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The Lyapunov quantities and the center conditions for a class of bidimensional polynomial systems of differential equations with nonlinearities of the fourth degree

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Abstract. For the autonomous bidimensional polynomial systems of differential equations with nonlinearities of the fourth degree the $GL(2,\mathbb{R})$-invariant recurrence equations for determination of the Lyapunov quantities were established. Moreover, the general form of Lyapunov quantities for the mentioned systems is obtained. For a class of such systems the necessary and sufficient $GL(2,\mathbb{R})$-invariant conditions for the existence of center are given.

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Let us consider the system of differential equations with nonlinearities of the fourth degree

$$\frac{dx}{dt} = \mathbf{P}_1(x,y) + \mathbf{P}_4(x,y) = \mathbf{P}(x,y), \quad \frac{dy}{dt} = \mathbf{Q}_1(x,y) + \mathbf{Q}_4(x,y) = \mathbf{Q}(x,y), \qquad (1)$$

where $\mathbf{P}_i(x,y)$, $\mathbf{Q}_i(x,y)$ are homogeneous polynomials of degree i in x and y with real coefficients.

The goal of this paper is to determine the invariant recurrence formulas for construction of the Lyapunov quantities for the system of differential equations with nonlinearities of the fourth degree and to establish the invariant center conditions for a class of these systems. The center-focus problem is one of the most important problem in the Qualitative Theory of Differential Equations. This problem is completely solved only for the bidimensional quadratic systems and for the systems with nonlinearities of the third degree [1–3]. Also, this problem was solved for some classes of cubic differential systems [4–7]. In [8] the center problem for a linear center perturbed by homogeneous polynomials, more exactly for the systems of the form

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -x + \mathbf{Q}_4(x,y)$$

was solved. In [9], the authors give some sufficient conditions for the integrability in polar coordinates of a bidimensional polynomial systems with linear part of center type and non-linear part given by homogeneous polynomials of the fourth degree.

Also they establish a conjecture that if it turns to be true then the integrable cases they found are the only possible ones. In [10] the author gives some center conditions for a class of bidimensional polynomial systems of the fourth degree.

1 Definitions and notations

The system (1) can be written in the following coefficient form:

$$\begin{aligned}\frac{dx}{dt} &= \mathrm{c}x + \mathrm{d}y + \mathrm{g}x^4 + 4\mathrm{h}x^3y + 6\mathrm{k}x^2y^2 + 4\mathrm{l}xy^3 + \mathrm{m}y^4,\\ \frac{dy}{dt} &= \mathrm{e}x + \mathrm{f}y + \mathrm{n}x^4 + 4\mathrm{p}x^3y + 6\mathrm{q}x^2y^2 + 4\mathrm{r}xy^3 + \mathrm{s}y^4.\end{aligned} \tag{2}$$

We denote by A the 14-dimensional coefficient space of the system (1), by $\mathbf{a} \in A$ the vector of coefficients $\mathbf{a} = (\mathrm{c}, \mathrm{d}, \mathrm{e}, \mathrm{f}, \mathrm{g}, \mathrm{h}, \mathrm{k}, \mathrm{l}, \mathrm{m}, \mathrm{n}, \mathrm{p}, \mathrm{q}, \mathrm{r}, \mathrm{s})$, by $\boldsymbol{q} \in \mathcal{Q} \subseteq Aff(2,\mathbb{R})$ a nondegenerate linear transformation of the phase plane of system (1), by $\mathbf{q}$ the transformation matrix and by $r_{\boldsymbol{q}}(\mathbf{a})$ the linear representation of the coefficients of transformed system in the space A.

Definition 1 (see [11, 12])**.** *A polynomial $\mathcal{K}(\mathbf{a}, \mathbf{x})$ in coefficients of system* (1) *and coordinates of the vector $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ is called a comitant of system* (1) *with respect to the group $\mathcal{Q}$ if there exists a function $\lambda : \mathcal{Q} \to \mathbb{R}$ such that*

$$\mathcal{K}(r_{\boldsymbol{q}}(\mathbf{a}), \mathbf{q}\mathbf{x}) \equiv \lambda(\boldsymbol{q})\mathcal{K}(\mathbf{a}, \mathbf{x})$$

for every $\boldsymbol{q} \in \mathcal{Q}$, $\mathbf{a} \in A$ and $\mathbf{x} \in \mathbb{R}^2$.

If $\mathcal{Q}$ is the group $GL(2,\mathbb{R})$ of nondegenerate linear transformations

$$\mathbf{u} = \mathbf{q}\mathbf{x}, \quad \Delta_{\mathbf{q}} = \det \mathbf{q} \neq 0 \tag{3}$$

of the phase plane of system (1), where $\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix}$ is a vector of new phase variables and $\mathbf{q} = \begin{pmatrix} q_1^1 & q_2^1 \\ q_1^2 & q_2^2 \end{pmatrix}$ is the transformation matrix, then the comitant is called $GL(2,\mathbb{R})$-comitant or center-affine comitant. In what follows only $GL(2,\mathbb{R})$-comitants are considered. If a comitant does not depend on coordinates of the vector $\mathbf{x}$, then it is called invariant.

The function $\lambda(\boldsymbol{q})$ is called a multiplicator. It is known [11] that the function $\lambda(\boldsymbol{q})$ has the form $\lambda(\boldsymbol{q}) = \Delta_{\mathbf{q}}^{-\chi}$, where χ is an integer, which is called the weight of the comitant $\mathcal{K}(\mathbf{a}, \mathbf{x})$. If $\chi = 0$, then the comitant is called absolute, otherwise it is called relative.

We say that a comitant $\mathcal{K}(\mathbf{a}, \mathbf{x})$ has the character $(\rho;\ \chi;\ \delta)$ if it has the weight χ, the degree δ with respect to the coefficients of the system (1) and the degree ρ with respect to the coordinates of the vector $\mathbf{x}$.

Definition 2 (see [13]). *Let φ and ψ be homogeneous polynomials in coordinates of the vector* $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ *of the degrees ρ_1 and ρ_2, respectively. The polynomial*

$$(\varphi, \psi)^{(j)} = \frac{(\rho_1 - j)!(\rho_2 - j)!}{\rho_1!\rho_2!} \sum_{i=0}^{j} (-1)^i \binom{j}{i} \frac{\partial^j \varphi}{\partial x^{j-i} \partial y^i} \frac{\partial^j \psi}{\partial x^i \partial y^{j-i}}$$

is called the transvectant of index j of polynomials φ and ψ.

Using this formula we have the following remarks.

