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Properties of accessible subrings of pseudonormed rings when taking quotient rings

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Abstract. Let (R, ξ) and $(\bar{R}, \bar{\xi})$ be pseudonormed rings, $\varphi : R \to \bar{R}$ be a ring isomorphism. We prove that $\varphi : (R, \xi) \to (\bar{R}, \bar{\xi})$ is a superposition of a finite number of semi-isometric isomorphisms if and only if it is a narrowing on an accessible subring of some isometric homomorphism.

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We will say that a pseudonormed ring is a ring R which may be non-associative and has a pseudonorm (see [1], Definition 2.3.1).

The following isomorphism theorem is widely applied in the general algebra and, in particular, in the ring theory:

Theorem 1. *If A is a subring of a ring R and I is an ideal of the ring R then the quotient rings $A/(A \bigcap I)$ and $(A + I)/I$ are isomorphic rings. In particular, if $A \bigcap I = 0$, then the ring A is isomorphic to the ring $(A+I)/I$, i.e. the rings A and $(A + I)/I$ possess identical algebraic properties.*

Since it is necessary to take into account properties of pseudonorms when studying the pseudonormed rings then one needs to consider isomorphisms which keep pseudonorms. Such isomorphisms are called isometric isomorphisms.

The isomorphism theorem does not always take place for pseudonormed rings. The following theorem was proved in the work [2]:

Theorem 2. *Let (R, ξ) and $(\bar{R}, \bar{\xi})$ be pseudonormed rings, $\varphi : R \to \bar{R}$ be a ring isomorphism. The inequality $\bar{\xi}(\varphi(r)) \leq \xi(r)$ is satisfied for all $r \in R$ if and only if:*

– there exists a pseudonormed ring $(\widehat{R}, \widehat{\xi})$ such that (R, ξ) is a subring of the pseudonormed ring $(\widehat{R}, \widehat{\xi})$;

– the isomorphism φ can be extended up to an isometric homomorphism $\widehat{\varphi}$: $\left(\widehat{R}, \widehat{\xi}\right) \to \left(\bar{R}, \bar{\xi}\right)$ of the pseudonormed rings, i. e. $\bar{\xi}\left(\widehat{\varphi}\left(\widehat{r}\right)\right) = \inf \left\{\widehat{\xi}\left(\widehat{r} + a\right) | a \in \ker \widehat{\varphi}\right\}$ for all $\widehat{r} \in \widehat{R}$.

As it's shown in Theorem 2 it is impossible to tell anything more than the validity of the inequality $\bar{\xi}(\varphi(r)) \leq \xi(r)$ in the case when A is a subring of a pseudonormed ring (R, ξ).

The case when A is an ideal of a pseudonormed ring (R, ξ) was studied in the work [2], the case when A is a one-sided ideal of a pseudonormed ring (R, ξ) was studied in the work [3].

The following definition was introduced in [2]:

Definition 1. *Let (R, ξ) and $(\bar{R}, \bar{\xi})$ be pseudonormed rings and $\varphi : R \to \bar{R}$ be a ring isomorphism. The isomorphism $\varphi : (R, \xi) \to (\bar{R}, \bar{\xi})$ is called a semi-isometric isomorphism if there exists a pseudonormed ring $(\widehat{R}, \widehat{\xi})$ such that the following conditions are valid:*

1) the ring R is an ideal in the ring $\widehat{R}$;

2) $\widehat{\xi}(r) = \xi(r)$ for any $r \in R$;

3) the isomorphism φ can be extended up to an isometric homomorphism $\widehat{\varphi} : (\widehat{R}, \widehat{\xi}) \to (\bar{R}, \bar{\xi})$ of the pseudonormed rings.

The following theorem was proved in [2]:

Theorem 3. *Let (R, ξ) and $(\bar{R}, \bar{\xi})$ be pseudonormed rings and $\varphi : R \to \bar{R}$ be a ring isomorphism. Then the isomorphism $\varphi : (R, \xi) \to (\bar{R}, \bar{\xi})$ is a semi-isometric isomorphism of the pseudonormed rings iff the inequalities $\xi(a \cdot b) \leq \bar{\xi}(\varphi(a)) \cdot \xi(b)$, $\xi(b \cdot a) \leq \bar{\xi}(\varphi(a)) \cdot \xi(b)$ and $\bar{\xi}(\varphi(a)) \leq \xi(a)$ are true for any $a, b \in R$.*

This paper is a continuation of [2] and [3] and it's devoted to the study of the case when A is an accessible subring of a pseudonormed ring (R, ξ) (see Definition 2). It's shown that a ring isomorphism is a superposition of semi-isometric isomorphisms iff it is a narrowing on the accessible subring A of some isometric homomorphism.

Definition 2. *As usual, a subring A of a rings R is called an accessible subring of the stage no more than n of the ring R if there exists a chain $A = R_0 \subseteq R_1 \subseteq R_2 \subseteq \ldots \subseteq R_n = R$ of subrings of the ring R such that R_i is an ideal in R_{i+1} for $i = 0, 1, \ldots n-1$. Further we shall designate it as $A = R_0 \lhd R_1 \lhd R_2 \lhd \ldots \lhd R_n = R$.*

Proposition 1. *Let: 1) $(\widehat{R}, \widehat{\xi})$ be a pseudonormed ring; 2) R be an ideal in $\widehat{R}$; 3) $\widehat{I}$ be a closed ideal in $(\widehat{R}, \widehat{\xi})$ and $I = \widehat{I} \bigcap R$; 4) $\widetilde{I} = [I]_{(\widehat{R}, \widehat{\xi})}$ and $\widetilde{R} = R + \widetilde{I}$; 5) $\bar{\varepsilon} : R/I \to (R + \widehat{I})/I$ be the natural embedding; 6) $\widehat{\omega} : \widehat{R} \to \widehat{R}/I$ and $\widetilde{\omega} : \widehat{R}/I \to \widehat{R}/\widetilde{I}$ be canonical homomorphisms. Then $\widetilde{\omega}|_{R/I} : (\bar{R}, \bar{\xi}) = (R, \widehat{\xi}|_R)/I \to (\widetilde{R}, \widehat{\xi}|_{\widetilde{R}})/\widetilde{I} = (\bar{\bar{R}}, \bar{\bar{\xi}})$ is an isometric isomorphism.*

Proof. Let's consider the following diagram 1.

$$
\begin{array}{ccccc}
R \subseteq & & \widetilde{R} = R + \widetilde{I} \subseteq & \widehat{R} \\
\widehat{\omega}|_R \downarrow & & \widehat{\omega}|_{\widetilde{R}} \downarrow & \widehat{\omega} \downarrow \\
R/I & \xrightarrow{\bar{\varepsilon}} & \widetilde{R}/I & \subseteq \widehat{R}/I \\
\| & & \widetilde{\omega}|_{\widetilde{R}/I} \downarrow & \widetilde{\omega} \downarrow \\
R/I & \xrightarrow{\widetilde{\omega}|_{R/I}} & \widetilde{R}/\widetilde{I} & \subseteq \widehat{R}/\widetilde{I}
\end{array}
$$

As $I \subseteq \widetilde{I}$ then $inf\{\widehat{\xi}(r+i)|i \in I\} \geq inf\{\widehat{\xi}(r+i)|i \in \widetilde{I}\}$ for any $r \in R$. Therefore $\bar{\xi}(\bar{r}) \geq \bar{\bar{\xi}}(\widetilde{\omega}(\bar{r}))$ for any $\bar{r} \in \bar{R}$.

We show that the reverse inequality is true.

