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# Radicals and generalizations of derivations

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**Abstract.** By results of Slin'ko and of Anderson, the locally nilpotent and nil radicals of algebras over a field of characteristic 0 are preserved by derivations. This note deals with radical preservation by various generalizations of derivations.

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## 1 Introduction

It was shown by Slin'ko [17] that if $d$ is a derivation on an associative algebra $A$ over a field of characteristic 0, then $d(\mathcal{L}(A)) \subseteq \mathcal{L}(A)$ and $d(\mathcal{N}(A)) \subseteq \mathcal{N}(A)$, where $\mathcal{L}$ and $\mathcal{N}$ are, respectively, the locally nilpotent and nil radical classes. This generalized a similar result proved earlier by Anderson [3] for a restricted class of algebras. The behaviour of the Jacobson radical is quite different; e.g. if $K$ is a field, the Jacobson radical of the ring $K[[X]]$ of formal power series is the principal ideal generated by $X$, and this is not invariant under formal differentiation.

A contrasting result for algebras over a field of prime characteristic was obtained by Krempa [13]: a hereditary radical class $\mathcal{R}$ is preserved by all derivations of all algebras if and only if $\mathcal{R}$ consists of (hereditarily) idempotent algebras.

In this note we shall examine several generalizations of derivations and their effects on certain radicals, mostly $\mathcal{L}$ and $\mathcal{N}$, and also their effects on idempotent ideals. Idempotent ideals are invariant under ordinary derivations, there are plenty of radical classes consisting of idempotent rings (including the class of *all* idempotent rings) and even the prime radical of a ring can be idempotent, so idempotent ideals are pertinent to our investigation.

Confining attention to algebras over fields (as in [3, 13] and [17]) avoids some complications, notably with ideal structure, but leaves some interesting questions unexamined. We shall prove a number of results about (additively) torsion-free rings $A$ by using, or first proving, the results in the special case of an algebra over a field of characteristic 0 and extending them to the general case by means of the divisible hull $D(A)$ of $A$. It is possible to extend some results without using $D(A)$, though not all, but we use a uniform approach.

All our rings and algebras are associative, but similar questions could be pursued for non-associative structures of various kinds. Indeed Krempa's investigations in [13] were more broadly based, and among other things he established a strong connection between derivations and the ADS condition for Lie algebras.

Now for the types of mappings whose effects we shall study.

A *derivation* on a ring is an additive endomorphism $d$ such that $d(ab) = d(a)b + ad(b)$ for all $a, b$.

A *higher derivation* is a sequence $(d_0, d_1, \dots, d_n, \dots)$ of additive endomorphisms such that for each $n$ we have $d_n(ab) = \sum\limits_{i+j=n} d_i(a)d_j(b)$ for all $a, b$ (so that in particular, $d_0$ is a ring endomorphism).

For ring endomorphisms $\alpha, \beta$, an $(\alpha, \beta)$-*derivation* is an additive endomorphism $d$ such that $d(ab) = d(a)\beta(b) + \alpha(a)d(b)$ for all $a, b$. (Thus for a higher derivation, as $d_1(ab) = d_1(a)d_0(b) + d_0(a)d_1(b)$ for all $a, b$, $d_1$ is a $(d_0, d_0)$-derivation).

Finally, a D-*structure* for a ring $A$ with identity 1 and a monoid $G$ with identity $e$ is a family of mappings $\sigma_{x,y} : A \to A$, where $x, y \in G$, satisfying

**Condition (A)**

(0) For each $x \in G$ and $a \in R$, we have $\sigma_{x,y}(a) = 0$ for almost all $y \in G$.

$(i)$ Each $\sigma_{x,y}$ is an additive endomorphism.

$(ii)$ $\sigma_{x,y}(ab) = \sum\limits_{z \in G} \sigma_{x,z}(a)\sigma_{z,y}(b)$.

$(iii)$ $\sigma_{xy,z} = \sum\limits_{uv=z} \sigma_{x,u} \circ \sigma_{y,v}$.

$(iv_1)$ $\sigma_{x,y}(1) = 0$ if $x \neq y$; $(iv_2)$ $\sigma_{x,x}(1) = 1$;

$(iv_3)$ $\sigma_{e,x}(a) = 0$ if $x \neq e$; $(iv_4)$ $\sigma_{e,e}(a) = a$.

For unexplained terms and ideas, see [9] for rings and radicals, [8] for abelian groups.

## 2 Known results

The first result is well known and elementary.

**Proposition 1.** *If $I$ is an idempotent ideal of a ring $R$ and $d$ is a derivation on $R$ then $d(I) \subseteq I$.*

The following two results were proved for algebras over fields of characteristic 0, but they can be extended to all rings that are additively torsion-free, as we shall see in the next section.