Remark 1 (see [14]). *If polynomials φ and ψ are $GL(2,\mathbb{R})$-comitants of system* (1) *with the characters $(\rho_\varphi;\ \chi_\varphi;\ \delta_\varphi)$ and $(\rho_\psi;\ \chi_\psi;\ \delta_\psi)$, respectively, then the transvectant of index $j \le \min\{\rho_\varphi, \rho_\psi\}$ is a $GL(2,\mathbb{R})$-comitant of system* (1) *with the character $(\rho_\varphi + \rho_\psi - 2j;\ \chi_\varphi + \chi_\psi + j;\ \delta_\varphi + d_\psi)$. If $j > \min\{\rho_\varphi, \rho_\psi\}$, then $(\varphi, \psi)^{(j)} = 0$.*

Remark 2. *If homogeneous polynomials f, g, φ and ψ have the degrees m, n, μ and 0 (m, n, $\mu \in \mathbb{N}^*$), respectively, with respect to x and y and $l, q \in \mathbb{N}$, $\alpha \in \mathbb{R}$, then*

a) $(\alpha f, g)^{(k)} = (f, \alpha g)^{(k)} = \alpha (f, g)^{(k)}$, ***b)*** $(f^q, f)^{(2l+1)} = 0$,

c) $(f + g, \varphi)^{(k)} = (f, \varphi)^{(k)} + (g, \varphi)^{(k)}$, ***d)*** $(\psi, f)^{(k)} = 0$,

e) $(f \cdot g, \varphi)^{(1)} = \dfrac{m}{m+n}(f, \varphi)^{(1)} g + \dfrac{n}{m+n}(g, \varphi)^{(1)} f.$

Remark 3. *If homogeneous polynomials f and φ have the degrees $m \in N^*$ and 2, respectively, with respect to x and y, then*

$$((f, \varphi)^{(1)}, \varphi)^{(1)} = \frac{m-1}{m}(f, \varphi)^{(2)} \varphi - \frac{1}{2} f (\varphi, \varphi)^{(2)}.$$

The $GL(2,\mathbb{R})$-comitants of the first degree with respect to the coefficients of the system (1) have the form

$$R_i = \mathbf{P}_i(x,y)y - \mathbf{Q}_i(x,y)x,\ \ S_i = \frac{1}{i}\left(\frac{\partial \mathbf{P}_i(x,y)}{\partial x} + \frac{\partial \mathbf{Q}_i(x,y)}{\partial y}\right),\ \ i = 1, 4. \quad (4)$$

By using the comitants R_i and S_i, $i = 1, 4$, the system (1) can be written [15] in the form

$$\begin{aligned} \frac{dx}{dt} &= \ \ \frac{1}{2}\frac{\partial R_1}{\partial y} + \frac{1}{2} S_1 x + \frac{1}{5}\frac{\partial R_4}{\partial y} + \frac{4}{5} S_4 x, \\ \frac{dy}{dt} &= -\frac{1}{2}\frac{\partial R_1}{\partial x} + \frac{1}{2} S_1 y - \frac{1}{5}\frac{\partial R_4}{\partial x} + \frac{4}{5} S_4 y. \end{aligned} \quad (5)$$

For every homogeneous $GL(2,\mathbb{R})$-comitant $\mathcal{K}(x,y)$ with degree $s \in \mathbb{N}^*$ of the system (1) from (5) we obtain the total derivative of $\mathcal{K}(x,y)$ with respect to t [16]:

$$\frac{d\mathcal{K}}{dt} = \frac{\partial \mathcal{K}}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial \mathcal{K}}{\partial y} \cdot \frac{dy}{dt} = \frac{\partial \mathcal{K}}{\partial x}\left(\frac{1}{2}\frac{\partial R_1}{\partial y} + \frac{1}{2} S_1 x + \frac{1}{5}\frac{\partial R_4}{\partial y} + \frac{4}{5} S_4 x\right) +$$

$$+\frac{\partial \mathcal{K}}{\partial y}\left(-\frac{1}{2}\frac{\partial R_1}{\partial x}+\frac{1}{2}S_1 y-\frac{1}{5}\frac{\partial R_4}{\partial x}+\frac{4}{5}S_4 y\right)= \tag{6}$$
$$=s(\mathcal{K},R_1)^{(1)}+\frac{s}{2}\mathcal{K}S_1+s(\mathcal{K},R_4)^{(1)}+\frac{4s}{5}\mathcal{K}S_4,$$

where $(\mathcal{K},R_i)^{(1)}$ is a Jacobian (the transvectant of the first index) of $GL(2,\mathbb{R})$-comitants $\mathcal{K}$ and R_i. The representation (6) shows that the derivative with respect to t of every homogeneous $GL(2,\mathbb{R})$-comitant with the degree $s\geq 1$ of the system (1) is a $GL(2,\mathbb{R})$-comitant too.

By using the comitants R_i and S_i $(i=1,4)$, and the notion of the transvectant the following $GL(2,\mathbb{R})$-comitants and invariants of the system (1) were constructed (in the list below, the bracket "$\llbracket$" is used in order to avoid placing the otherwise necessary parenthesis "("):

$$I_1=S_1,\quad I_2=(R_1,R_1)^{(2)},\quad I_3=\llbracket S_4,R_1)^{(2)},R_1)^{(1)},(S_4,R_1)^{(2)})^{(1)},$$
$$I_4=\llbracket R_4,R_1)^{(2)},R_1)^{(2)},R_1)^{(1)},((R_4,R_1)^{(2)},R_1)^{(2)})^{(1)},$$
$$K_1=(S_4,R_1)^{(1)},\quad K_2=((S_4,R_1)^{(2)},R_1)^{(1)},\quad K_3=(R_4,S_4)^{(3)},$$
$$K_4=(K_3^2,S_4)^{(3)},\quad K_5=((K_3,S_4)^{(2)},R_1)^{(2)}$$
$$J_1=((R_4,R_4)^{(4)},R_1)^{(2)},\quad J_2=((R_4,S_4)^{(3)},R_1)^{(2)},\quad J_3=((S_4,S_4)^{(2)},R_1)^{(2)},$$
$$J_4=\llbracket R_4,R_4)^{(2)},R_1)^{(2)},R_1)^{(2)},R_1)^{(2)},\quad J_5=\llbracket R_4,S_4)^{(2)},R_1)^{(2)},R_1)^{(2)},$$
$$J_6=(K_4,K_5)^{(1)}.$$

2 Lyapunov quantities for bidimensional polynomial systems of differential equations with nonlinearities of the fourth degree with $S_1=0,\ I_2\neq 0$

We will consider the system (1) with the conditions $S_1=0, I_2>0$. These conditions mean that the eigenvalues of the Jacobian matrix at the singular point $(0,0)$ are pure imaginary, i.e., the system has the center or a weak focus at $(0,0)$. In these conditions the system (1) can be reduced, via a linear transformation and time rescaling, to the system

$$\frac{dx}{dt}=y+\mathbf{P}_4(x,y),\qquad \frac{dy}{dt}=-x+\mathbf{Q}_4(x,y), \tag{7}$$

which can be written in the form

$$\frac{dx}{dt}=\frac{1}{2}\frac{\partial R_1}{\partial y}+\frac{1}{5}\frac{\partial R_4}{\partial y}+\frac{4}{5}S_4 x,\qquad \frac{dy}{dt}=-\frac{1}{2}\frac{\partial R_1}{\partial x}-\frac{1}{5}\frac{\partial R_4}{\partial x}+\frac{4}{5}S_4 y, \tag{8}$$

where $R_1=x^2+y^2$.

Let us consider the formal power series of the form

$$F(x,y)=x^2+y^2+\sum_{j=3}^{\infty}F_j(x,y)$$

where for each j, $F_j(x,y)$ is a homogeneous polynomial of degree j, so that the derivative of $F(x,y)$ along the solutions of the system (7) (or (8)) satisfies

$$\frac{dF(x,y)}{dt} = \sum_{k=2}^{\infty} G_{2k}(x^2+y^2)^k,$$

where G_{2k} are the polynomials in the coefficients of the system (7), called *Lyapunov quantities* [17].