Let $\bar{r}$ be any element in the ring $\bar{R} = R/I$ and ε be any positive number. If $r \in R$ is an element such that $\bar{r} = r + I$ then there exists an element $\widetilde{i}_0 \in \widetilde{I}$ such that $\bar{\bar{\xi}}(\widetilde{\omega}(\bar{r})) + \frac{\varepsilon}{2} \geq \widehat{\xi}(r + \widetilde{i}_0)$. Since $\widetilde{i}_0 \in \widetilde{I} = [I]_{(\widehat{R},\widehat{\xi})}$ then there exists an element $i_0 \in I$ such that $\widehat{\xi}(i_0 - \widetilde{i}_0) < \frac{\varepsilon}{2}$. Hence we have the inequality

$$\bar{\xi}(\bar{r}) = inf\{\widehat{\xi}(r+i)|i \in I\} \leq \widehat{\xi}(r+i_0) = \widehat{\xi}(r + \widetilde{i}_0 - \widetilde{i}_0 + i_0) \leq$$
$$\widehat{\xi}(r + \widetilde{i}_0) + \widehat{\xi}(i_0 - \widetilde{i}_0) < \bar{\bar{\xi}}(\widetilde{\omega}(\bar{r})) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \bar{\bar{\xi}}(\widetilde{\omega}(\bar{r})) + \varepsilon.$$

Passing to the limit in these inequalities when $\varepsilon \to 0$, we obtain $\bar{\xi}(\bar{r}) \leq \bar{\bar{\xi}}(\widetilde{\omega}(\bar{r}))$.

Thus it follows from the inequalities $\bar{\xi}(\bar{r}) \geq \bar{\bar{\xi}}(\widetilde{\omega}(\bar{r}))$ and $\bar{\xi}(\bar{r}) \leq \bar{\bar{\xi}}(\widetilde{\omega}(\bar{r}))$ we have the equality $\bar{\xi}(\bar{r}) = \bar{\bar{\xi}}(\widetilde{\omega}(\bar{r}))$, i.e. $\widetilde{\omega}|_{R/I} : (\bar{R}, \bar{\xi}) = (R, \widehat{\xi}|_R)/I \to (\widetilde{R}, \widehat{\xi}|_{\widetilde{R}})/\widetilde{I} = (\bar{\bar{R}}, \bar{\bar{\xi}})$ is an isometric isomorphism.

The proposition is proved.

Theorem 4. *Let (R, ξ) and $(\bar{R}, \bar{\xi})$ be pseudonormed rings and $\varphi : R \to \bar{R}$ be a ring isomorphism. Then the following statements are equivalent:*

1. There exists a pseudonormed ring $(\widehat{R}, \widehat{\xi})$ such that (R, ξ) is an accessible subring of the stage no more than n of the pseudonormed ring $(\widehat{R}, \widehat{\xi})$ and the isomorphism φ can be extended up to an isometric homomorphism $\widehat{\varphi} : (\widehat{R}, \widehat{\xi}) \to (\bar{R}, \bar{\xi})$.

2. φ is a superposition of n semi-isometric isomorphisms, i.e. there exist pseudonormed rings $(R, \xi) = (R_0, \xi_0), (R_1, \xi_1), \ldots, (R_n, \xi_n) = (\bar{R}, \bar{\xi})$ and semi-isometric isomorphisms $\varphi_i : (R_i, \xi_i) \to (R_{i+1}, \xi_{i+1})$ for $i = 0, 1, \ldots, n-1$ such that $\varphi = \varphi_n \circ \varphi_{n-1} \circ \ldots \circ \varphi_0$.

Proof 1 ⇒ 2. Let $R = \widehat{R}_0 \lhd \widehat{R}_1 \lhd \widehat{R}_2 \lhd \ldots \lhd \widehat{R}_n = \widehat{R}$ be a chain of subrings such that $\widehat{R}_i$ is an ideal in $\widehat{R}_{i+1}$ for $i = 0, 1, \ldots n-1$ and the isomorphism $\varphi : R \to \bar{R}$ can be extended up to an isometric homomorphism $\widehat{\varphi} : (\widehat{R}, \widehat{\xi}) \to (\bar{R}, \bar{\xi})$.

If $\widehat{I} = \ker \widehat{\varphi}$ and $\widetilde{\omega} : R_{k+1} \to R_{k+1}/\widehat{I}$ is the canonical homomorphism (i.e. $\widetilde{\omega}(r) = r + \widehat{I}$) then there exists an isometric isomorphism $\eta : (\widehat{R_n}, \widehat{\xi_n})/\widehat{I} \to (\bar{R}, \bar{\xi})$ such that $\widehat{\varphi} = \eta \circ \widetilde{\omega}$.

Let's consider the following diagram 2 (mappings entering into the diagram are defined below).

$$\begin{array}{cccccccccc}
R = \widehat{R}_0 & \lhd & \ldots & \lhd & \widehat{R}_k & \lhd & \widehat{R}_{k+1} & = \widehat{R}_{k+1} = & & \widehat{R} \\
\| & & & & \Big\downarrow{\scriptstyle \omega|_{\widehat{R}_k}} & & \Big\downarrow{\scriptstyle \omega} & & & \\
R & \xrightarrow{\varphi_0} & \ldots & \xrightarrow{\varphi_{k-1}} & \widehat{R}_k/I & \lhd & \widehat{R}_{k+1}/I & \Big\downarrow{\scriptstyle \widetilde{\omega}} & & \Big\downarrow{\scriptstyle \widehat{\varphi}} \\
\Big\downarrow{\scriptstyle \varphi} & & & & \Big\downarrow{\scriptstyle \varphi_k} & & \Big\downarrow{\scriptstyle \widehat{\omega}} & & & \\
\bar{R} & & = & & \widehat{R}_{k+1}/\widehat{I} & = & \widehat{R}_{k+1}/\widehat{I} & = \widehat{R}_{k+1}/\widehat{I} & \xrightarrow{\eta} & \bar{R}
\end{array}$$

The further proof will be done by induction on the number n.

If $n = 1$ then (R, ξ) is an accessible subring of the stage 1 (i.e. it is an ideal) of the pseudonormed ring $(\widehat{R}, \widehat{\xi})$ and the isomorphism φ can be extended up to an isometric homomorphism $\widehat{\varphi} : (\widehat{R}, \widehat{\xi}) \to (\bar{R}, \bar{\xi})$, and hence $\varphi : (R, \xi) \to (\bar{R}, \bar{\xi})$ is a semi-isometric isomorphism.

Let's assume that the theorem is true for $n = k$, and let $n = k+1$. Since $\widehat{R}_k$ and $\widehat{I}$ are ideals in $\widehat{R}_{k+1}$ then $I = \widehat{R}_k \bigcap \widehat{I}$ is an ideal in $\widehat{R}_{k+1}$ too.

In the beginning let's consider the case when $I = \widehat{R}_k \bigcap \widehat{I}$ is a closed ideal in $(\widehat{R}_{k+1}, \widehat{\xi})$. If $\omega : \widehat{R}_{k+1} \to \widehat{R}_{k+1}/I$ is the canonical homomorphism, then $\omega|_{\widehat{R}_k} : (\widehat{R}_k, \widehat{\xi}|_{\widehat{R}_k}) \to (\widehat{R}_k, \widehat{\xi}|_{\widehat{R}_k})/I$ is an isometric homomorphism. As $\widehat{R}_0 \bigcap \ker \omega|_{\widehat{R}_k} = \widehat{R}_0 \bigcap I = \widehat{R}_0 \bigcap \widehat{I} = \widehat{R}_0 \bigcap \ker \widehat{\varphi} = \ker \varphi = \{0\}$ and $\widehat{R}_k = \widehat{R}_k \bigcap \widehat{R} = \widehat{R}_k \bigcap (R + \widehat{I}) = R + (\widehat{R}_k \bigcap \widehat{I}) = R + I$ then $\omega|_{\widehat{R}_0} : \widehat{R}_0 \to \widehat{R}_k/I$ is an isomorphism and by the assumption $\omega|_{\widehat{R}_0}$ is a superposition of k semi-isometric isomorphisms, i.e. there are pseudonormed rings $(R, \xi) = (R_0, \xi_0), (R_1, \xi_1), \ldots, (R_k, \xi_k) = (\widehat{R}_k, \widehat{\xi}|_{R_k})/I$ and isometric isomorphisms $\varphi_i : (R_i, \xi_i) \to (R_{i+1}, \xi_{i+1})$ for $i = 0, 1, \ldots, k-1$ such that $\omega|_{\widehat{R}_0} = \varphi_{k-1} \circ \varphi_{k-2} \circ \ldots \circ \varphi_0$.