**Theorem 1.** *(Anderson [3]) Let $A$ be an algebra over a field $K$ of characteristic 0 with DCC on ideals. For every hereditary radical class $\mathcal{R}$ we have $d(\mathcal{R}(A)) \subseteq \mathcal{R}(A)$ for all $K$-linear derivations $d$ on $A$.*

**Theorem 2.** *(Slin'ko [17]) Let $\mathcal{L}(A), \mathcal{N}(A)$ denote, respectively, the locally nilpotent and nil radicals of an algebra $A$ over a field $K$ of characteristic 0. Then $d(\mathcal{L}(A)) \subseteq \mathcal{L}(A)$ and $d(\mathcal{N}(A)) \subseteq \mathcal{N}(A)$ for all $K$-linear derivations $d$ on $A$.*

The situation with algebras over a field of positive characteristic is rather different.

**Theorem 3.** *(Krempa [13]) Let $\mathcal{V}$ be a variety of algebras over a field of prime characteristic p which is closed under tensoring by commutative-associative algebras. Let $\mathcal{R}$ be a hereditary radical class in $\mathcal{V}$. Then $d(\mathcal{R}(A)) \subseteq \mathcal{R}(A)$ for all derivations d of all algebras $A \in \mathcal{V}$ if and only if $\mathcal{R}$ consists of idempotent algebras.*

The varieties of associative and commutative-associate algebras satisfy the conditions of $\mathcal{V}$ in this theorem.

## 3 Some results involving additive structure

For an (additively written) abelian group $G$, a positive integer $n$ and a prime $p$, let

$$nG = \{nx : x \in G\}; \quad G[n] = \{x \in G : nx = 0\}; \quad G_p = \bigcup_{n \in \mathbb{Z}^+} G[p^n].$$

All of the indicated subsets are subgroups, and if $G$ is the additive group of a ring they are all ideals. Moreover, if $G$ is a torsion group then $G = \bigoplus_p G_p$ (where the sum is taken over all primes $p$) and if $G$ is the additive group of a torsion ring this is also a ring direct sum. In general $\bigoplus_p G_p$ is the *torsion subgroup* of $G$, which we shall call $T(G)$. When $G$ is the additive group of a ring, $T(G)$ is an ideal, which we shall call the *torsion ideal*. In what follows, when referring to additive aspects of rings, we shall not distinguish notationally between a ring and its additive group. Thus, for instance, if $A$ is a ring then $A[n] = \{a \in A : na = 0\} \triangleleft A$.

**Proposition 2.** *Let $A$ be a ring, $I = nA$, $A[n]$, $A_p$ or $T(A)$. If $d$ is a derivation on $A$, then $d(I) \subseteq I$ and we get a derivation $\overline{d}$ on $A/I$ by defining $\overline{d}(a + I) = d(a) + I$ for all $a \in A$.*

*Proof.* Since $d$ is an additive endomorphism we have $d(I) \subseteq I$ so $\overline{d}$ is well-defined. The rest is straightforward. □

**Proposition 3.** *If $A$ is a torsion ring and $d$ is a derivation on $A$, then for each prime $p$, the restriction of $d$ defines a derivation $d_p$ of $A_p$. Conversely, if $e_p$ is a derivation on $A_p$ for each $p$, then we get a derivation $e$ on $A$ by defining $e(\sum_p a_p) = \sum_p e_p(a_p)$, where $a_p$ is the component of $a$ in $A_p$ for each $p$.*

*Proof.* The first part follows from Proposition 2. For the second part, if $a = \sum a_p, b = \sum b_p \in A$, then

$$e(ab) = e\left(\sum a_p b_p\right) = \sum e_p(a_p b_p) = \sum \left(e_p(a_p) b_p + a_p e_p(b_p)\right)$$
$$= \sum e_p(a_p) \sum b_p + \sum a_p \sum e_p(b_p) = e(a)b + ae(b),$$

and clearly $e(a + b) = e(a) + e(b)$. □

**Corollary 1.** *Let $A$ be a torsion ring, $\mathcal{R}$ a radical class of rings. Then $\mathcal{R}(A)$ is preserved by all derivations on $A$ if and only if for every $p$, $\mathcal{R}(A_p)$ is preserved by all derivations on $A_p$.*

*Proof.* First note that $\mathcal{R}(A) = \bigoplus_p \mathcal{R}(A_p)$. If $\mathcal{R}(A)$ is preserved by derivations and $\delta$ is a derivation on $A_p$, then $\delta$ extends to a derivation $d$ on $A$, so $d(\mathcal{R}(A)) \subseteq \mathcal{R}(A)$. Also $d(A_p) \subseteq A_p$. Hence

$$\delta(\mathcal{R}(A_p)) = \delta(A_p \cap \mathcal{R}(A)) = d(A_p \cap \mathcal{R}(A)) \subseteq A_p \cap \mathcal{R}(A) = \mathcal{R}(A_p).$$

If the action of $\mathcal{R}$ is preserved by derivations in all the $A_p$ and $e$ is any derivation on $A$, then

$$e(\mathcal{R}(A)) = e(\bigoplus_p \mathcal{R}(A_p)) = \bigoplus_p e_p(\mathcal{R}(A_p)) \subseteq \bigoplus_p \mathcal{R}(A_p) = \mathcal{R}(A).$$

□

Thus the radical-preservation problem for torsion rings reduces to that for $p$-rings. A $p$-ring $R$ satisfying the stronger condition $pR = 0$ is an algebra over the field $\mathbb{Z}_p$ and all its ring ideals are $\mathbb{Z}_p$-algebra ideals. It is not known whether the preservation property for $\mathbb{Z}_p$-algebras (for some or all radicals) has much influence on that for $p$-rings generally. We shall prove one theorem related to this question.