For establishing the center conditions for the system (7) we will determine Lyapunov quantities. The polynomials $F_j(x,y)$ and the constants G_{2k} can be determined from the identity:

$$\frac{\partial\left(x^2+y^2+\sum\limits_{j=3}^{\infty}F_j(x,y)\right)}{\partial x}(y+\mathbf{P}_4(x,y))+$$
$$+\frac{\partial\left(x^2+y^2+\sum\limits_{j=3}^{\infty}F_j(x,y)\right)}{\partial y}(-x+\mathbf{Q}_4(x,y)) \equiv \sum_{k=2}^{\infty} G_{2k}(x^2+y^2)^k. \tag{9}$$

Because for the system (7) $R_1 = x^2+y^2$ and by using (8), the identity (9) can be written in the form:

$$\frac{\partial\left(R_1+\sum\limits_{j=3}^{\infty}F_j(x,y)\right)}{\partial x}\left(\frac{1}{2}\frac{\partial R_1}{\partial y}+\frac{1}{5}\frac{\partial R_4}{\partial y}+\frac{4}{5}S_4 x\right)+$$
$$+\frac{\partial\left(R_1+\sum\limits_{j=3}^{\infty}F_j(x,y)\right)}{\partial y}\left(-\frac{1}{2}\frac{\partial R_1}{\partial x}-\frac{1}{5}\frac{\partial R_4}{\partial x}+\frac{4}{5}S_4 y\right) \equiv \sum_{k=2}^{\infty} G_{2k}R_1^k. \tag{10}$$

Next, we analyze the identity (10) which is more general than the identity (9), taking $S_1=0,\ I_2=(R_1,R_1)^{(2)}\neq 0$. By using the notion of the transvectant and Euler formula, the left side of the identity (10) can be written into the form:

$$\frac{1}{5}\left(\frac{\partial R_1}{\partial x}\cdot\frac{\partial R_4}{\partial y}-\frac{\partial R_1}{\partial y}\cdot\frac{\partial R_4}{\partial x}\right)+\frac{4}{5}S_4\left(\frac{\partial R_1}{\partial x}\cdot x-\frac{\partial R_1}{\partial y}\cdot y\right)+$$
$$+\frac{1}{2}\sum_{j=3}^{\infty}\left(\frac{\partial F_j(x,y)}{\partial x}\cdot\frac{\partial R_1}{\partial y}-\frac{\partial F_j(x,y)}{\partial y}\cdot\frac{\partial R_1}{\partial x}\right)+$$
$$+\frac{1}{5}\sum_{j=3}^{\infty}\left(\frac{\partial F_j(x,y)}{\partial x}\cdot\frac{\partial R_4}{\partial y}-\frac{\partial F_j(x,y)}{\partial y}\cdot\frac{\partial R_4}{\partial x}\right)+$$
$$+\frac{4}{5}S_4\sum_{j=3}^{\infty}\left(\frac{\partial F_j(x,y)}{\partial x}\cdot x+\frac{\partial F_j(x,y)}{\partial y}\cdot y\right)=$$

$$= 2(R_1, R_4)^{(1)} + 2 \cdot \frac{4}{5} R_1 S_4 + \sum_{j=3}^{\infty} j \cdot (F_j, R_1)^{(1)} + \sum_{j=3}^{\infty} j \cdot (F_j, R_4)^{(1)} + \frac{4}{5} \sum_{j=3}^{\infty} j \cdot F_j S_4,$$

and the identity (10) is reduced to the form:

$$\sum_{j=3}^{\infty} j \cdot (F_j, R_1)^{(1)} + \sum_{j=2}^{\infty} j \cdot W(F_j) \equiv \sum_{k=2}^{\infty} G_{2k} R_1^k, \tag{11}$$

where $F_2 = R_1$, $W(F_j) = (F_j, R_4)^{(1)} + \frac{4}{5} F_j S_4$.

Equaling in (11) polynomials with the same degree with respect to the coordinates of the vector (x, y), the identity (11) can be reduced to the system of differential equations in partial derivatives:

$$\begin{aligned}
&3(F_3, R_1)^{(1)} = 0,\\
&4(F_4, R_1)^{(1)} = G_4 R_1^2,\\
&5(F_5, R_1)^{(1)} + 2W(F_2) = 0,\\
&6(F_6, R_1)^{(1)} + 3W(F_3) = G_6 R_1^3,\\
&7(F_7, R_1)^{(1)} + 4W(F_4) = 0,\\
&8(F_8, R_1)^{(1)} + 5W(F_5) = G_8 R_1^4,\\
&9(F_9, R_1)^{(1)} + 6W(F_6) = 0,\\
&10(F_{10}, R_1)^{(1)} + 7W(F_7) = G_{10} R_1^5,\\
&11(F_{11}, R_1)^{(1)} + 8W(F_8) = 0,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&j(F_j, R_1)^{(1)} + (j-3)W(F_{j-3}) = \begin{cases} 0, & \text{for } j = 2l+1,\ l \in \mathbb{N}^*, \\ G_j R_1^{\frac{j}{2}}, & \text{for } j = 2l+2,\ l \in \mathbb{N}^*, \end{cases}\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned} \tag{12}$$

Equations of the form $j(F_j, R_1)^{(1)} = 0$, in the case when j is an odd number, have the solution $F_j \equiv 0$ in the class of homogeneous polynomials with real coefficients. In the case when j is an even number, the equations $j(F_j, R_1)^{(1)} = G_j R_1^{\frac{j}{2}}$ admit the solution of the form $F_j = C R_1^{\frac{j}{2}}$ and then $G_j = 0$, where C is an arbitrary real constant. Assuming $C = 0$, we can consider in this case that $F_j \equiv 0$. From the first equation of the system (12), it follows that $F_3 \equiv 0$. This implies $W(F_3) \equiv 0$ and so, $F_6 \equiv 0$ and $G_6 = 0$. In turn, $F_6 \equiv 0$ implies $W(F_6) \equiv 0$, and then $F_9 \equiv 0$ and so on. From the second equation of the system (12), it follows that $F_4 \equiv 0$ and $G_4 = 0$. From $F_4 \equiv 0$, it turns out that $W(F_4) \equiv 0$ and then $F_7 \equiv 0$. In turn, $F_7 \equiv 0$ implies $W(F_7) \equiv 0$ and then $F_{10} \equiv 0$ and $G_{10} = 0$, and so on. Basing on those mentioned, the system (12) is reduced to the following system:

$$5(F_5, R_1)^{(1)} + 2W(F_2) = 0,$$

$$
\begin{aligned}
&8(F_8,R_1)^{(1)}+5W(F_5)=G_8R_1^4,\\
&11(F_{11},R_1)^{(1)}+8W(F_8)=0,\\
&14(F_{14},R_1)^{(1)}+11W(F_{11})=G_{14}R_1^7,\\
&17(F_{17},R_1)^{(1)}+14W(F_{14})=0,\\
&20(F_{20},R_1)^{(1)}+17W(F_{17})=G_{20}R_1^{10},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&(3m+2)(F_{3m+2},R_1)^{(1)}+(3m-1)W(F_{3m-1})=\\
&\qquad=\begin{cases} 0, & \text{for } m=2l-1,\ l\in\mathbb{N}^*,\\[2ex] G_{3m+2}R_1^{\frac{3m+2}{2}}, & \text{for } m=2l,\ l\in\mathbb{N}^*,\end{cases}\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
\tag{13}
$$