As $I = \widehat{I} \bigcap R_k = (\ker \widehat{\varphi}) \bigcap R_k = \ker(\widehat{\varphi}|_{R_k})$ and $\overline{R} = \varphi(R) = \widehat{\varphi}(R)$ then $\widehat{R}_k + \widehat{I} = \widehat{R}_0 + \widehat{I} = \widehat{R}_{k+1}$, and so $\varphi_k = \widehat{\omega}|_{\widehat{R}_k/I} : \widehat{R}_k/I \to \widehat{R}_{k+1}/\widehat{I}$ is an isomorphism.

Since $\widehat{R}_k/I$ is an ideal in $\widehat{R}_{k+1}/I$ then $\varphi_k : (\widehat{R}_k, \widehat{\xi}|_{\widehat{R}_k})/I \to (\widehat{R}_{k+1}, \widehat{\xi})/\widehat{I}$ is a semi-isometric isomorphism. Hence $\eta \circ \varphi_k : (\widehat{R}_k, \widehat{\xi}|_{\widehat{R}_k})/I \to (\overline{R}_{k+1}, \overline{\xi})$ is a semi-isometric isomorphism, and $(\eta \circ \varphi_k) \circ \varphi_{k-1} \circ \varphi_{k-2} \circ \ldots \circ \varphi_0 = \eta \circ \varphi_k \circ \omega|_{\widehat{R}_0} = \eta \circ \widetilde{\omega}|_{R_0} = \widehat{\varphi}|_{R_0} = \varphi$, i.e. the isomorphism φ is a superposition of $k+1$ semi-isometric isomorphisms in the case when I is a closed ideal in $(\widehat{R}_{k+1}, \widehat{\xi})$.

Let's consider now the case when $I = \widehat{R}_k \bigcap \widehat{I}$ is non-closed ideal in $(\widehat{R}_{k+1}, \widehat{\xi})$. Let's designate $\widetilde{I} = [I]_{(\widehat{R}_{k+1}, \widehat{\xi})}$ and consider the diagram 3 which is obtained by adding one line to the diagram 2 (definitions of unknown by now rings and mappings see below).

$$
\begin{array}{ccccccccccc}
R = \widehat{R}_0 & \lhd & \ldots & \lhd & \widehat{R}_k & \lhd & \widehat{R}_{k+1} & = & \widehat{R}_{k+1} & = & \widehat{R} \\
\| & & & & {\scriptstyle \omega|_{\widehat{R}_k}}\big\downarrow & & {\scriptstyle \omega}\big\downarrow & & & & \\
R & \xrightarrow{\varphi_0} & \ldots & \xrightarrow{\varphi_{k-1}} & \widehat{R}_k/I & \lhd & \widehat{R}_{k+1}/I & & & & \\
& & & & {\scriptstyle \bar{\eta}}\big\downarrow & & {\scriptstyle \omega'}\big\downarrow & & & & \\
{\scriptstyle \varphi}\big\downarrow & & & & (\widehat{R}_k + \widetilde{I})/\widetilde{I} & \lhd & \widehat{R}_{k+1}/\widetilde{I} & & \big\downarrow{\scriptstyle \widetilde{\omega}} & & \big\downarrow{\scriptstyle \widehat{\varphi}} \\
& & & & {\scriptstyle \varphi'_k}\big\downarrow & & {\scriptstyle \bar{\omega}}\big\downarrow & & & & \\
\bar{R} & & = & & \widehat{R}_{k+1}/\widehat{I} & = & \widehat{R}_{k+1}/\widehat{I} & = & \widehat{R}_{k+1}/\widehat{I} & \xrightarrow{\eta} & \bar{R}
\end{array}
$$

As $\widehat{R}_k$ is an ideal in $\widehat{R}_{k+1}$ then $I = \widehat{R}_k \bigcap \widehat{I}$ is an ideal in $\widehat{R}$, and hence $\widetilde{I}$ is a closed ideal in $(\widehat{R}, \widehat{\xi}) = (\widehat{R}_{k+1}, \widehat{\xi})$. Then $(\widehat{R}_{k+1}, \widehat{\xi})/\widetilde{I}$ and $(\widehat{R}_k + \widetilde{I}, \widehat{\xi}|_{\widehat{R}_k + \widetilde{I}})/\widetilde{I}$ are pseudonormed rings. If $\omega : \widehat{R} \to \widehat{R}/I$, $\omega' : \widehat{R}/I \to \widehat{R}/\widetilde{I}$ and $\bar{\omega} : \widehat{R}/\widetilde{I} \to \widehat{R}/\widehat{I}$ are the canonical homomorphisms then $\widetilde{\omega} = \overline{\omega} \circ \omega' \circ \omega$. As $(\widehat{R}_k + \widetilde{I})/\widetilde{I}$ is an ideal in $\widehat{R}_{k+1}/\widetilde{I}$ then $\varphi'_k = \bar{\omega}|_{(\widehat{R}_k+\widetilde{I})/\widetilde{I}} : (\widehat{R}_k + \widetilde{I}, \widehat{\xi}|_{\widehat{R}_k+\widetilde{I}})/\widetilde{I} \to (\widehat{R}_{k+1}, \widehat{\xi})/\widetilde{I}$ is a semi-isometric isomorphism.

According to Proposition 1 $\bar{\eta} = \omega'|_{(\widehat{R}_k/I)} : (\widehat{R}_k, \widehat{\xi}|_{\widehat{R}_k})/I \to (\widehat{R}_k + \widetilde{I}, \widehat{\xi}|_{\widehat{R}_k+\widetilde{I}})/\widetilde{I}$ is an isometric isomorphism and hence $\overline{\eta} \circ \omega|_{\widehat{R}_k} : (\widehat{R}_k, \widehat{\xi}|_{\widehat{R}_k}) \to (\widehat{R}_k + \widetilde{I}, \widehat{\xi}|_{\widehat{R}_k+\widetilde{I}})/\widetilde{I}$ is an isometric homomorphism.

By the induction hypothesis, there exist pseudonormed rings $(R, \xi) = (R_0, \xi_0)$, $(R_1, \xi_1), \ldots, (R_k, \xi_k) = (\widehat{R}_k, \widehat{\xi}|_{\widehat{R}_k})/I$ and semi-isometric isomorphisms $\varphi_i : (R_i, \xi_i) \to (R_{i+1}, \xi_{i+1})$ for $i = 0, 1, 2, \ldots, k-1$ such that $\overline{\eta} \circ \omega|_{\widehat{R}_0} = \varphi_{k-1} \circ \varphi_{k-2} \circ \ldots \circ \varphi_0$.