**Proposition 4.** *For every $p$-ring $A$ we have $pA \subseteq \mathcal{L}(A) \subseteq \mathcal{N}(A)$, whence $\mathcal{L}(A/pA) = \mathcal{L}(A)/pA$ and $\mathcal{N}(A/pA) = \mathcal{N}(A)/pA$*

*Proof.* We only have to show that $pA$ is locally nilpotent. For this it suffices to prove that if $S$ is a finite sub*set* of $pA$ then there is a positive integer $m$ such that all products of elements of $S$ with $m$ or more factors are zero. (This is straightforward but tedious to prove by brute force; it is also contained in Theorem 4.1.5, p.186 of [9].) If $a, b \in A$, then $(pa)b = \underbrace{(a + a + \cdots + a)}_{p \text{ terms}} b = \underbrace{ab + ab + \cdots + ab}_{p \text{ terms}} = p(ab)$ and similarly $a(pb) = p(ab)$. Hence $pa \cdot pb = p(a \cdot pb) = p(p(ab)) = p^2ab$ and so on. If $a_1, a_2, \ldots, a_n \in A$, then for $y_1, y_2, \ldots, y_m \in \{a_1, a_2, \ldots, a_n\}$ we have $py_1 \cdot py_2 \cdot \ldots \cdot py_m = p^m y_1 y_2 \ldots y_m = 0$ if $p^m \geq \max\{o(a_1), o(a_2), \ldots, o(a_n)\}$, where $o(a_i)$ is the (additive) order of $a_i$ for each $i$. □

In fact the same proof shows that if $\mathcal{R}$ is any radical class with $\mathcal{L} \subseteq \mathcal{R}$, then $\mathcal{R}(A/pA) = \mathcal{R}(A)/pA$. This gives us

**Theorem 4.** *Let $d$ be a derivation on a $p$-ring $A$, $\overline{d}$ the induced derivation on $A/pA$. Let $\mathcal{R}$ be a radical class containing $\mathcal{L}$. If $d(\mathcal{R}(A) \subseteq \mathcal{R}(A)$, then $\overline{d}(\mathcal{R}(A/pA)) \subseteq \mathcal{R}(A/pa)$.*

Now let $A$ be a torsion-free ring. Its *divisible hull* $D(A)$ is a minimal divisible group containing $A$. For each $a \in A$ and each non-zero integer $n$ there is an element $\alpha \in D(A)$ such that $n\alpha = a$, and as $D(A)$ is torsion-free, $\alpha$ is unique. It is therefore natural to give $\alpha$ the name $\frac{a}{n}$. Then $\frac{a}{n} = \frac{b}{m}$ if and only if $ma = nb$. In $D(A)$ we similarly define elements $\frac{x}{k}$ for $x \in D(A)$ and non-zero $k \in \mathbb{Z}$. We get a ring on $D(A)$

by defining $\frac{a}{n}\frac{c}{k} = \frac{ac}{nk}$ and this ring has a subring $\left\{\frac{a}{1} : a \in A\right\}$ which we identify with $A$. We make $D(A)$ into an algebra over the field $\mathbb{Q}$ by defining $\frac{m}{n}x = \frac{mx}{n}$ for $m, n, k \in \mathbb{Z}$, $x \in D(A)$. In particular, $\frac{m}{n}\frac{a}{k} = \frac{ma}{nk}$ for $a \in A$. For all this cf. Theorem 119.1, p.284 of [8], Vol. II.

**Proposition 5.** *Let $A$ be a torsion-free ring. Then $\mathcal{L}(D(A)) = D(\mathcal{L}(A))$ and $\mathcal{N}(D(A)) = D(\mathcal{N}(A))$.*

*Proof.* We shall prove the result for $\mathcal{L}$. The proof for $\mathcal{N}$ is similar but simpler.

Let $I = \mathcal{L}(A)$. For $n \in \mathbb{Z}^+$ let $I_n = \{a \in A : na \in I\}$. Then $I_n \triangleleft A$. If $a_1, a_2, \ldots, a_k \in I_n$ then $na_1, na_2, \ldots, na_k$ are in the locally nilpotent ideal $I$, so there is a positive integer $\ell$ such that every $\ell$-fold product of $na_i$s is zero. Such a product has the form $n^\ell c_1c_2 \ldots c_\ell$, so since $A$ is torsion-free, $c_1c_2 \ldots c_\ell = 0$. But the $c_j$ are arbitrary elements of $\{a_1, a_2, \ldots, a_k\}$, so by Theorem 4.1.5 of [9] referred to above, $I_n$ is locally nilpotent, whence $I_n \subseteq I$ and thus $I_n = I$. This being so for every $n$, $I$, as an additive subgroup, is *pure* in $A$. If $a \in A$, $c \in I$, $m, n$ are non-zero integers and $\frac{a}{n} = \frac{c}{m}$, then $ma = nc \in I$, so $a \in I$. Thus without ambiguity we can identify $D(I)$ with the obvious subring of $D(A)$. It is easily seen that $D(I) \triangleleft D(A)$.