From the system (13) it follows that only the homogeneous polynomials $F_{3m-1}(\mathbf{a},\mathbf{x})$, $m\in\mathbb{N}^*$ and the Lyapunov quantities $G_{6l+2}(\mathbf{a})$, $l\in\mathbb{N}^*$ participate in solving the center-focus problem for the system (1). By solving consecutively the equations of the system (13) the polynomials F_5, F_8, F_{11}, F_{14}, F_{17}, F_{20}, $\ldots$, and respectively the Lyapunov quantities G_8, G_{14}, G_{20}, $\ldots$, are determined.

$$
F_5=\sum_{j=0}^{2}\frac{2\cdot 5!\cdot 2^{j+1}\cdot R_1^j\cdot [\![W(F_2),\overbrace{R_1)^{(2)},\ldots,R_1)^{(2)}}^{j},R_1)^{(1)}}{(4-2j)!\cdot\prod\limits_{i=0}^{j}\Big((5-2i)^2\cdot(R_1,R_1)^{(2)}\Big)},
$$

$$
F_8=\sum_{j=0}^{3}\frac{5\cdot 8!\cdot 2^{j+1}\cdot R_1^j\cdot [\![W(F_5),\overbrace{R_1)^{(2)},\ldots,R_1)^{(2)}}^{j},R_1)^{(1)}}{(7-2j)!\cdot\prod\limits_{i=0}^{j}\Big((8-2i)^2\cdot(R_1,R_1)^{(2)}\Big)},
$$

$$
F_{11}=\sum_{j=0}^{5}\frac{8\cdot 11!\cdot 2^{j+1}\cdot R_1^j\cdot [\![W(F_8),\overbrace{R_1)^{(2)},\ldots,R_1)^{(2)}}^{j},R_1)^{(1)}}{(10-2j)!\cdot\prod\limits_{i=0}^{j}\Big((11-2i)^2\cdot(R_1,R_1)^{(2)}\Big)},
$$

$$
F_{14}=\sum_{j=0}^{6}\frac{11\cdot 14!\cdot 2^{j+1}\cdot R_1^j\cdot [\![W(F_{11}),\overbrace{R_1)^{(2)},\ldots,R_1)^{(2)}}^{j},R_1)^{(1)}}{(13-2j)!\cdot\prod\limits_{i=0}^{j}\Big((14-2i)^2\cdot(R_1,R_1)^{(2)}\Big)},
$$

$$
F_{17}=\sum_{j=0}^{8}\frac{14\cdot 17!\cdot 2^{j+1}\cdot R_1^j\cdot [\![W(F_{14}),\overbrace{R_1)^{(2)},\ldots,R_1)^{(2)}}^{j},R_1)^{(1)}}{(16-2j)!\cdot\prod\limits_{i=0}^{j}\Big((17-2i)^2\cdot(R_1,R_1)^{(2)}\Big)},
$$

$$F_{20}=\sum_{j=0}^{9}\frac{17\cdot 20!\cdot 2^{j+1}\cdot R_1^j\cdot [\![W(F_{17}),\overbrace{R_1)^{(2)},\dots,R_1)^{(2)}}^{j},R_1)^{(1)}}{(19-2j)!\cdot\prod\limits_{i=0}^{j}\Big((20-2i)^2\cdot(R_1,R_1)^{(2)}\Big)},$$

....

$$F_{3m+2}=$$

$$=\sum_{j=0}^{[\frac{3m+1}{2}]}\frac{(3m-1)\cdot(3m+2)!\cdot 2^{j+1}\cdot R_1^j\cdot [\![W(F_{3m-1}),\overbrace{R_1)^{(2)},\dots,R_1)^{(2)}}^{j},R_1)^{(1)}}{(3m-2j+1)!\cdot\prod\limits_{i=0}^{j}\Big((3m-2i+2)^2\cdot(R_1,R_1)^{(2)}\Big)},\quad(14)$$

....

where $m\in\mathbb{N}^*$, $W(F_i)=(F_i,R_4)^{(1)}+\dfrac{4}{5}F_iS_4$.

$$G_8=\frac{5\cdot 8!\cdot 2^4\cdot [\![W(F_5),\overbrace{R_1)^{(2)},\dots,R_1)^{(2)}}^{4}}{\prod\limits_{i=0}^{3}\Big((8-2i)^2\cdot(R_1,R_1)^{(2)}\Big)},$$

$$G_{14}=\frac{11\cdot 14!\cdot 2^7\cdot [\![W(F_{11}),\overbrace{R_1)^{(2)},\dots,R_1)^{(2)}}^{7}}{\prod\limits_{i=0}^{6}\Big((14-2i)^2\cdot(R_1,R_1)^{(2)}\Big)},$$

$$G_{20}=\frac{17\cdot 20!\cdot 2^{10}\cdot [\![W(F_{17}),\overbrace{R_1)^{(2)},\dots,R_1)^{(2)}}^{10}}{\prod\limits_{i=0}^{9}\Big((20-2i)^2\cdot(R_1,R_1)^{(2)}\Big)},$$

....

$$G_{6l+2}=$$

$$=\frac{(6l-1)\cdot(6l+2)!\cdot 2^{3l+1}\cdot [\![W(F_{6l-1}),\overbrace{R_1)^{(2)},\dots,R_1)^{(2)}}^{3l+1}}{\prod\limits_{i=0}^{3l}\Big((6l-2i+2)^2\cdot(R_1,R_1)^{(2)}\Big)},\quad(15)$$

....

where $l\in\mathbb{N}^*$, $W(F_i)=(F_i,R_4)^{(1)}+\dfrac{4}{5}F_iS_4$.

Next we show that the polynomials F_{3m+2} (14) and Lyapunov quantities G_{6l+2} (15) satisfy the equations of system (13). Replacing in the right side of (13) the expresion for F_{3m+2} (14) and by using Remarks 1, 2 and 3 we obtain:

$$(3m+2)(3m-1)(3m+2)!\times$$

$$\times \sum_{j=0}^{[\frac{3m+1}{2}]} \frac{2^{j+1}\cdot\left(R_1^j\cdot[\![W(F_{3m-1}),\overbrace{R_1)^{(2)},\ldots,R_1)^{(2)}}^{j},R_1)^{(1)},R_1\right)^{(1)}}{(3m-2j+1)!\cdot\prod\limits_{i=0}^{j}\left((3m-2i+2)^2\cdot(R_1,R_1)^{(2)}\right)}+$$

$$+(3m-1)W(F_{3m-1})=$$

> applying Remark 2. **e)**, taking $f=R_1^j$,
> $g=[\![W(F_{3m-1}),\overbrace{R_1)^{(2)},\ldots,R_1)^{(2)}}^{j},R_1)^{(1)}$ and $\varphi=R_1$, we obtain

$$=(3m+2)(3m-1)(3m+2)!\times$$

$$\times \sum_{j=0}^{[\frac{3m+1}{2}]} \frac{2^{j+1}}{(3m-2j+1)!\cdot\prod\limits_{i=0}^{j}\left((3m-2i+2)^2\cdot(R_1,R_1)^{(2)}\right)}\times$$

$$\times\left[\frac{2j}{3m+2}(R_1^j,R_1)^{(1)}\cdot[\![W(F_{3m-1}),\overbrace{R_1)^{(2)},\ldots,R_1)^{(2)}}^{j},R_1)^{(1)}+\right.$$

