notice that $(A, \mathcal{T}_{Bohr})$ has a fundamental system of neighborhoods of zero consisting of all subgroups of finite index.
Then:

(i) ${}_{\operatorname{End} A}A$ is a simple module.

(ii) Every $(\operatorname{End} A, \mathcal{T}_d)$-module topology on ${}_{\operatorname{End} A}A$ is Hausdorff or discrete.

(iii) Every endomorphism α of $(A, \mathcal{T}_{Bohr})$ is contintuous.

(iv) ${}_{(\operatorname{End} A, \mathcal{T}_d)}(A, \mathcal{T}_{Bohr})$ is a topological module.

(v) $\mathcal{T}_{Bohr} \leq \mathcal{T}$ for each Hausdorff $(\operatorname{End} A, \mathcal{T}_d)$-module topology $\mathcal{T}$ on A.

(vi) Every nondiscrete Hausdorff topological module ${}_{(\operatorname{End} A, \mathcal{T}_d)}(A, \mathcal{T})$ has no nontrivial convergent sequence.

(vii) Every compact subspace of ${}_{(\operatorname{End} A, \mathcal{T}_d)}(A, \mathcal{T}_{Bohr})$ is finite.

(viii) The topology $\mathcal{T}_{Bohr}$ is maximal in the set of all nondiscrete Hausdorff $(\operatorname{End} A, \mathcal{T}_d)$-module topologies on A.

Proofs:

(i) Indeed, let $0 \neq a \in A$ and $b \in A$. There exists $\alpha \in \operatorname{End} A$ such that $\alpha a = b$. Therefore ${}_{\operatorname{End} A}A$ is a simple module.

(ii) Follows from (i).

(iii) This property was proved in [4], p. 39 for arbitrary abelian groups. We recall here the proof: If H is a subgroup of finite index of A, then $\alpha^{-1}(H)$ is a subgroup of finite index of A.

(iv) Follows from (iii).

(v) Indeed, let H be a subgroup of A of finite index. Let $H \oplus H' = A$. Put $\alpha \in \operatorname{End} A$, $\alpha\restriction_H = 0, \alpha\restriction_{H'} = 1_{H'}$. Then α is a continuous endomorphism of $(A, \mathcal{T})$. It follows that $H = \ker \alpha$ is closed in $(A, \mathcal{T})$. Since H has a finite index, it is open in $(A, \mathcal{T})$. We have proved that $\mathcal{T}_{Bohr} \leq \mathcal{T}$.

(vi) Assume the contrary. Let $\{a_n\}_{n\in\mathbb{N}}$ be a convergent sequence and let $\lim\limits_{n\to\infty} a_n = a$. Then $\lim\limits_{n\to\infty}(a_n - a) = 0$. Therefore we can assume without loss of generality that $a = 0$.

Since $\{a_n\}_{n\in\mathbb{N}}$ is a nontrivial sequence there exists $k_1 \in \mathbb{N}$ such that $a_{k_1} \neq 0$. The group A has a structure of a vector $\mathbb{F}_p$-space. Assume that the vectors $a_{k_1}, \cdots, a_{k_{n-1}}$, where $k_1 < \cdots < k_{n-1}$, are linearly independent. Since the subgroup B generated by the elements $a_{k_1}, \cdots, a_{k_{n-1}}$ is finite, there exists $k_n \in \mathbb{N}$ such that $k_{n-1} < k_n$ and $a_{k_n} \notin B$. Since $\lim\limits_{n\to\infty} a_{k_n} = 0$, we can assume without loss of generality that $\{a_n\}_{n\in\mathbb{N}}$ is a linearly independent system. Let $0 \neq b \in A$ and let $\alpha \in \operatorname{End} A, \alpha a_n = b$ for every $n \in \mathbb{N}$. Since α is a continuous endomorphism of $A, 0 = \lim\limits_{n\to\infty} \alpha a_n = b$, a contradiction.

(vii) Assume on the contrary that ${}_{(\operatorname{End} A, \mathcal{T}_d)}(A, \mathcal{T}_{Bohr})$ contains an infinite compact subset K. Since K is countable, it contains nontrivial convergent sequence. A contradiction with (vi).

(viii) Assume the contrary: let $\mathcal{T}$ be a nondiscrete Hausdorff $(\operatorname{End} A, \mathcal{T}_d)$-module topology and $\mathcal{T} \geq \mathcal{T}_{Bohr}, \mathcal{T} \neq \mathcal{T}_{Bohr}$. Let H be a subgroup of A such that $H \in \mathcal{T}, H \notin \mathcal{T}_{Bohr}$. Let $H \oplus H' = A$. Then H and H' are infinite and countable. Let α be an isomorphism of H on H' and $\beta \in \operatorname{End} A, \beta(h \oplus h') = \alpha(h)(h, h' \in H)$. There exists a neighborhood U of zero of ${}_{(\operatorname{End} A, \mathcal{T}_d)}(A, \mathcal{T})$ such that $U \subseteq H$ and $\beta(U) = \alpha(U) \subseteq H$. Thus $\alpha(U) = 0$, a contradiction.

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Mihail Ursul
Department of Mathematics and Computer Science
University of Technology, Lae, Papua New Guinea
E-mail: *mihail.ursul@gmail.com*

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Adela Tripe
Department of Mathematics and Computer Science
University of Oradea, Oradea, Romania
E-mail: *adela.tripe@gmail.com*