If $\frac{c_1}{k_1}, \frac{c_2}{k_2}, \ldots, \frac{c_t}{k_t} \in D(I)$ ($c_j \in I, k_j \in \mathbb{Z}$), then long enough products of $c_j$s are zero. But such products are multiples, by non-zero integers, of products of $\frac{c_j}{k_j}$s of the same length. It follows that $D(I)$ is locally nilpotent and thus $D(I) \subseteq \mathcal{L}(D(A))$.

Let $J/D(I)$ be a locally nilpotent ideal of $D(A)/D(I)$. Then $J$ is a locally nilpotent ideal of $D(A)$, so $J \cap A$ is a locally nilpotent ideal of $A$ and hence $J \cap A \subseteq I$. If $\frac{g}{s} \in J$, where $g \in A$, $s \in \mathbb{Z}$, then $g = s\frac{g}{s} \in J \cap A \subseteq I$, so $\frac{g}{s} \in D(I)$ and so $J/D(I) = 0$. Thus $\mathcal{L}(D(A))/D(I) = 0$. It follows that $\mathcal{L}(D(A)) \subseteq D(I)$, so the two ideals are equal, i.e. $\mathcal{L}(D(A)) = D(\mathcal{L}(A))$. □

It follows that $\mathcal{L}(A) = A \cap \mathcal{L}(D(A))$ and $\mathcal{N}(A) = A \cap \mathcal{N}(D(A))$.

Note that the corresponding result for the Jacobson radical is false. For instance, if $A = \left\{\frac{2n}{2m+1} : n, m \in \mathbb{Z}\right\}$, then $\mathbb{Q}$ is a divisible hull for $A$, $A$ is its own Jacobson radical and $\mathbb{Q}$ has zero radical.

**Lemma 1.** *If $G$ is a torsion-free abelian group, each of its endomorphisms has a unique extension to an endomorphism of $D(G)$ and this is a $\mathbb{Q}$-linear transformation of $D(A)$ as a $\mathbb{Q}$-vector space.*

*Proof.* For an endomorphism $f$ of $G$ define $\hat{f} : D(G) \to D(G)$ by setting $\hat{f}\left(\frac{a}{n}\right) = \frac{f(a)}{n}$ for all $a \in G$, $n \in \mathbb{Z}\backslash\{0\}$. Then $\hat{f}$ is well-defined, as if $\frac{a}{n} = \frac{b}{m}$, then $mf(a) = f(ma) = f(nb) = nf(b)$, i.e. $\frac{f(a)}{n} = \frac{f(b)}{m}$. Then for all $a$, $c \in G$, $n$, $k \in \mathbb{Z}\backslash\{0\}$

we have $\hat{f}\left(\frac{a}{n}+\frac{c}{k}\right) = \hat{f}\left(\frac{ka+nc}{nk}\right) = \frac{f(ka+nc)}{nk} = \frac{kf(a)+nf(c)}{nk} = \frac{kf(a)}{nk} + \frac{nf(c)}{nk} = \frac{f(a)}{n}+\frac{f(c)}{k} = \hat{f}\left(\frac{a}{n}\right)+\hat{f}\left(\frac{c}{k}\right)$. Also $\hat{f}\left(\frac{m}{n}\frac{a}{k}\right) = \hat{f}\left(\frac{ma}{nk}\right) = \frac{f(ma)}{nk} = \frac{mf(a)}{nk} = \frac{m}{n}\frac{f(a)}{k} = \frac{m}{n}\hat{f}\left(\frac{a}{k}\right)$ for $a \in A, m, n, k \in \mathbb{Z}$. If $\tilde{f}$ is any extension of $f$, then $G \subseteq Ker(\hat{f}-\tilde{f})$, so $Im(\hat{f}-\tilde{f})$ is a torsion group and hence zero. □

**Corollary 2.** *Let $A$ be a torsion-free ring.*
*(i) Every derivation $d$ on $A$ has a unique extension to $D(A)$ and this is $\mathbb{Q}$-linear.*
*(ii) Every higher derivation on $A$ has a unique extension to $D(A)$ and all its maps are $\mathbb{Q}$-linear.*
*(ii) If $\alpha$ and $\beta$ are endomorphisms of $A$, then every $(\alpha,\beta)$-derivation on $A$ has a unique extension to an $(\hat{\alpha},\hat{\beta})$-derivation on $D(A)$ and this is $\mathbb{Q}$-linear.*

*Proof.* All the maps involved in (i), (ii) and (iii) are additive endomorphisms of $A$, and so have unique extensions to additive endomorphisms of $D(A)$. We just need to show that these endomorphisms have all other properties required of them.