$$\left.+\frac{3m-2j+2}{3m+2}R_1^j\cdot[\![W(F_{3m-1}),\overbrace{R_1)^{(2)},\ldots,R_1)^{(2)}}^{j},R_1)^{(1)},R_1)^{(1)}\right]+$$

$$+(3m-1)W(F_{3m-1})=$$

> according to Remark 2. **b)**, the first term in square brackets is equal to zero, because $(R_1^j,R_1)^{(1)}=0$. For the second term, by applying Remark 3, taking $f=[\![W(F_{3m-1}),\overbrace{R_1)^{(2)},\ldots,R_1)^{(2)}}^{j}$ and $\varphi=R_1$, we obtain

$$=(3m+2)(3m-1)(3m+2)!\times$$

$$\times \sum_{j=0}^{[\frac{3m+1}{2}]} \frac{2^{j+1}}{(3m-2j+1)!\cdot\prod\limits_{i=0}^{j}\left((3m-2i+2)^2\cdot(R_1,R_1)^{(2)}\right)}\times$$

$$\times\left[\frac{(3m-2j+1)(3m-2j+2)}{(3m-2j+2)(3m+2)}R_1^{j+1}\cdot[\![W(F_{3m-1}),\overbrace{R_1)^{(2)},\ldots,R_1)^{(2)}}^{j},R_1)^{(2)}-\right.$$

$$- \frac{3m-2j+2}{2(3m+2)} R_1^j \cdot (R_1,R_1)^{(2)} \cdot [\![W(F_{3m-1}), \overbrace{R_1)^{(2)},\ldots,R_1)^{(2)}}^{j} \Bigg] +$$

$$+(3m-1)W(F_{3m-1}) =$$

$$= (3m-1)(3m+2)! \times$$

$$\times \left[\sum_{j=0}^{\left[\frac{3m+1}{2}\right]} \frac{2^{j+1} \cdot (3m-2j+1) \cdot R_1^{j+1} \cdot [\![W(F_{3m-1}), \overbrace{R_1)^{(2)},\ldots,R_1)^{(2)}}^{j}, R_1)^{(2)}}{(3m-2j+1)! \cdot \prod\limits_{i=0}^{j} \left((3m-2i+2)^2 \cdot (R_1,R_1)^{(2)} \right)} - \right.$$

$$\left. - \sum_{j=0}^{\left[\frac{3m+1}{2}\right]} \frac{2^{j+1} \cdot (3m-2j+2) \cdot R_1^j \cdot (R_1,R_1)^{(2)} \cdot [\![W(F_{3m-1}), \overbrace{R_1)^{(2)},\ldots,R_1)^{(2)}}^{j}}{2(3m-2j+1)! \cdot \prod\limits_{i=0}^{j} \left((3m-2i+2)^2 \cdot (R_1,R_1)^{(2)} \right)} \right] +$$

$$+(3m-1)W(F_{3m-1}) =$$

$$\boxed{\text{because for } j=0\text{, the term obtained from the second sum is equal to } -(3m-1)W(F_{3m-1})\text{, we get}}$$

$$= (3m-1)(3m+2)! \times$$

$$\times \left[\sum_{j=0}^{\left[\frac{3m+1}{2}\right]} \frac{2^{j+1} \cdot (3m-2j+1) \cdot R_1^{j+1} \cdot [\![W(F_{3m-1}), \overbrace{R_1)^{(2)},\ldots,R_1)^{(2)}}^{j+1}}{(3m-2j+1)! \cdot \left[(R_1,R_1)^{(2)} \right]^{j+1} \cdot \prod\limits_{i=0}^{j} (3m-2i+2)^2} - \right.$$

$$\left. - \sum_{j=1}^{\left[\frac{3m+1}{2}\right]} \frac{2^{j} \cdot (3m-2j+3) \cdot R_1^j \cdot [\![W(F_{3m-1}), \overbrace{R_1)^{(2)},\ldots,R_1)^{(2)}}^{j}}{(3m-2j+2)! \cdot (3m-2j+3) \cdot \left[(R_1,R_1)^{(2)} \right]^{j} \cdot \prod\limits_{i=0}^{j-1} (3m-2i+2)^2} \right] =$$

$$\boxed{\text{by changing the sum index in the second sum, we obtain}}$$

$$= (3m-1)(3m+2)! \times$$

$$\times \left[\sum_{j=0}^{\left[\frac{3m+1}{2}\right]} \frac{2^{j+1} \cdot (3m-2j+1) \cdot R_1^{j+1} \cdot [\![W(F_{3m-1}), \overbrace{R_1)^{(2)},\ldots,R_1)^{(2)}}^{j+1}}{(3m-2j+1)! \cdot \left[(R_1,R_1)^{(2)} \right]^{j+1} \cdot \prod\limits_{i=0}^{j} (3m-2i+2)^2} - \right.$$

$$\left. - \sum_{j=0}^{\left[\frac{3m+1}{2}\right]-1} \frac{2^{j+1} \cdot (3m-2j+1) \cdot R_1^{j+1} \cdot [\![W(F_{3m-1}), \overbrace{R_1)^{(2)},\ldots,R_1)^{(2)}}^{j+1}}{(3m-2j+1)! \cdot \left[(R_1,R_1)^{(2)} \right]^{j+1} \cdot \prod\limits_{i=0}^{j} (3m-2i+2)^2} \right] =$$

$$= (3m-1)(3m+2)!\times$$

$$\times\frac{2^{\left[\frac{3m+3}{2}\right]}\cdot\left(3m-2\left[\frac{3m+1}{2}\right]+1\right)\cdot R_1^{\left[\frac{3m+3}{2}\right]}\cdot [\![W(F_{3m-1}),\overbrace{R_1)^{(2)},\dots,R_1)^{(2)}}^{\left[\frac{3m+3}{2}\right]}}{\left(3m-2\left[\frac{3m+1}{2}\right]+1\right)!\cdot\left[(R_1,R_1)^{(2)}\right]^{\left[\frac{3m+3}{2}\right]}\cdot\prod\limits_{i=0}^{\left[\frac{3m+1}{2}\right]}(3m-2i+2)^2}. \tag{16}$$

If m is an odd number, i.e. $m=2l-1,\ l\in\mathbb{N}^*$, the expression (16) is written in the form:

$$\frac{(6l-4)(6l-1)!\cdot 2^{3l}\cdot 0\cdot R_1^{3l}\cdot [\![W(F_{6l-4}),\overbrace{R_1)^{(2)},\dots,R_1)^{(2)}}^{3l}}{\left[(R_1,R_1)^{(2)}\right]^{3l}\cdot\prod\limits_{i=0}^{3l-1}(6l-2i-1)^2},$$

where the transvectant

$$[\![W(F_{6l-4}),\overbrace{R_1)^{(2)},\dots,R_1)^{(2)}}^{3l}$$

is equal to 0, because the degree of comitant $W(F_{6l-4})$ with respect to the coordinates of the vector $\mathbf{x}$ is equal to $6l-1$, but the total index of transvectants with R_1 is equal to $6l$.