Since $\eta, \overline{\eta}$ are isometric isomorphisms and φ'_k is a semi-isometric isomorphism then $\varphi''_k = \eta \circ \varphi'_k \circ \overline{\eta} : (\widehat{R}_k, \widehat{\xi}|_{\widehat{R}_k})/I \to (\overline{R}, \bar{\xi})$ is a semi-isometric isomorphism, at that $\varphi = \widehat{\varphi}|_R = \eta \circ \widetilde{\omega}|_R = \eta \circ \overline{\omega} \circ \omega' \circ \omega|_R = \eta \circ \varphi'_k \circ \overline{\eta} \circ \omega|_R = \varphi''_k \circ \overline{\eta} \circ \omega|_R = \varphi''_k \circ \varphi_{k-1} \circ \varphi_{k-2} \circ \ldots \circ \varphi_0$, i.e. the isomorphism φ is a superposition of $k+1$ semi-isometric isomorphisms in the case when I is a non-closed ideal in $(\widehat{R}_{k+1}, \widehat{\xi})$.

Thus we have proved that 2 follows from 1 for any natural number n.

Proof 2 $\Rightarrow$ 1. Let's assume there are pseudonormed rings

$$(R, \xi) = (R_0, \xi_0), (R_1, \xi_1), (R_2, \xi_2) \ldots, (R_n, \xi_n) = (\bar{R}, \bar{\xi})$$

and semi-isometric isomorphisms $\varphi_i : (R_i, \xi_i) \to (R_{i+1}, \xi_{i+1})$ for $i = 0, 1, \ldots, n-1$ such that φ is the superposition of these semi-isometric isomorphisms, i.e. $\varphi = \varphi_{n-1} \circ \varphi_{n-2} \circ \ldots \circ \varphi_0$.

For any $0 \leq i \leq j \leq n$ we consider the isomorphism $f_{i,j}$ such that $f_{i,j} = \varphi_{j-1} \circ \ldots \circ \varphi_i : R_i \to R_j$ for $i < j$ and $f_{i,i} : R_i \to R_i$ is the identical mapping.

The further proof will be done in some stages.

I. The construction of the ring $\widehat{R}$ and checking of some its algebraic properties.

Let's define on the set $\widehat{R} = \{(r_0, r_1, \ldots, r_n) \mid r_i \in R_i, i = 0, 1, \ldots, n\}$ the operations of addition and multiplication as follows:

$$(a_0, a_1, \ldots, a_n) + (b_0, b_1, \ldots, b_n) = (a_0 + b_0, a_1 + b_1, \ldots, a_n + b_n)$$

and

$$(a_0, a_1, \ldots, a_n) \cdot (b_0, b_1, \ldots, b_n) = (r_0, r_1, \ldots, r_n),$$

where $r_i = a_i \cdot b_i$ for $i \in \{0, n\}$ and $r_i = a_i \cdot b_i + (f_{0,i}(a_0) - a_i) \cdot \varphi_i^{-1}(b_{i+1}) + \varphi_i^{-1}(a_{i+1}) \cdot (f_{0,i}(b_0) - b_i)$ for $1 \leq i \leq n-1$.

As the mappings $\varphi_i : R_i \to R_{i+1}$ and $f_{0,i} : R_0 \to R_i$ are isomorphisms then it's easily checked that:

I.1. $\widehat{R}$ is a non-associative ring with respect to these operations (even if the initial rings are associative).

I.2. For any $0 \leq k < n$ the set $\widehat{R}_k = \{(r_0, \ldots, r_n) \in \widehat{R} \mid r_i = 0 \text{ if } i > k\}$ is an ideal in the ring $\widehat{R}_{k+1} = \{(r_0, \ldots, r_n) \in \widehat{R} \mid r_i = 0 \text{ if } i > k+1\}$.

I.3. $\widehat{R}_0 = \{(r_0, \ldots, r_n) \in \widehat{R} \mid r_i = 0 \text{ if } i \geq 1\}$ is an accessible subring of the stage no more than n in the ring $\widehat{R}_n = \widehat{R}$;

I.4. The mapping $\psi : \widehat{R}_0 \to R_0 = R$ which transfers the element $(a, 0, \ldots, 0) \in \widehat{R}_0$ into the element $a \in R_0$ is isomorphic.

I.5. From the definition of the operations of addition and multiplication in $\widehat{R}$ it follows that $\widehat{I} = \{(0, r_1, \dots r_n) \mid r_i \in R_i, i = 1, \dots, n\}$ is an ideal in the ring $\widehat{R}$ and $\widehat{R}_0 \bigcap \widehat{I} = \{0\}$ and $\widehat{R}_0 + \widehat{I} = \widehat{R}$.

I.6. If $\widehat{\varphi} : \widehat{R} \to \bar{R}$ is a mapping such that $\widehat{\varphi}(r_0, r_1, \dots, r_n) = \varphi(r_0)$ for any $(r_0, r_1, \dots, r_n) \in \widehat{R}$ then $\widehat{\varphi} : \widehat{R} \to \bar{R}$ is a ring homomorphism, and besides $\ker \widehat{\varphi} = \widehat{I}$ and $\widehat{\varphi}|_R = \varphi$.

Identifying any elements $(a, 0, \dots, 0) \in \widehat{R}_0$ with the elements $a \in R_0$, we shall identify the ring $\widehat{R}_0$ with the ring R_0. Therefore we can consider that $R = R_0$ is an accessible subring of the stage no more than n of the ring $\widehat{R}_n = \widehat{R}$.

II. The definition of a pseudonorm $\widehat{\xi}$ on the ring $\widehat{R}$ and checking of some properties of the pseudonormed ring $(\widehat{R}, \widehat{\xi})$.

Let's define $\widehat{\xi}((r_0, r_1, \dots, r_n)) = \sum\limits_{i=0}^{n-1} \xi_i(r_i - \varphi_i^{-1}(r_{i+1})) + \xi_n(r_n)$.

II.1. Let's check that $\widehat{\xi}$ is a pseudonorm on the ring $\widehat{R}$.

It's easy follows from the definition of the function $\widehat{\xi}$ that $\widehat{\xi}((-r_0, -r_1, \dots, -r_n)) = \widehat{\xi}((r_0, r_1, \dots, r_n)) \geq 0$ for any $(r_0, r_1, \dots, r_n) \in \widehat{R}$ and $\widehat{\xi}((r_0, r_1, \dots, r_n)) = 0$ if and only if $(r_0, r_1, \dots, r_n) = (0, 0, \dots, 0)$.

Let $a = (a_0, a_1, \dots, a_n) \in \widehat{R}$ and $b = (b_0, b_1, \dots, b_n) \in \widehat{R}$. Then

$$\widehat{\xi}(a+b) = \sum_{i=0}^{n-1} \xi_i(a_i + b_i - \varphi_i^{-1}(a_{i+1} + b_{i+1})) + \xi_n(a_n + b_n) \leq$$

$$\sum_{i=0}^{n-1} (\xi_i(a_i - \varphi_i^{-1}(a_{i+1})) + \xi_i(b_i - \varphi_i^{-1}(b_{i+1}))) + \xi_n(a_n) + \xi_n(b_n) = \widehat{\xi}(a) + \widehat{\xi}(b).$$

If $r = (r_0, r_1, \dots, r_n) = a \cdot b = (a_0, a_1, \dots, a_n) \cdot (b_0, b_1, \dots, b_n)$ then $r_0 = a_0 \cdot b_0$, $r_n = a_n \cdot b_n$, $r_i = a_i \cdot b_i + (f_{0,i}(a_0) - a_i) \cdot \varphi_i^{-1}(b_{i+1}) + \varphi_i^{-1}(a_{i+1}) \cdot (f_{0,i}(b_0) - b_i)$ for $i \in \{1, 2, \dots, n-1\}$ and

$$\widehat{\xi}(a \cdot b) = \widehat{\xi}((r_0, r_1, \dots, r_n)) = \xi_n(r_n) + \sum_{i=0}^{n-1} \xi_i(r_i - \varphi_i^{-1}(r_{i+1})).$$

Let's consider each term of this sum. It's obvious that $\xi_n(r_n) \leq \xi_n(a_n) \cdot \xi_n(b_n)$.

