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Stochastic Games on Markov Processes with Final Sequence of States

Alexandru Lazari

Abstract. In this paper a class of stochastic games, defined on Markov processes with final sequence of states, is investigated. In these games each player, knowing the initial distribution of the states, defines his stationary strategy, represented by one proper transition matrix. The game is started by first player and, at every discrete moment of time, the stochastic system passes to the next state according to the strategy of the current player. After the last player, the first player acts on the system evolution and the game continues in this way until, for the first time, the given final sequence of states is achieved. The player who acts the last on the system evolution is considered the winner of the game. In this paper we prove that the distribution of the game duration is a homogeneous linear recurrence and we determine the initial state and the generating vector of this recurrence. Based on these results, we develop polynomial algorithms for determining the main probabilistic characteristics of the game duration and the win probabilities of players. Also, using the signomial and geometric programming approaches, the optimal cooperative strategies that minimize the expectation of the game duration are determined.

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1 Introduction and Problem Formulation

Let L be a discrete stochastic system with finite set of states V, $|V| = \omega$. At every discrete moment of time $t \in \mathbb{N}$, the state of the system is $v(t) \in V$. The system L starts its evolution from the state v with the probability $p^*(v)$, for all $v \in V$, where $\sum_{v\in V} p^*(v) = 1$.

Also, the transition from one state $u \in V$ to another state $v \in V$ is performed according to the probability $p(u,v) \in [0,1]$ such that $\sum_{v\in V} p(u,v) = 1$, $\forall u \in V$. Additionally we assume that a sequence of states $X = (x_1, x_2, \ldots, x_m) \in V^m$ is given and the stochastic system stops transitions as soon as the states $x_1, x_2, \ldots, x_m$ are reached consecutively in given order. The time T when the system stops is called evolution time of the system L with given final sequence of states X.

The stochastic system L, described above, represents a Markov process with final sequence of states X. Several interpretations of these Markov processes were analyzed in 1981 by Leo J. Guibas and Andrew M. Odlyzko in [10] and by G. Zbaganu in 1992 in [9]. Various problems, related to such systems, have been studied in [1]–[6].

Also, in these papers, polynomial algorithms for determining the main probabilistic characteristics (expectation, variance, mean square deviation, n-order moments) of evolution time of the given stochastic system L were proposed.

Next, in this paper, a generalization of this problem is studied. The following game is considered. Initially, each player $\mathcal{P}_\ell$ defines his stationary strategy, represented by one transition matrix $(p^{(\ell)}(u,v))_{u,v\in V}$, $\ell=\overline{0,r-1}$. The initial distribution of states is established according to the given distribution $(p^*(v))_{v\in V}$.

The game is started by first player $\mathcal{P}_0$. At every moment of time, the stochastic system passes consecutively to the next state according to the strategy of the current player. After the last player $\mathcal{P}_{r-1}$, the first player $\mathcal{P}_0$ acts on the system evolution and the game continues in this way until the given final sequence of states X is achieved. The player $\mathcal{P}_{T mod\ r}$ who acts the last on the system evolution is considered the winner of the game.

Our goal is to study the duration T of this game, knowing the initial distribution of states $p^{*(\ell)}=(p^{*(\ell)}(v))_{v\in V}$, the stationary strategy $P^{(\ell)}=(p^{(\ell)}(u,v))_{u,v\in V}$ of each player $\mathcal{P}_\ell$, $\ell=\overline{0,r-1}$, and the final sequence of states X of the stochastic system L. We will prove that the distribution of the game duration T is a homogeneous linear recurrence ([2], [7]) and we will develop a polynomial algorithm to determine the initial state and the generating vector of this recurrence. Having the generating vector and the initial state of the recurrence, we can use the related algorithm from [2], which was mentioned above, for determining the main probabilistic characteristics of the game duration. Also, based on these results, we will show how to determine the win probabilities of players.

2 Scientific Prerequisites

The developed algorithms for probabilistic characterization of the game duration and for determining the win probabilities of players are based on the theory of homogeneous linear recurrences.

2.1 Main Properties of Homogeneous Linear Recurrences

In this section we remind several properties of these recurrences, proved and described in [1], [2] and [6], that will be helpful in the following analysis from this paper.

The sequence $a=\{a_n\}_{n=0}^{\infty}$ represents a homogeneous linear m-recurrence on the set K if $\exists q=(q_k)_{k=0}^{m-1}\in K^m$ such that $a_n=\sum\limits_{k=0}^{m-1} q_k a_{n-1-k}$, $\forall n\geq m$, where q is the generating vector and $I_m^{[a]}=(a_n)_{n=0}^{m-1}$ is the initial state of the sequence a. The recurrence a is called non-degenerate when $|q_{m-1}|\neq 0$ and degenerate otherwise. Also, a is a homogeneous linear recurrence on the set K if $\exists m\in\mathbb{N}^*$ such that a is a homogeneous linear m-recurrence on the set K.