If m is an even number, i.e. $m=2l,\ l\in\mathbb{N}^*$, the expression (16) is written in the form:

$$\frac{(6l-1)(6l+2)!\cdot 2^{3l+1}\cdot R_1^{3l+1}\cdot [\![W(F_{6l-1}),\overbrace{R_1)^{(2)},\dots,R_1)^{(2)}}^{3l+1}}{\left[(R_1,R_1)^{(2)}\right]^{3l+1}\cdot\prod\limits_{i=0}^{3l}(6l-2i+2)^2}=G_{6l+2}\cdot R_1^{3l+1}, \tag{17}$$

where G_{6l+2} coincides with the expression (15). So, for establishing the Lyapunov quantities for the system (1) with the conditions $S_1=0, I_2\neq 0$, the formulas (14) and (15) can be used.

Notice that, when $m=2l-1,\ l\in\mathbb{N}^*$, the respective equations of the system (13) have a unique solution with respect to F_{3m+2}, i.e. in this case F_{3m+2} are determined unambiguously. In the case $m=2l,\ l\in\mathbb{N}^*$, the solutions of respective equations of the system (13) with respect to F_{3m+2} are determined up to a term of the form $CR_1^{\frac{3m+2}{2}}$, where C is an arbitrary real constant. This implies that Lyapunov quantities $G_{6l+2},\ l\in\mathbb{N}^*$, are not determined unambiguously.

Notice that the numerators in formulas (14) and (15) are expressed by transvectants constructed by using the comitants R_1, R_4 and S_4, but the denominators represent the powers of invariant $I_2=(R_1,R_1)^{(2)}$. Based on Remark 1, it follows that the numerators in formulas (14) and (15) are $GL(2,\mathbb{R})$-comitants for the system

(1). Since the $GL(2,\mathbb{R})$-comitants in (15) does not depend on the coordinates of the vector $\mathbf{x}$ it follows they are $GL(2,\mathbb{R})$-invariants for the system (1).

On the above analysis, it results that the system (1), with the conditions $S_1 = 0, I_2 \neq 0$ and all Lyapunov quantities (15) being equal to zero, admits first formal integral of the form:

$$F(x,y) = \sum_{m=0}^{\infty} F_{3m+2}(x,y),$$

where $F_2(x,y) = R_1$, but $F_{3m+2}(x,y),\ m \in \mathbb{N}^*$ are expressions (14).

3 The center conditions for the class of bidimensional polynomial systems of differential equations with nonlinearities of the fourth degree with $S_1 = 0,\ I_2 > 0,\ I_3 = I_4 = 0$

Let us consider the bidimensional polynomial system of differential equations with nonlinearities of the fourth degree (1).

By using the comitants R_i and S_i $(i = \overline{1,4})$ the system (1) can be written in the form (5).

We will consider the system (5) (or (1)) with the conditions $S_1 = 0$, $I_2 > 0$ which has a center or a weak focus at $(0,0)$.

Remark 4. *If $R_4 \cdot S_4 \equiv 0$ then the system (5) (or (1)) with $S_1 = 0$ and $I_2 > 0$ has a singular point of the center type at the origin of coordinates.*

Indeed, if $R_4 \equiv 0$, then the system (5) has the invariant algebraic curve

$$\Phi(x,y) = 32R_1 \cdot K_2 + 8I_2 \cdot K_1 - 5I_2^2 = 0$$

and the first integral

$$|\Phi|^{\frac{2}{3}} \cdot |R_1|^{-1} = c_1,$$

where c_1 is a real constant.

If $S_4 \equiv 0$, then the system (5) has the first integral:

$$5R_1 + 2R_4 = c_2,$$

where c_2 is a real constant.

For the system (1) with $S_1 = 0$, $I_2 > 0$ and $I_3 = I_4 = 0$ the $GL(2,\mathbb{R})$-invariant conditions for distinguishing between center and focus were established.

Theorem 1. *The system (1) with the conditions $S_1 = 0$, $I_2 > 0$ and $I_3 = I_4 = 0$ has the center at the origin of coordinates if and only if the following conditions are fulfilled*

$$G_8 = G_{26} = G_{32} = G_{38} = 0,$$

where G_8, G_{26}, G_{32} and G_{38} are Lyapunov quantities given in (15).

Moreover, the above conditions are equivalent to the following invariant ones:

$$J_5 = J_6 = 0.$$

Proof. *Necessity.* The system (1) (or (2)) with $S_1 = 0$, $I_2 > 0$ can be reduced by a centeraffine transformation and time scaling to the form

$$\begin{aligned} \frac{dx}{dt} &= y + \mathrm{g}x^4 + 4\mathrm{h}x^3y + 6\mathrm{k}x^2y^2 + 4\mathrm{l}xy^3 + \mathrm{m}y^4, \\ \frac{dy}{dt} &= -x + \mathrm{n}x^4 + 4\mathrm{p}x^3y + 6\mathrm{q}x^2y^2 + 4\mathrm{r}xy^3 + \mathrm{s}y^4. \end{aligned} \tag{18}$$

By a transformation of rotation, in the system (18) can be obtained the equality

$$\mathrm{h} + \mathrm{q} = 0. \tag{19}$$

By using the substitutions

$$\begin{aligned} &\mathrm{g} = \frac{4P+5H}{5}, \quad \mathrm{h} = \frac{10K+6Q}{10}, \quad \mathrm{k} = \frac{30L+12R}{30}, \quad \mathrm{l} = \frac{5M+S}{5}, \quad \mathrm{m} = N, \\ &\mathrm{n} = -G, \quad \mathrm{p} = \frac{P-5H}{5}, \quad \mathrm{q} = \frac{12Q-30K}{30}, \quad \mathrm{r} = \frac{6R-10L}{10}, \quad \mathrm{s} = \frac{4S-5M}{5} \end{aligned}$$

and using (19), the system (18) can be reduced to the form

$$\begin{aligned} \frac{dx}{dt} &= y + \frac{5H+4P}{5}x^4 + 4Kx^3y + \frac{30L+12R}{5}x^2y^2 + \frac{20M+4S}{5}xy^3 + Ny^4, \\ \frac{dy}{dt} &= -x - Gx^4 + \frac{4P-20H}{5}x^3y - 6Kx^2y^2 + \frac{12R-20L}{5}xy^3 + \frac{4S-5M}{5}y^4, \end{aligned} \tag{20}$$

for which

$$\begin{aligned} R_1 &= x^2 + y^2, \\ R_4 &= Gx^5 + 5Hx^4y + 10Kx^3y^2 + 10Lx^2y^3 + 5Mxy^4 + Ny^5, \\ S_4 &= Px^3 + 3Rxy^2 + Sy^3, \\ I_3 &= (P+R)^2 + S^2, \\ I_4 &= (G+2K+M)^2 + (H+2L+N)^2. \end{aligned}$$

So, $I_3 = 0$ implies $S = 0$ and $R = -P$, and $I_4 = 0$, implies $G = -2K - M$ and $N = -2L - H$, i.e., the system (1) with $S_1 = 0$, $I_2 > 0$ and $I_3 = I_4 = 0$ can be reduced to the form

$$\begin{aligned} \frac{dx}{dt} &= y + \frac{5H+4P}{5}x^4 + 4Kx^3y + \frac{30L-12P}{5}x^2y^2 + 4Mxy^3 - (H+2L)y^4, \\ \frac{dy}{dt} &= -x + (2K+M)x^4 + \frac{4P-20H}{5}x^3y - 6Kx^2y^2 - \frac{12P+20L}{5}xy^3 - My^4, \end{aligned} \tag{21}$$

for which

$$\begin{aligned} R_4 &= -(2K+M)x^5 + 5Hx^4y + 10Kx^3y^2 + 10Lx^2y^3 + 5Mxy^4 - (H+2L)y^5, \\ S_4 &= Px^3 - 3Pxy^2. \end{aligned}$$