Let $h_i = a_i - \varphi_i^{-1}(a_{i+1})$ and $h'_i = b_i - \varphi_i^{-1}(b_{i+1})$ for $i \in \{0, 1, \dots, n-1\}$; $h_n = a_n$ and $h'_n = b_n$. Taking in consideration the definitions of mapping $f_{i,j}$ by induction on the number $j - i$ it's easy proved that

$$f_{i,j}(a_i) - a_j = f_{i,j}(a_i) - \varphi_{j-1}(\varphi_{j-1}^{-1}(a_j)) =$$
$$f_{i,j}(a_i) - f_{i,j}(\varphi_i^{-1}(a_{i+1})) + f_{i,j}(\varphi_i^{-1}(a_{i+1})) - f_{j-1,j}(\varphi_{j-1}^{-1}(a_j)) =$$
$$f_{i,j}(a_i - \varphi_i^{-1}(a_{i+1})) + f_{i,j}(\varphi_i^{-1}(a_{i+1})) - f_{j-1,j}(\varphi_{j-1}^{-1}(a_j)) = f_{i,j}(h_i) +$$
$$f_{i,j}(\varphi_i^{-1}(a_{i+1})) - f_{j-1,j}(\varphi_{j-1}^{-1}(a_j)) = \ldots = f_{i,j}(h_i) + f_{i+1,j}(h_{i+1}) + \ldots + f_{j-1,j}(h_{j-1})$$

for any $0 \leq i < j \leq n$. Then for $i \in \{1, 2, \dots, n-2\}$ we have

$\xi_i(r_i - \varphi_i^{-1}(r_{i+1})) = \xi_i(a_i \cdot b_i + (f_{0,i}(a_0) - a_i) \cdot \varphi_i^{-1}(b_{i+1}) + \varphi_i^{-1}(a_{i+1}) \cdot (f_{0,i}(b_0) - b_i) - \varphi_i^{-1}(a_{i+1} \cdot b_{i+1} + (f_{0,i+1}(a_0) - a_{i+1}) \cdot \varphi_{i+1}^{-1}(b_{i+2}) + \varphi_{i+1}^{-1}(a_{i+2}) \cdot (f_{0,i+1}(b_0) - b_{i+1}))) = \xi_i(a_i \cdot b_i + \sum\limits_{k=0}^{i-1} f_{k,i}(h_k) \cdot \varphi_i^{-1}(b_{i+1}) + \varphi_i^{-1}(a_{i+1}) \cdot \sum\limits_{k=0}^{i-1} f_{k,i}(h'_k) - \varphi_i^{-1}(a_{i+1}) \cdot \varphi_i^{-1}(b_{i+1}) - \varphi_i^{-1}(\sum\limits_{k=0}^{i} f_{k,i+1}(h_k) \cdot \varphi_{i+1}^{-1}(b_{i+2}) + \varphi_{i+1}^{-1}(a_{i+2}) \cdot \sum\limits_{k=0}^{i} f_{k,i+1}(h'_k))) = \xi_i(a_i \cdot b_i + \sum\limits_{k=0}^{i-1} f_{k,i}(h_k) \cdot$

$\varphi_i^{-1}(b_{i+1}-\varphi_{i+1}^{-1}(b_{i+2}))+\varphi_i^{-1}(a_{i+1}-\varphi_{i+1}^{-1}(a_{i+2}))\cdot\sum\limits_{k=0}^{i-1}f_{k,i}(h'_k)-(a_i-h_i)\cdot(b_i-h'_i)-$ $h_i\cdot\varphi_i^{-1}(b_{i+1}-h'_{i+1})-\varphi_i^{-1}(a_{i+1}-h_{i+1})\cdot h'_i)=\xi_i(\sum\limits_{k=0}^{i-1}f_{k,i}(h_k)\cdot\varphi_i^{-1}(h'_{i+1})+\varphi_i^{-1}(h_{i+1})\cdot$ $\sum\limits_{k=0}^{i-1}f_{k,i}(h'_k)+h_i\cdot(b_i-\varphi_i^{-1}(b_{i+1}))+(a_i-\varphi_i^{-1}(a_{i+1}))\cdot h'_i-h_i\cdot h'_i+h_i\cdot\varphi_i^{-1}(h'_{i+1})+$ $\varphi_i^{-1}(h_{i+1})\cdot h'_i)=\xi_i(\sum\limits_{k=0}^{i-1}f_{k,i}(h_k)\cdot\varphi_i^{-1}(h'_{i+1})+\varphi_i^{-1}(h_{i+1})\cdot\sum\limits_{k=0}^{i-1}f_{k,i}(h'_k)+h_i\cdot h'_i+h_i\cdot$ $\varphi_i^{-1}(h'_{i+1})+\varphi_i^{-1}(h_{i+1})\cdot h'_i)$.

If $i=n-1$ then
$\xi_{n-1}(r_{n-1}-\varphi_{n-1}^{-1}(r_n))=\xi_{n-1}(a_{n-1}\cdot b_{n-1}+(f_{0,n-1}(a_0)-a_{n-1})\cdot\varphi_{n-1}^{-1}(b_n)+\varphi_{n-1}^{-1}(a_n)\cdot$ $(f_{0,n-1}(b_0)-b_{n-1})-\varphi_{n-1}^{-1}(a_n\cdot b_n))=\xi_{n-1}(a_{n-1}\cdot b_{n-1}+\sum\limits_{k=0}^{n-2}f_{k,n-1}(h_k)\cdot\varphi_{n-1}^{-1}(h'_n)+$ $\varphi_{n-1}^{-1}(h_n)\cdot\sum\limits_{k=0}^{n-2}f_{k,n-1}(h'_k)-(a_{n-1}-h_{n-1})\cdot(b_{n-1}-h'_{n-1}))=\xi_{n-1}(\sum\limits_{k=0}^{n-2}f_{k,n-1}(h_k)\cdot$ $\varphi_{n-1}^{-1}(h'_n)+\varphi_{n-1}^{-1}(h_n)\cdot\sum\limits_{k=0}^{n-2}f_{k,n-1}(h'_k)+h_{n-1}\cdot(h'_{n-1}+\varphi_{n-1}^{-1}(h'_n))+(h_{n-1}+\varphi_{n-1}^{-1}(h_n))\cdot$ $h'_{n-1}-h_{n-1}\cdot h'_{n-1})=\xi_{n-1}(\sum\limits_{k=0}^{n-2}f_{k,n-1}(h_k)\cdot\varphi_{n-1}^{-1}(h'_n)+\varphi_{n-1}^{-1}(h_n)\cdot\sum\limits_{k=0}^{n-2}f_{k,n-1}(h'_k)+$ $h_{n-1}\cdot h'_{n-1}+h_{n-1}\cdot\varphi_{n-1}^{-1}(h'_n)+\varphi_{n-1}^{-1}(h_n)\cdot h'_{n-1})$.