(ii) Let $(d_0, d_1, \ldots, d_n \ldots)$ be a higher derivation on $A$. For each $n$ let $\hat{d_n}$ be the extension of $d_n$ to $D(A)$ as in the lemma. For each $a, b \in A$ and non-zero $k, \ell \in \mathbb{Z}$, we have $\hat{d_n}\left(\frac{a}{k}\frac{b}{\ell}\right) = \hat{d_n}\left(\frac{ab}{k\ell}\right) = \frac{d_n(ab)}{k\ell} = \frac{\sum_{i+j=n} d_i(a)d_j(b)}{k\ell} = \sum_{i+j=n}\frac{d_i(a)}{k}\frac{d_j(b)}{\ell} = \sum_{i+j=n}\hat{d_i}\left(\frac{a}{k}\right)\hat{d_j}\left(\frac{b}{\ell}\right)$.

Similar arguments show that extensions of ring endomorphisms and extensions of derivations are derivations.

(iii) Let $d$ be an $(\alpha,\beta)$-derivation on $A$. Then for all $a, b \in A$ and non-zero $k, \ell \in \mathbb{Z}$, we have

$$\hat{d}\left(\frac{a}{k}\right)\hat{\beta}\left(\frac{b}{\ell}\right)+\hat{\alpha}\left(\frac{a}{k}\right)\hat{d}\left(\frac{b}{\ell}\right) = \frac{d(a)}{k}\frac{\beta(b)}{\ell}+\frac{\alpha(a)}{k}\frac{d(b)}{\ell} = \frac{d(a)\beta(b)+\alpha(a)d(b)}{k\ell}$$
$$= \frac{d(ab)}{k\ell} = \hat{d}\left(\frac{a}{k}\frac{b}{\ell}\right).$$

□

Note that not every derivation on $D(A)$ is an extension of one on $A$: consider inner derivations, for example.

Now if $A$ is a torsion-free ring, $d$ a derivation on $A$, then by Corollary 2 $d$ extends to a $\mathbb{Q}$-linear derivation $\hat{d}$ on $D(A)$, so

$$d(\mathcal{L}(A)) = d(A \cap \mathcal{L}(D(A))) = \hat{d}(A \cap \mathcal{L}(D(A))) \subseteq \hat{d}(\mathcal{L}(D(A))) \subseteq \mathcal{L}(D(A))$$

and $d(\mathcal{L}(A)) \subseteq A$, so

$$d(\mathcal{L}(A)) \subseteq A \cap \mathcal{L}(D(A)) = \mathcal{L}(A).$$

We can argue similarly for $\mathcal{N}(A)$. Thus we have

**Theorem 5.** *If $d$ is a derivation on a torsion-free ring $A$ then $d(\mathcal{L}(A)) \subseteq \mathcal{L}(A)$ and $d(\mathcal{N}(A)) \subseteq \mathcal{N}(A)$.*

## 4 Preservation by higher derivations

**Proposition 6.** *Let $(d_0, d_1, \dots, d_n, \dots)$ be a higher derivation on a ring $A$, $I$ an idempotent ideal of $A$ with $d_0(I) \subseteq I$. Then $d_n(I) \subseteq I$ for all $n$.*

*Proof.* If $d_n(I) \subseteq I$ then for all $a, b \in I$ we have

$$d_{n+1}(ab) = d_0(a)d_{n+1}(b) + d_1(a)d_n(b) + d_2(a)d_{n-1}(b) + \cdots + d_{n-1}(a)d_2(b) + d_n(a)d_1(b) + d_{n+1}(a)d_0(b) \in I$$

if $d_0(I), d_1(I), \dots, d_n(I) \subset I$. □

**Theorem 6.** *Let $A$ be a torsion-free ring, $(d_0, d_1, \dots, d_n, \dots)$ a higher derivation on $A$. If $d_0$ is an automorphism, then $d_n(\mathcal{L}(A)) \subseteq \mathcal{L}(A)$ and $d_n(\mathcal{N}(A)) \subseteq \mathcal{N}(A)$ for all $n$.*

*Proof.* We first treat the case where $A$ is an algebra over a field of characteristic 0. Note that $\mathcal{L}(A)$ and $\mathcal{N}(A)$ (where $A$ is treated as a ring) are algebra ideals (as happens with all radicals) and so coincide with these radicals of $A$ treated as an algebra (see [7]).