Applying the formulas (14) and (15) for the system (21) we obtain the following expresions for Lyapunov quantities $G_8,\ G_{14},\ G_{20}$:

$$\begin{aligned}
G_8 &= (K+M)P = J_5/4,\\
G_{14} &= J_5\left(405I_2J_1 - 2160I_2J_2 + 952I_2J_3 + 2025J_4\right)/14400,\\
G_{20} &= J_5\big(2815560I_2^2J_1^2 - 19591875I_2^2J_1J_2 + 63518400I_2^2J_2^2 + 8637786I_2^2J_1J_3 -\\
&\quad 58484160I_2^2J_2J_3 + 14084096I_2^2J_3^2 + 13454100I_2J_1J_4 - 71938125I_2J_2J_4 +\\
&\quad 29031030I_2J_3J_4 - 3118500J_4^2\big)/414720000.
\end{aligned}$$

Since the condition $G_8 = 0$ for the system (21) is equivalent to the $GL(2,\mathbb{R})$- invariant condition $J_5 = 0$, we obtain the first $GL(2,\mathbb{R})$- invariant necessary condition to have a center at the origin of coordinates of system (1) with $S_1 = 0,\ I_2 > 0$ and $I_3 = I_4 = 0$.

So we have that $G_8 = 0$ implies $G_{14} = G_{20} = 0$. Because for the system (21) $G_8 = (K+M)P$, then the condition $G_8 = 0$ implies $P = 0$ or $K + M = 0$.

If, $P = 0$, then the comitant $S_4 \equiv 0$. In this case, by Remark 4., the system has center at the origin of coordinates.

So, next we consider the situation when $K + M = 0$. In this case, the system (21) is reduced to the system:

$$\begin{aligned}
\frac{dx}{dt} &= y + \frac{5H+4P}{5}x^4 + 4Kx^3y + \frac{30L-12P}{5}x^2y^2 - 4Kxy^3 - (H+2L)y^4,\\
\frac{dy}{dt} &= -x + Kx^4 + \frac{4P-20H}{5}x^3y - 6Kx^2y^2 - \frac{12P+20L}{5}xy^3 + Ky^4.
\end{aligned} \tag{22}$$

For the system (22) the Lyapunov quantities G_{26}, G_{32}, G_{38}, calculated by using the formulas (14) and (15), have the following form:

$$\begin{aligned}
G_{26} &= F_0F_1F_2F_3F_4/84000000\\
G_{32} &= G_{26}(922393092509I_2J_1 - 7764307622400I_2J_2 + 4866278972800I_2J_3 +\\
&\quad 3192990020695J_4)/3146766336000 +\\
&\quad 3F_0F_2F_3F_4(H+L)T_1/36700160000 +\\
&\quad F_0F_1F_3F_4(H+L)T_2/3369074688000 -\\
&\quad 221F_0F_1F_2F_4(H+L)T_3/23506452480000 -\\
&\quad 19F_0F_1F_2F_3(H+L)T_4/580123856076800,\\
G_{38} &= G_{26}\big(126033098843417720962811 3I_2^2J_1^2 - 156502278147003176194590 0I_2^2J_1J_2 +\\
&\quad 39610069368448344439363 20I_2^2J_1J_3 - 816812053926570075225 6\cdot 10^3I_2^2J_2J_3 +\\
&\quad 236923223606813101639680 0I_2^2J_3^2 + 102456066236057734244739 80I_2J_1J_4 -\\
&\quad 540613501307535389829450 0I_2J_2J_3 + 191790006077592063945936 00I_2J_3J_4 +\\
&\quad 199950356936752778428220 75J_4^2\big)/833778038297581977600000 -\\
&\quad F_0F_2F_3F_4(H+L)\big(79683781250(H+L)^4 + 16596426225(H+L)^2T_1 -\\
&\quad 142466T_1^2\big)/465032131379200000 -\\
&\quad F_0F_1F_3F_4(H+L)\big(5162357307858086250(H+L)^4 +
\end{aligned}$$

$$56310112366375(H+L)^2T_2 - 29394738T_2^2\big)/250457744498759156367360000+$$
$$F_0F_1F_2F_4(H+L)\big(24262059975447656250(H+L)^4+$$
$$11785658137723675(H+L)^2T_3 + 12640691034T_3^2\big)/86567256533369880576000000+$$
$$F_0F_1F_2F_3(H+L)\big(3648566975734071058003814 7(H+L)^4 - 1810577808T_4^2+$$
$$22352124982450552136(H+L)^2T_4)/153408002525451763169034240000 0,$$

where polynomials $F_i, i = \overline{0,4}$, and $T_j, j = \overline{1,4}$, have the forms

$$F_0 = K(-3H^2 + 16K^2 + 18HL - 27L^2)P,$$
$$F_1 = 45H + 45L + 8P,$$
$$F_2 = 35H + 35L + 24P,$$
$$F_3 = 85H + 85L + 24P,$$
$$F_4 = 665H + 665L + 116P,$$
$$T_1 = -2051H^2 + 1584K^2 - 4894HL - 1259L^2,$$
$$T_2 = -373481H^2 + 994704K^2 - 1244314HL + 123871L^2,$$
$$T_3 = -105177H^2 + 36368K^2 - 228538HL - 86993L^2,$$
$$T_4 = -215747339H^2 + 134963680K^2 - 498976518HL - 148265499L^2.$$

If $F_0 = 0$, then the Lyapunov quantities G_{26}, G_{32} and G_{38} are equal to zero.

If $F_0 \neq 0$, then $G_{26} = 0$ if and only if $F_1F_2F_3F_4 = 0$. If at least two of polynomials $F_i, i = \overline{1,4}$, are equal to zero, then $H + L = 0$ and $P = 0$ which implies $G_{32} = G_{38} = 0$. Moreover, this implies also $F_0 = 0$.

We claim that even the equality with zero of only one of the polynomials $F_i, i = \overline{1,4}$, together with $G_{32} = G_{38} = 0$ also implies $F_0 = 0$. For the vanishing of G_{26}, we consider the following four cases:

1. $F_1 = 45H + 45L + 8P = 0$ with $F_2, F_3, F_4 \neq 0$,

2. $F_2 = 35H + 35L + 24P = 0$ with $F_1, F_3, F_4 \neq 0$,

3. $F_3 = 85H + 85L + 24P = 0$ with $F_1, F_2, F_4 \neq 0$ and

4. $F_4 = 665H + 665L + 116P = 0$ with $F_1, F_2, F_3 \neq 0$.

Case 1. Let $F_1 = 45H + 45L + 8P = 0$ and $F_2, F_3, F_4 \neq 0$. In this case

$$G_{32} = 3F_0F_2F_3F_4(H+L)T_1/36700160000$$

and for the vanishing of G_{32} we have the following subcases:

1.1. $H + L = 0$ and

1.2. $T_1 = 0$.

Subcase 1.1. If $H + L = 0$ then together with the condition $F_1 = 45H + 45L + 8P = 0$ it leads to $P = 0$, which implies the comitant $S_4 \equiv 0$. In this case the system has a center at the origin of coordinates.