Since the isomorphism $\varphi_i:(R_i,\xi_i)\to(R_{i+1},\xi_{i+1})$ is a semi-isometric then according to Theorem 3 the following inequalities are true:

$$\frac{\xi_i(a_i\cdot b_i)}{\xi_i(b_i)}\le\xi_{i+1}(\varphi_i(a_i))\le\xi_i(a_i)\text{ and }\frac{\xi_i(a_i\cdot b_i)}{\xi_i(a_i)}\le\xi_{i+1}(\varphi_i(b_i))\le\xi_i(b_i).$$

It's follows from the definition of the isomorphisms $f_{k,i}$:

$$\xi_i(f_{k,i}(h_k))\le\xi_k(h_k)\text{ and }\xi_i(f_{k,i}(h'_k))\le\xi_k(h'_k)$$

for any $0\le k\le i\le n$. Then for $i\in\{1,2,\dots,n-1\}$ we have
$\xi_i(\sum\limits_{k=0}^{i-1}f_{k,i}(h_k)\cdot\varphi_i^{-1}(h'_{i+1})+\varphi_i^{-1}(h_{i+1})\cdot\sum\limits_{k=0}^{i-1}f_{k,i}(h'_k)+h_i\cdot h'_i+h_i\cdot\varphi_i^{-1}(h'_{i+1})+\varphi_i^{-1}(h_{i+1})\cdot$ $h'_i)\le\sum\limits_{k=0}^{i-1}\xi_i(f_{k,i}(h_k))\cdot\xi_{i+1}(h'_{i+1})+\sum\limits_{k=0}^{i-1}\xi_{i+1}(h_{i+1})\cdot\xi_i(f_{k,i}(h'_k))+\xi_i(h_i)\cdot\xi_i(h'_i)+\xi_i(h_i)\cdot$ $\xi_{i+1}(h'_{i+1})+\xi_{i+1}(h_{i+1})\cdot\xi_i(h'_i)\le\sum\limits_{k=0}^{i-1}\xi_k(h_k)\cdot\xi_{i+1}(h'_{i+1})+\sum\limits_{k=0}^{i-1}\xi_{i+1}(h_{i+1})\cdot\xi_k(h'_k)+$ $\xi_i(h_i)\cdot\xi_i(h'_i)+\xi_i(h_i)\cdot\xi_{i+1}(h'_{i+1})+\xi_{i+1}(h_{i+1})\cdot\xi_i(h'_i)$.

If $i=0$ then
$\xi_0(r_0-\varphi_1^{-1}(r_1))=\xi_0(a_0\cdot b_0-\varphi_0^{-1}(a_1\cdot b_1+(\varphi_0(a_0)-a_1)\cdot\varphi_1^{-1}(b_2)+\varphi_1^{-1}(a_2)\cdot(\varphi_0(b_0)-$ $b_1)))=\xi_0(a_0\cdot b_0-\varphi_0^{-1}(a_1)\cdot\varphi_0^{-1}(b_1)-(a_0-\varphi_0^{-1}(a_1))\cdot\varphi_0^{-1}(\varphi_1^{-1}(b_2))-\varphi_0^{-1}(\varphi_1^{-1}(a_2))\cdot$ $(b_0-\varphi_0^{-1}(b_1)))=\xi_0(a_0\cdot b_0-(a_0-h_0)\cdot(b_0-h'_0)-h_0\cdot\varphi_0^{-1}(b_1-h'_1)-\varphi_0^{-1}(a_1-h_1)\cdot h'_0)=$ $\xi_0(h_0\cdot h'_0+h_0\cdot\varphi_0^{-1}(h'_1)+\varphi_0^{-1}(h_1)\cdot h'_0)\le\xi_0(h_0)\cdot\xi_0(h'_0)+\xi_0(h_0)\cdot\xi_1(h'_1)+\xi_1(h_1)\cdot\xi_0(h'_0)$.

It follows from the proven inequalities that

$$\widehat{\xi}(a\cdot b) \le \xi_0(h_0)\cdot\xi_0(h'_0)+\xi_0(h_0)\cdot\xi_1(h'_1)+\xi_1(h_1)\cdot\xi_0(h'_0)+\sum_{i=1}^{n-1}\Big(\sum_{k=0}^{i-1}\xi_k(h_k)\cdot\xi_{i+1}(h'_{i+1})+$$

$$\sum_{k=0}^{i-1}\xi_{i+1}(h_{i+1})\cdot\xi_k(h'_k)+\xi_i(h_i)\cdot\xi_i(h'_i)+\xi_i(h_i)\cdot\xi_{i+1}(h'_{i+1})+\xi_{i+1}(h_{i+1})\cdot\xi_i(h'_i)\Big)+$$

$$\xi_n(a_n)\cdot\xi_n(b_n)=\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\xi_i(h_i)\cdot\xi_j(h'_j)+\xi_n(a_n)\cdot\sum_{j=0}^{n-1}\xi_j(h'_j)+\sum_{i=0}^{n-1}\xi_i(h_i)\cdot\xi_n(b_n)$$

$$+\xi_n(a_n)\cdot\xi_n(b_n)=\left(\sum_{i=0}^{n-1}\xi_i(h_i)+\xi_n(a_n)\right)\cdot\left(\sum_{j=0}^{n-1}\xi_j(h'_j)+\xi_n(b_n)\right)=\widehat{\xi}(a)\cdot\widehat{\xi}(b).$$

Thus we have shown the inequality $\widehat{\xi}(a\cdot b)\le\widehat{\xi}(a)\cdot\widehat{\xi}(b)$ for any $a,b\in\widehat{R}$. Therefore $(\widehat{R},\widehat{\xi})$ ia a pseudonormed ring.

II.2. Since $\widehat{\xi}(r,0,\ldots,0)=\xi_0(r-0)+\xi_1(0)+\ldots+\xi_n(0)=\xi(r)$ for any $r\in R$ and any element $r\in R$ is identifying with the element $(r,0,\ldots,0)\in\widehat{R}_0$ then $\widehat{\xi}|_R=\xi$.

II.3. Let's show that $\widehat{\varphi}:(\widehat{R},\widehat{\xi})\to(\bar{R},\bar{\xi})$ is an isometric homomorphism, i.e. $\bar{\xi}\left(\widehat{\varphi}\left(\widehat{r}\right)\right)=\inf\left\{\widehat{\xi}\left(\widehat{r}+\widehat{a}\right)\middle|\widehat{a}\in\ker\widehat{\varphi}\right\}$ for all $\widehat{r}\in\widehat{R}$. Let $\widehat{r}=(r_0,r_1,\ldots,r_n)\in\widehat{R}$ and $\widehat{b}=(0,f_{0,1}(r_0)-r_1,\ldots,f_{0,n}(r_0)-r_n)$. Then $\widehat{b}\in\widehat{I}$ and so

$$\inf\left\{\widehat{\xi}(\widehat{r}+\widehat{a})\middle|\widehat{a}\in\ker\widehat{\varphi}\right\}\le\widehat{\xi}(\widehat{r}+\widehat{b})=\widehat{\xi}((r_0,r_1,\ldots,r_n)+$$

$$(0,f_{0,1}(r_0)-r_1,\ldots,f_{0,n}(r_0)-r_n))=\widehat{\xi}((r_0,f_{0,1}(r_0),\ldots,f_{0,n}(r_0)))=$$

$$\xi_0(r_0-\varphi_0^{-1}(f_{0,1}(r_0)))+\xi_1(f_{0,1}(r_0)-$$

$$\varphi_1^{-1}(f_{0,2}(r_0)))+\ldots+\xi_{n-1}(f_{0,n-1}(r_0)-\varphi_{n-1}^{-1}(f_{0,n}(r_0)))+\xi_n(f_{0,n}(r_0))=$$

$$\xi_0(0)+\xi_1(0)+\ldots+\xi_{n-1}(0)+\xi_n(\varphi(r_0))=\bar{\xi}(\varphi(r_0))=\bar{\xi}(\widehat{\varphi}(\widehat{r})).$$