It has been proved by many authors e.g. Heerema [11], Miller [15], Abu-Saymeh [1],[2], Mirzavaziri [16], Hazewinkel [10]) that in the circumstances of the theorem, if $d_0 = id$ then each $d_n$ $(n \geq 1)$ is a linear combination of compositions of derivations, whence the result follows from Theorem 2. In general we have

$$\begin{aligned} d_0^{-1} \circ d_n(ab) =& d_0^{-1}(d_0(a)d_n(b) + d_1(a)d_{n-1}(b) + \cdots + d_{n-1}(a)d_1(b) + \\ d_n(a)d_0(b)) =& d_0^{-1} \circ d_0(a)d_0^{-1} \circ d_n(b) + d_0^{-1} \circ d_1(a)d_0^{-1} \circ d_{n-1}(b) + \cdots + \\ & d_0^{-1} \circ d_{n-1}(a)d_0^{-1} \circ d_1(b) + d_0^{-1} \circ d_n(a)d_0^{-1} \circ d_0(b) \end{aligned}$$

for all $n \geq 1$, so $(d_0^{-1} \circ d_0, d_0^{-1} \circ d_1, \dots, d_0^{-1} \circ d_n, \dots)$ is a higher derivation with the identity as its zeroth term, whence $d_0^{-1} \circ d_n(\mathcal{L}(A)) \subseteq \mathcal{L}(A)$ for all $n$. But $\mathcal{L}(A)$ is invariant under automorphisms, so

$$d_n(\mathcal{L}(A)) = d_0 \circ d_0^{-1} \circ d_n(\mathcal{L}(A)) \subseteq d_0(\mathcal{L}(A)) = \mathcal{L}(A).$$

The argument for $\mathcal{N}(A)$ is the same.

Now turning to a general torsion-free ring $A$, by Corollary 2 (ii) we can extend our higher derivation uniquely to a higher derivation $(\hat{d_0}, \hat{d_1}, \dots, \hat{d_n}, \dots)$ of $D(A)$, which is an algebra over the field $\mathbb{Q}$ of rational numbers. It is easy to see that if $d_0$ is an automorphism of $A$, then $\hat{d_0}$ is an automorphism of $D(A)$. Hence by Proposition 5 and the first part of the proof we have

$$\hat{d_n}(\mathcal{L}(D(A)) = \hat{d_n}(D(\mathcal{L}(A))) \subseteq D(\mathcal{L}(A)) \quad \text{for every } n.$$

Thus if $a \in \mathcal{L}(A)$, then

$$d_n(a) = \hat{d_n}\left(\frac{a}{1}\right) \in D(\mathcal{L}(A)) \cap A = \mathcal{L}(A)$$

for each $n$.

Again, the argument for $\mathcal{N}$ is the same. □

A natural question is whether for a higher derivation $(d_0, d_1, \ldots, d_n, \ldots)$, in particular on a torsion-free ring, if $d_0$ preserves one of our radicals the latter must be preserved by every $d_n$. We have an example of similar behaviour in a ring with prime characteristic $p$; the radical involved is not $\mathcal{L}$ or $\mathcal{N}$, but it is a hereditary supernilpotent radical.

**Example 1.** (Cf. Krempa [12]) Let $\mathcal{U}$ be the upper radical class defined by the field $K_p$ with $p$ elements. We get a higher derivation $(d_0, d_1, \ldots, d_n, \ldots)$ on $K_p[X]$ by defining $d_i(a_0 + a_1X + \cdots + a_kX^k) = a_iX^i$ for all $i$. Now $\mathcal{U}$ is special, so if $\alpha \in \mathcal{U}(K_p[X])$ then $\alpha$ is taken to 0 by each homomorphism from $K_p[X]$ to $K_p$. In particular $d_0(\alpha) = 0$ (as the function which assigns the zeroth coefficient is a homomorphism). Thus $d_0(\mathcal{U}(K_p[X])) \subseteq \mathcal{U}(K_p[X])$. But $X - X^p \in \mathcal{U}(K_p[X])$ and $d_1(X - X^p) = X$. If $X$ were in $\mathcal{U}(K_p[X])$ then the principal ideal $(X)$ would be in $\mathcal{U}$. But $K_p$ is a homomorphic image of $(X)$ *via* $X \mapsto 1$. Thus $X \notin \mathcal{U}(K_p[X])$ so $d_1(\mathcal{U}(K_p[X])) \nsubseteq \mathcal{U}(K_p[X])$.

For *commutative* rings we have a preservation result which does not depend on additive properties.

**Theorem 7.** *Let $A$ be a commutative ring, $(d_0, d_1, \ldots, d_n, \ldots)$ a higher derivation on $A$. Then $d_n(\mathcal{L}(A)) \subseteq \mathcal{L}(A)$ and $d_n(\mathcal{N}(A)) \subseteq \mathcal{N}(A)$ for all $n$.*

*Proof.* Since $A$ is commutative, $\mathcal{L}(A) = \mathcal{N}(A) =$ the set of nilpotent elements of $A$. The correspondence $a \mapsto \sum_{n=0}^{\infty} d_n(a)X^n$ defines a homomorphism $f : A \to A[[X]]$ (the formal power series ring). If $a$ is nilpotent then so is $f(a)$ and then, by commutativity, so are its coefficients. (This is presumably well known. Here is an outline of a proof. If $(\sum_{n=0}^{\infty} a_nX^n)^m = 0$, then $a_0^m = 0$. By commutativity, $\sum_{n=1}^{\infty} a_nX^n = \sum_{n=0}^{\infty} a_nX^n - a$ is also nilpotent, whence $a_1$ is nilpotent, and so on.) Thus each $d_n(a)$ is nilpotent and therefore in $\mathcal{L}(A)$. □

Presumably this result does not hold in the absence of any restriction on $A$, though we do not have an example to show this. The following example shows that higher derivations do not necessarily take nilpotent elements to nilpotent elements.