Subcase 1.2. If $T_1 = 0$, then G_{38}, up to a numerical factor, has the form $G_{38} = F_0F_2F_3F_4(H+L)^5$. Notice that the Lyapunov quantity G_{38} can be nonzero and this implies that the condition $G_{38} = 0$ is a necessary condition for the existence of a center at the origin of coordinates. The condition $G_{38} = 0$ implies $H + L = 0$ then together with the condition $F_1 = 45H + 45L + 8P = 0$ it leads to $P = 0$. In this case the system has a center at the origin of coordinates.

So, in this case for the existence of a center at the origin of coordinates of the phase plane of system (22) the vanishing of Lyapunov quantities G_{26}, G_{32} and G_{38} is necessary , which implies

$$F_0 = K(-3H^2 + 16K^2 + 18HL - 27L^2)P = 0.$$

This condition is equivalent with the following invariant condition

$$J_6 = 16K(-3H^2 + 16K^2 + 18HL - 27L^2)P^5 = 0.$$

Cases 2, 3 and 4 can be analyzed by the same way described above and it leads to the same result. So, we obtain that for the existence of a center at the origin of coordinates of the phase plane of system (21) the realization of the conditions:

$$G_8 = G_{26} = G_{32} = G_{38} = 0$$

is necessary, which leads to the invariant conditions:

$$J_5 = J_6 = 0.$$

Sufficiency. In proving the necessity, it was established that the condition

$$KP\left[(16K^2 - 3(H - 3L)^2\right] = 0 \tag{23}$$

is the necessary one for the existence of a center at the origin of coordinates for the system (22). Next we prove the sufficiency of this condition. Condition (23) is verified if one of the following equalities is fulfilled:

$$(i)\ P = 0; \quad (ii)\ K = 0; \quad (iii)\ K = \frac{\sqrt{3}}{4}(H - 3L); \quad (iv)\ K = -\frac{\sqrt{3}}{4}(H - 3L).$$

Case (i). If $P = 0$, then $S_4 \equiv 0$ and the point $(0; 0)$ is a singular point of center type for the system (22). This case was analyzed above.

Case (ii). If $K = 0$, then in this case the system (22) takes the form:

$$\begin{aligned} \frac{dx}{dt} &= y + \frac{5H + 4P}{5}x^4 + \frac{30L - 12P}{5}x^2y^2 - (H + 2L)y^4, \\ \frac{dy}{dt} &= -x + \frac{4P - 20H}{5}x^3y - \frac{12P + 20L}{5}xy^3. \end{aligned} \tag{24}$$

For the system (24), the condition

$$\mathbf{Q}(-x; y)\mathbf{P}(x; y) = -\mathbf{P}(-x; y)\mathbf{Q}(x; y) \tag{25}$$

is fulfilled, i.e. the straight line defined by the equation $x = 0$ is a symmetry axis for the system (24). So, the point $(0; 0)$ is a singular point of center type for the system (24), i.e. for the system (22) with $K = 0$.

Case (iii). If $K = \frac{\sqrt{3}}{4}(H - 3L)$, then the system (22) takes the form

$$\begin{aligned} \frac{dx}{dt} &= y + \frac{5H+4P}{5}x^4 + (\sqrt{3}H - 3\sqrt{3}L)x^3y + \frac{30L-12P}{5}x^2y^2 - \\ &\quad (\sqrt{3}H - 3\sqrt{3}L)xy^3 - (H+2L)y^4, \\ \frac{dy}{dt} &= -x + \frac{\sqrt{3}H - 3\sqrt{3}L}{4}x^4 + \frac{4P-20H}{5}x^3y - \frac{3\sqrt{3}H - 9\sqrt{3}L}{2}x^2y^2 - \\ &\quad \frac{12P+20L}{5}xy^3 + \frac{\sqrt{3}H - 3\sqrt{3}L}{4}y^4. \end{aligned} \tag{26}$$

The trajectories of the system (26) are symmetric with respect to the straight line defined by the equation $x - \sqrt{3}y = 0$. With the rotation of axes

$$x_1 = x\cos\alpha + y\sin\alpha, \quad y_1 = -x\sin\alpha + y\cos\alpha \tag{27}$$

with the angle $\alpha = -\frac{\pi}{3}$, the system (26) becomes as follows:

$$\begin{aligned} \frac{dx_1}{dt} &= y_1 - \frac{5H+45L+16P}{20}{x_1}^4 + \frac{-45H+75L+24P}{10}{x_1}^2{y_1}^2 + \frac{7H-L}{4}{y_1}^4, \\ \frac{dy_1}{dt} &= -x_1 + \frac{5H+45L-4P}{5}{x_1}^3y_1 + \frac{15H-25L+12P}{5}x_1{y_1}^3. \end{aligned} \tag{28}$$

For the system (28) the condition (25) is verified in coordinates of x_1 and y_1, i.e. the straight line defined by the equation $x_1 = 0$ is a symmetry axis for the system (28). Therefore, it follows that the straight line defined by the equation $x - \sqrt{3}y = 0$ is the symmetry axis for the system (26). So, the point $(0;0)$ is a singular point of center type for the system (26), or for the system (22) with $K = \frac{\sqrt{3}}{4}(H - 3L)$.

Case (iv). If $K = -\frac{\sqrt{3}}{4}(H - 3L)$, then the system (22) takes the form

$$\begin{aligned} \frac{dx}{dt} &= y + \frac{5H+4P}{5}x^4 - (\sqrt{3}H - 3\sqrt{3}L)x^3y + \frac{30L-12P}{5}x^2y^2 + \\ &\quad (\sqrt{3}H - 3\sqrt{3}L)xy^3 - (H+2L)y^4, \\ \frac{dy}{dt} &= -x - \frac{\sqrt{3}H - 3\sqrt{3}L}{4}x^4 + \frac{4P-20H}{5}x^3y + \frac{3\sqrt{3}H - 9\sqrt{3}L}{2}x^2y^2 - \\ &\quad \frac{12P+20L}{5}xy^3 - \frac{\sqrt{3}H - 3\sqrt{3}L}{4}y^4. \end{aligned} \tag{29}$$

The trajectories of system (29) are symmetric with respect to the straight line defined by the equation $x + \sqrt{3}y = 0$. With the rotation of axes (27) with the angle $\alpha = \frac{\pi}{3}$, the system (29) becomes like the system (28), for which the line defined by the equation $x_1 = 0$ is a symmetry axis. So, the point $(0;0)$ is a singular point of center type for the system (29), or for the system (22) with $K = -\frac{\sqrt{3}}{4}(H - 3L)$.

In such a way the conditions

$$G_8 = G_{26} = G_{32} = G_{38} = 0 \tag{30}$$

or the invariant conditions

$$J_5 = J_6 = 0 \tag{31}$$

are sufficient conditions for the existence of a singular point of center type at the origin of coordinates for the system (21). Because G_8, G_{26}, G_{32}, G_{38}, J_5, J_6 are $GL(2,\mathbb{R})$-invariants and the system (21) was obtained from system (1), with conditions $S_1 = 0$, $I_2 > 0$, $I_3 = I_4 = 0$, by linear transformation and time scaling, it follows that the conditions (30) and (31) are necessary and sufficient for the existence of a singular point of center type at the origin of coordinates for the system (1) with $S_1 = 0$, $I_2 > 0$ and $I_3 = I_4 = 0$.

Theorems 1 is proved.

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