On the other hand, since $f_{0,n}=\varphi$ and $\xi_i(d_i)\ge\xi_n(f_{i,n}(d_n))$ for every $d_i\in R_i$ and any $i\in\{0,1,\ldots,n\}$ then for every element $\widehat{a}=(o,a_1,\ldots,a_n)\in\widehat{I}$ we have

$$\widehat{\xi}(\widehat{r}+\widehat{a})=\widehat{\xi}((r_0,r_1+a_1,\ldots,r_n+a_n)=\xi_0(r_0-\varphi_0^{-1}(r_1+a_1))+$$

$$\sum_{i=1}^{n-1} \xi_i(r_i + a_i - \varphi_i^{-1}(r_{i+1} + a_{i+1})) + \xi_n(r_n + a_n) \geq \xi_n(f_{0,n}(r_0) - f_{0,n}(\varphi_0^{-1}(r_1 + a_1)))+$$

$$\sum_{i=1}^{n-1} \xi_n(f_{i,n}(r_i+a_i) - f_{i,n}(\varphi_i^{-1}(r_{i+1}+a_{i+1}))) + \xi_n(r_n+a_n) = \xi_n(f_{0,n}(r_0) - f_{1,n}(r_1+a_1))+$$

$$\sum_{i=1}^{n-1} \xi_n(f_{i,n}(r_i + a_i) - f_{i+1,n}(r_{i+1} + a_{i+1})) + \xi_n(r_n + a_n) \geq$$

$$\xi_n\left(f_{0,n}(r_0) - f_{1,n}(r_1 + a_1) + \sum_{i=1}^{n-1}(f_{i,n}(r_i + a_i) - f_{i+1,n}(r_{i+1} + a_{i+1})) + r_n + a_n\right) =$$

$$\xi_n\left(f_{0,n}(r_0)\right) = \xi_n(\varphi(r_0)) = \bar{\xi}(\widehat{\varphi}(\widehat{r})).$$

Since $\widehat{a} \in \widehat{I}$ is any element then $\inf\left\{\widehat{\xi}\left(\widehat{r} + \widehat{a}\right) \middle| \widehat{a} \in \ker \widehat{\varphi}\right\} \geq \bar{\xi}(\widehat{\varphi}(\widehat{r}))$ and so $\inf\left\{\widehat{\xi}\left(\widehat{r} + \widehat{a}\right) \middle| \widehat{a} \in \ker \widehat{\varphi}\right\} = \bar{\xi}(\widehat{\varphi}(\widehat{r}))$. Therefore $\widehat{\varphi} : (\widehat{R}, \widehat{\xi}) \to (\bar{R}, \bar{\xi})$ is an isometric homomorphism.

The theorem is completely proved.

Designation 1. *Let R be a ring. Put $R^1 = R$ and for any natural number n define R^n as the subgroup generated by the set $\{a \cdot b | a \in R^s, b \in R^t, 0 < s, t < n, s + t = n\}$. It's easy to note that R^n is an ideal in the ring R.*

Definition 3. *A ring R is called a nilpotent ring if $R^n = 0$ for some natural number n. The minimal one from these natural numbers is called the index of nilpotence.*

Theorem 5. *Let (R, ξ) and $(\bar{R}, \bar{\xi})$ be associative pseudonormed rings, $\varphi : R \to \bar{R}$ be a ring isomorphism and $R^n = 0$. Then the following statements are equivalent:*

1. $\bar{\xi}(\varphi(r)) \leq \xi(r)$ for any $r \in R$.

2. φ is a superposition of n semi-isometric isomorphisms, i.e. there exist pseudonormed rings $(R, \xi) = (R_0, \xi_0), (R_1, \xi_1), \ldots, (R_n, \xi_n) = (\bar{R}, \bar{\xi})$ and semi-isometric isomorphisms $\varphi_i : (R_i, \xi_i) \to (R_{i+1}, \xi_{i+1})$ for $i = 0, 1, \ldots, n-1$ such that $\varphi = \varphi_n \circ \varphi_{n-1} \circ \ldots \circ \varphi_0$.

3. There exists a non-associative pseudonormed ring $(\widehat{R}, \widehat{\xi})$ such that (R, ξ) is an accessible subring of the stage no more than n of the pseudonormed ring $(\widehat{R}, \widehat{\xi})$ and the isomorphism φ can be extended up to an isometric homomorphism $\widehat{\varphi} : (\widehat{R}, \widehat{\xi}) \to (\bar{R}, \bar{\xi})$.

Proof 1 ⇒ 2.

Let $R_k = R$ for $k = 0, 1, \cdots, n-1$ and $R_n = \bar{R}$; let $\varphi_{n-1} = \varphi : R \to \bar{R}$ and $\varphi_k = \varepsilon : R \to R$ be the identical mapping for $k = 0, 1, \cdots, n-2$; let $\xi_0(r) = \xi(r)$, $\xi_n(\bar{r}) = \bar{\xi}(\bar{r})$, $\xi_{n-1}(r) = \bar{\xi}(\varphi(r))$ and

$$\xi_k(r) = sup\left\{\bar{\xi}(\varphi(r)), \frac{\xi_{k-1}(r\cdot a)}{\xi_{k-1}(a)}, \frac{\xi_{k-1}(a\cdot r)}{\xi_{k-1}(a)}\Big| a\in R\backslash\{0\}\right\}$$

for $k = 1, 2, \cdots, n-2$.

Let's prove by induction on the number k that each function ξ_k is a pseudonorm on the ring R_k.

It's obvious that $\xi_k(-r) = \xi_k(r) \geq 0$ for any $r \in R_k$ and $\xi_k(r) = 0$ if and only if $r = 0$. Let's show the validity of inequalities $\xi_k(r_1 + r_2) \leq \xi_k(r_1) + \xi_k(r_2)$ and $\xi_k(r_1 \cdot r_2) \leq \xi_k(r_1) \cdot \xi_k(r_2)$ for any $r_1, r_2 \in R_k$.

Indeed, for any $a \in R\backslash\{0\}$ we have

$$\frac{\xi_{k-1}((r_1+r_2)\cdot a)}{\xi_{k-1}(a)} \leq \frac{\xi_{k-1}(r_1\cdot a)}{\xi_{k-1}(a)} + \frac{\xi_{k-1}(r_2\cdot a)}{\xi_{k-1}(a)} \leq$$

$$sup\left\{\frac{\xi_{k-1}(r_1\cdot b)}{\xi_{k-1}(b)}\Big| b\in R\backslash\{0\}\right\} + sup\left\{\frac{\xi_{k-1}(r_2\cdot b)}{\xi_{k-1}(b)}\Big| b\in R\backslash\{0\}\right\} \leq \xi_k(r_1)+\xi_k(r_2),$$

$$\frac{\xi_{k-1}(a\cdot(r_1+r_2))}{\xi_{k-1}(a)} \leq \frac{\xi_{k-1}(a\cdot r_1)}{\xi_{k-1}(a)} + \frac{\xi_{k-1}(a\cdot r_2)}{\xi_{k-1}(a)} \leq$$

$$sup\left\{\frac{\xi_{k-1}(b\cdot r_1)}{\xi_{k-1}(b)}\Big| b\in R\backslash\{0\}\right\} + sup\left\{\frac{\xi_{k-1}(b\cdot r_2)}{\xi_{k-1}(b)}\Big| b\in R\backslash\{0\}\right\} \leq \xi_k(r_1)+\xi_k(r_2)$$

and

$$\bar{\xi}(\varphi(r_1+r_2)) = \bar{\xi}(\varphi(r_1)+\varphi(r_2)) \leq \bar{\xi}(\varphi(r_1)) + \bar{\xi}(\varphi(r_2)) \leq \xi_k(r_1)+\xi_k(r_2).$$

Therefore

$$\xi_k(r_1+r_2) = sup\left\{\bar{\xi}(\varphi(r_1+r_2)), \frac{\xi_{k-1}((r_1+r_2)\cdot a)}{\xi_{k-1}(a)}, \frac{\xi_{k-1}(a\cdot(r_1+r_2))}{\xi_{k-1}(a)}\Big| a\in R\backslash\{0\}\right\} \leq$$

$\xi_k(r_1)+\xi_k(r_2)$.