**Example 2.** *We use an example of [4]. Let $R$ be a ring with identity, $A = M_2(R)[X]$. We get a higher derivation on $A[X]$ by defining $d_n(c_0 + c_1X + \ldots) = c_nX^n$ for all $n$. Then $(e_{12} + (e_{11} - e_{22})X - e_{21}X^2)^2 = 0$, but $d_1(e_{12} + (e_{11} - e_{22})X - e_{21}X^2) = e_{11} - e_{22}$, which is a unit.*

Not much seems to be known about representing the terms of a general higher derivation by combinations of some kind of derivations. Loy [14] remarks that if $(d_0, d_1, \ldots, d_n, \ldots)$ is a higher derivation, $d_0$ is idempotent and $d_0 \circ d_n = d_n \circ d_0$ for all $n$, then the $d_n$ are expressible as linear combinations of compositions of $(d_0, d_0)-$derivations $\delta$ with $d_0 \circ \delta = \delta \circ d_0$.

Note that there are related results expressing the maps of certain D-structures in terms of endomorphisms and derivations of various kinds in Section 6 of [5] and Section 3 of [6].

## 5 Preservation by $(\alpha, \beta)-$derivations

It might be expected that ideals preserved by $\alpha$ and $\beta$ and by derivations might be preserved by $(\alpha, \beta)-$ derivations. The situation is more complicated, however. The case of idempotent ideal is easy.

**Proposition 7.** *If $I$ is an idempotent ideal of a ring $A$, $d$ an $(\alpha, \beta)-$derivation on $A$, where $\alpha(I) \subseteq I$ and $\beta(I) \subseteq I$, then $d(I) \subseteq I$.*

*Proof.* For $a, b \in I$ we have $d(ab) = d(a)\beta(b) + \alpha(a)d(b) \in I$ as $\beta(b), \alpha(a) \in I$. □

**Theorem 8.** *If $\alpha$ is an automorphism of a torsion-free ring $A$ then $d(\mathcal{L}(A)) \subseteq \mathcal{L}(A)$ and $d(\mathcal{N}(A)) \subseteq \mathcal{N}(A)$ for all $(\alpha, \alpha)-$derivations $d$ of $A$.*

*Proof.* The proof uses Corollary 2 and is like part of that of Theorem 6: $\alpha^{-1} \circ d$ is an ordinary derivation, so $\alpha^{-1} \circ d(\mathcal{L}(A)) \subseteq \mathcal{L}(A)$. Hence $d(\mathcal{L}(A)) \subseteq \alpha(\mathcal{L}(A)) \subseteq \mathcal{L}(A)$. The same argument gives the result for the nil radical. □

We do not know if there is an analogous theorem for $(\alpha, \beta)$-derivations when $\alpha$ and $\beta$ are *distinct* automorphisms. We do however have counterexamples when $\alpha$ and $\beta$ are non-automorphisms, distinct or not.

**Example 3.** *Let $K$ be a field (any characteristic),*

$$A = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} : a, b \in K \right\}$$

*and define $f, \delta : A \to A$ by setting $f\left(\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$, $\delta\left(\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}\right) = \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix}$ for all $a, b \in K$. Then $f$ is an endomorphism and $\delta$ is an $(f, f)$-derivation. We have $\mathcal{L}(A) = \mathcal{N}(A) = \begin{bmatrix} 0 & K \\ 0 & 0 \end{bmatrix}$ and the radicals are preserved by $f$ but not by $\delta$.*

**Example 4.** *For a field $K$ we consider the ring $\begin{bmatrix} K & K \\ 0 & K \end{bmatrix}$ of upper triangular $2 \times 2$ matrices. Let $\alpha\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$, $\beta\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$*

*and* $d\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} 0 & b \\ 0 & b \end{bmatrix}$ *for all* $a, b, c \in K$. *Clearly* $\alpha$ *and* $\beta$ *are endomorphisms. For all* $a, b, c, d, e$ *and* $f \in K$ *we have* $d\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right)\beta\left(\begin{bmatrix} d & e \\ 0 & f \end{bmatrix}\right) + \alpha\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right)d\left(\begin{bmatrix} d & e \\ 0 & f \end{bmatrix}\right) = \begin{bmatrix} 0 & b \\ 0 & b \end{bmatrix}\begin{bmatrix} f & 0 \\ 0 & f \end{bmatrix} + \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}\begin{bmatrix} 0 & e \\ 0 & e \end{bmatrix} = \begin{bmatrix} 0 & bf \\ 0 & bf \end{bmatrix} + \begin{bmatrix} 0 & ae \\ 0 & ae \end{bmatrix} = \begin{bmatrix} 0 & bf+ae \\ 0 & bf+ae \end{bmatrix} = d\left(\begin{bmatrix} ad & ae+bf \\ 0 & cf \end{bmatrix}\right) = d\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\begin{bmatrix} d & e \\ 0 & f \end{bmatrix}\right)$, *so* $d$ *is an* $(\alpha, \beta)-$*derivation. Now* $\mathcal{L}\left(\begin{bmatrix} K & K \\ 0 & K \end{bmatrix}\right) = \mathcal{N}\left(\begin{bmatrix} K & K \\ 0 & K \end{bmatrix}\right) = \begin{bmatrix} 0 & K \\ 0 & 0 \end{bmatrix}$ *and* $\alpha\left(\begin{bmatrix} 0 & K \\ 0 & 0 \end{bmatrix}\right) = \beta\left(\begin{bmatrix} 0 & K \\ 0 & 0 \end{bmatrix}\right) = 0$ *so both radicals are preserved by* $\alpha$ *and* $\beta$. *However, if* $b \neq 0$ *then* $d\left(\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & b \\ 0 & b \end{bmatrix} \notin \begin{bmatrix} 0 & K \\ 0 & 0 \end{bmatrix}$, *so the radicals are not preserved by* $d$.

## 6 Preservation by D-structures

Preservation by all mappings of an arbitrary D-structure is a very demanding condition. We shall see that even for algebras over a field of characteristic 0, the locally nilpotent and nil radicals need not be preserved. We begin the section however with a positive result.

**Theorem 9.** *Let* $\sigma$ *be a D-structure defined by a ring* $A$ *and a free monoid* $G = \{e, x, x^2, \dots, x^n, \dots\}$ *and write* $\sigma_{nm}$ *for* $\sigma_{x^n,x^m}$. *Suppose further that* $\sigma_{nm} = 0$ *for* $n < m$. *If* $I$ *is an idempotent ideal of* $A$ *and* $\sigma_{11}(I) \subseteq I$ *then* $\sigma_{ij}(I) \subseteq I$ *for all* $i, j$.

*Proof.* The conditions imposed imply that $\sigma_{11}$ is an endomorphism and $\sigma_{nn} = \sigma_{11}^n$ for all $n$ (see [5], Proposition 3.1 and (6.9)). Clearly we need only consider $\sigma_{ij}$ for $i \geq j$, and prove that $\sigma_{ij}(ab) \in I$ for all $a, b \in I$. It is given that $\sigma_{11}(I) \subseteq I$. Now for all $a, b \in I$ we have $\sigma_{10}(ab) = \sigma_{11}(a)\sigma_{10}(b) + \sigma_{10}(a)\sigma_{00}(b) \in I$, since $\sigma_{11}(I) \subseteq I$. Thus $\sigma_{1j}(I) \subseteq I$ for all $j \leq 1$. Now we proceed by induction.

Suppose $\sigma_{ij}(I) \subseteq I$ for all $j \leq i$ when $i < n$. Then $\sigma_{nn}(I) \subseteq I$ as $\sigma_{nn} = \sigma_{11}^n$. If $j < n$ then

$$\sigma_{nj}(ab) = \sum_{n \geq k \geq j} \sigma_{nk}(a)\sigma_{kj}(b) = \sigma_{nn}(a)\sigma_{nj}(b) + \sigma_{nj}(a)\sigma_{jj}(b) + \sum_{n > k > j} \sigma_{nk}(a)\sigma_{kj}(b).$$

But $\sigma_{nn}(a)$ and $\sigma_{jj}(b) \in I$ and for $k < n$ we have $\sigma_{kj}(b) \in I$ by the inductive hypothesis. Hence $\sigma_{nj}(I) \subseteq I$ for all $j \leq n$. We have proved that for every $i$ and for all $j \leq i$, we have $\sigma_{ij}(I) \subseteq I$, and this is what we need. ∎

It is not known how the mappings of a D-structure treat idempotent ideals in general.

Even in the presence of DCC for ideals, the mappings of a D-structure need not preserve the locally nilpotent or the nil radical of an algebra over a field of characteristic 0.

**Example 5.** *The ring* $\begin{bmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{bmatrix}$ *is a* $\mathbb{Q}-$*algebra and has DCC on ideals. Also* $\mathcal{L}\left(\begin{bmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{bmatrix}\right) = \mathcal{N}\left(\begin{bmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{bmatrix}\right) = \begin{bmatrix} 0 & \mathbb{Q} \\ 0 & 0 \end{bmatrix}$. *For the cyclic group* $G = \{e, x\}$ *of order* 2 *we get a D-structure as follows:* $\sigma_{x,x}\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$, $\sigma_{x,e}\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} 0 & c-a \\ 0 & b \end{bmatrix}$ *for all* $a, b, c \in \mathbb{Q}$, $\sigma_{e,e} = id, \sigma_{x,e} = 0$. *Then* $\sigma_{x,x}$ *preserves the radicals, but* $\sigma_{x,e}$ *does not.*

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