For any $a \in R\backslash\{0\}$ we have

$$\frac{\xi_{k-1}((r_1\cdot r_2)\cdot a)}{\xi_{k-1}(a)} = \frac{\xi_{k-1}(r_1\cdot(r_2\cdot a))}{\xi_{k-1}(r_2\cdot a))} \cdot \frac{\xi_{k-1}(r_2\cdot a)}{\xi_{k-1}(a)} \leq$$

$$sup\left\{\frac{\xi_{k-1}(r_1\cdot b)}{\xi_{k-1}(b)}\Big| b\in R\backslash\{0\}\right\} \cdot sup\left\{\frac{\xi_{k-1}(r_2\cdot c)}{\xi_{k-1}(c)}\Big| c\in R\backslash\{0\}\right\} \leq \xi_k(r_1)\cdot\xi_k(r_2),$$

$$\frac{\xi_{k-1}(a\cdot(r_1\cdot r_2))}{\xi_{k-1}(a)} \leq \frac{\xi_{k-1}(a\cdot r_1)}{\xi_{k-1}(a)} \cdot \frac{\xi_{k-1}((a\cdot r_1)\cdot r_2)}{\xi_{k-1}(a\cdot r_1)} \leq$$

$$sup\left\{\frac{\xi_{k-1}(b\cdot r_1)}{\xi_{k-1}(b)}\Big| b\in R\backslash\{0\}\right\} \cdot sup\left\{\frac{\xi_{k-1}(c\cdot r_2)}{\xi_{k-1}(c)}\Big| c\in R\backslash\{0\}\right\} \leq \xi_k(r_1)\cdot\xi_k(r_2)$$

and

$$\bar{\xi}(\varphi(r_1\cdot r_2)) = \bar{\xi}(\varphi(r_1)\cdot\varphi(r_2)) \leq \bar{\xi}(\varphi(r_1))\cdot\bar{\xi}(\varphi(r_2)) \leq \xi_k(r_1)\cdot\xi_k(r_2).$$

Therefore

$$\xi_k(r_1 \cdot r_2) = sup\left\{\bar{\xi}(\varphi(r_1 \cdot r_2)), \frac{\xi_{k-1}((r_1 \cdot r_2) \cdot a)}{\xi_{k-1}(a)}, \frac{\xi_{k-1}(a \cdot (r_1 \cdot r_2))}{\xi_{k-1}(a)} \Big| a \in R\backslash\{0\}\right\} \leq$$

$\xi_k(r_1) \cdot \xi_k(r_2)$.

Thus the function ξ_k is a pseudonorm on the ring R_k.

Let's prove that $\varphi_k : (R_k, \xi_k) \to (R_{k+1}, \xi_{k+1})$ is a semi-isometric isomorphism for $k = 0, 1, \cdots, n-2$.

Let's check the validity of inequality $\xi_{k+1}(\varphi_k(r)) \leq \xi_k(r)$.

Since

$$\bar{\xi}(\varphi(r)) \leq \xi_k(r), \frac{\xi_k(r \cdot a)}{\xi_k(a)} \leq \xi_k(r) \text{ and } \frac{\xi_k(a \cdot r)}{\xi_k(a)} \leq \xi_k(r)$$

for any $a \in R\backslash\{0\}$ then

$$sup\left\{\bar{\xi}(\varphi(r)), \frac{\xi_{k-1}(r \cdot a)}{\xi_{k-1}(a)}, \frac{\xi_{k-1}(a \cdot r)}{\xi_{k-1}(a)} \Big| a \in R\backslash\{0\}\right\} \leq \xi_k(r)$$

and

$$\xi_{k+1}(\varphi_k(r)) = \xi_{k+1}(\varepsilon(r)) = \xi_{k+1}(r) \leq \xi_k(r)$$

for any $r \in R_k$.

Let's show that the inequalities $\xi_k(r \cdot q) \leq \xi_{k+1}(\varphi_k(r)) \cdot \xi_k(q)$ and $\xi_k(q \cdot r) \leq \xi_{k+1}(\varphi_k(r)) \cdot \xi_k(q)$ are true.

Indeed, for any $q \neq 0$ we have

$$\frac{\xi_k(r \cdot q)}{\xi_k(q)} \leq sup\left\{\frac{\xi_k(r \cdot a)}{\xi_k(a)} \Big| a \in R\backslash\{0\}\right\} \leq$$

$$sup\left\{\bar{\xi}(\varphi(r)), \frac{\xi_k(r \cdot a)}{\xi_k(a)}, \frac{\xi_k(a \cdot r)}{\xi_k(a)} \Big| a \in R\backslash\{0\}\right\} = \xi_{k+1}(r)$$

and

$$\frac{\xi_k(q \cdot r)}{\xi_k(q)} \leq sup\left\{\frac{\xi_k(a \cdot r)}{\xi_k(a)} \Big| a \in R\backslash\{0\}\right\} \leq$$

$$sup\left\{\bar{\xi}(\varphi(r)), \frac{\xi_k(r \cdot a)}{\xi_k(a)}, \frac{\xi_k(a \cdot r)}{\xi_k(a)} \Big| a \in R\backslash\{0\}\right\} = \xi_{k+1}(r).$$

Thus

$$\xi_k(r \cdot q) \leq \xi_{k+1}(r) \cdot \xi_k(q) = \xi_{k+1}(\varepsilon(r)) \cdot \xi_k(q) = \xi_{k+1}(\varphi_k(r)) \cdot \xi_k(q)$$

and

$$\xi_k(q \cdot r) \leq \xi_{k+1}(r) \cdot \xi_k(q) = \xi_{k+1}(\varepsilon(r)) \cdot \xi_k(q) = \xi_{k+1}(\varphi_k(r)) \cdot \xi_k(q).$$

All conditions of Theorem 3 are satisfied. Therefore $\varphi_k : (R_k, \xi_k) \to (R_{k+1}, \xi_{k+1})$ is a semi-isometric isomorphism for $k = 0, 1, \cdots, n-2$.

Let's consider $\varphi_{n-1} : (R_{n-1}, \xi_{n-1}) \to (R_n, \xi_n)$. Since $\xi_{n-1}(r) = \bar{\xi}(\varphi(r))$ for any $r \in R$ that the isomorphism $\varphi_{n-1} = \varphi : (R_{n-1}, \xi_{n-1}) = (R, \xi_{n-1}) \to (R_n, \xi_n) = (\bar{R}, \bar{\xi})$ is isometric.

Therefore there exist pseudonormed rings $(R, \xi) = (R_0, \xi_0), (R_1, \xi_1), \dots, (R_n, \xi_n) = (\bar{R}, \bar{\xi})$ and semi-isometric isomorphisms $\varphi_i : (R_i, \xi_i) \to (R_{i+1}, \xi_{i+1})$ for $i = 0, 1, \dots, n-1$ such that $\varphi = \varphi_n \circ \varphi_{n-1} \circ \dots \circ \varphi_0$.

The implication $\mathbf{1} \Rightarrow \mathbf{2}$ is proved.

The implication $\mathbf{2} \Rightarrow \mathbf{3}$ follows from Theorem 4. The implication $\mathbf{3} \Rightarrow \mathbf{1}$ follows from Theorem 2.

The theorem is completely proved.

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