We denote by $Rol[K]$ (respectively $Rol[K][m]$) the set of non-degenerate homogeneous linear (m-)recurrences on the set K. The set $G[K](a)$ (respectively

$G[K][m](a)$) represents the set of generating vectors (of length m) of the sequence $a \in Rol[K]$ (respectively $a \in Rol[K][m]$).

The function $G^{[a]}(z) = \sum_{n=0}^{\infty} a_n z^n$ represents the generating function of the sequence $a = (a_n)_{n=0}^{\infty}$ and the function $G_t^{[a]}(z) = \sum_{n=0}^{t-1} a_n z^n$ represents the partial generating function of order t of the sequence a. We consider the unit characteristic polynomial $H_m^{[q]}(z) = 1 - zG_m^{[q]}(z)$. For an arbitrary non-zero α, the polynomial $H_{m,\alpha}^{[q]}(z) = \alpha H_m^{[q]}(z)$ represents a characteristic polynomial of the sequence a of order m. We denote by $H[K](a)$ (respectively $H[K][m](a)$) the set of characteristic polynomials (of order m) of the sequence $a \in Rol[K]$ (respectively $a \in Rol[K][m]$).

In the case when we will operate with arbitrary recurrence (not obligatory non-degenerate) for the corresponding set we will use the similar notation and will specify it with the mark "*", i.e. we will denote respectively sets by $Rol^*[K][m]$, $Rol^*[K]$, $G^*[K][m](a)$, $G^*[K](a)$, $H^*[K][m](a)$ and $H^*[K](a)$.

The sequence $a \in Rol^*[K]$ is called m-minimal on the set K if $a \in Rol^*[K][m]$ and $a \notin Rol^*[K][t]$, for all $t < m$. The number m is called the dimension of sequence a on the set K (denoted $dim[K](a) = m$).

Next, we will consider a subfield K of the field of complex numbers $\mathbb{C}$ and $a = \{a_n\}_{n=0}^{\infty} \subseteq \mathbb{C}$. The following Theorem allows us to determine the generating function $G^{[a]}(z)$ of an arbitrary homogeneous linear recurrence a on the set $\mathbb{C}$.

Theorem 1. *If $a \in Rol^*[\mathbb{C}][m]$ and $q \in G^*[\mathbb{C}][m](a)$, then*

$$G^{[a]}(z) = \frac{G_m^{[a]}(z) - \sum_{k=0}^{m-1} q_k z^{k+1} G_{m-1-k}^{[a]}(z)}{H_m^{[q]}(z)}.$$

Also the inverse theorem is true:

Theorem 2. *If $G^{[a]}(z) = \dfrac{A(z)}{B(z)}$ is a rational fraction, $B(z) = 1 - z\sum_{k=0}^{m-1} q_k z^k$ and $q_k \in K$, $k = \overline{0, m-1}$, then $a \in Rol^*[K][t+1]$ and $B(z) \in H^*[K][t+1](a)$, where $t = deg(A(z))$.*

The function $L.C.M.$ means the least common multiple of respective polynomials. An algebraic property of a linear combination is:

Theorem 3. *Let $a^{(j)} \in Rol[K]$, $P_j(z) \in H[K](a^{(j)})$, $\alpha_j \in \mathbb{C}$, $j = \overline{1, t}$. Then $a = \sum_{k=1}^{t} \alpha_k a^{(k)} \in Rol[K]$ and $P(z) = L.C.M.(P_1(z), P_2(z), \ldots, P_t(z)) \in H[K](a)$.*

A homogeneous linear recurrence property of polynomials is:

Theorem 4. *For each polynomial $P(X) \in \mathbb{C}[X]$ of degree $deg(P(X)) = m$, $c = (P(n))_{n=0}^{\infty} \in Rol[\mathbb{R}][m+1]$ and $Q(z) = (1-z)^{m+1} \in H[\mathbb{R}][m+1](c)$.*

The following Theorem shows us that the product of a homogeneous linear m-recurrence and a geometric progression is also a homogeneous linear m-recurrence:

Theorem 5. *We consider $a \in Rol[K][m]$, $b \in Rol[K][1]$, $(q_0) \in G[K][1](b)$ and $P(z) \in H[K][m](a)$. Then $ab = (a_n b_n)_{n=0}^{\infty} \in Rol[K][m]$ and $P(q_0 z) \in H[K][m](ab)$.*

The direct formula for homogeneous linear recurrences is given by the following theorem:

Theorem 6. *Let $a \in Rol[K][m]$, $q \in G[K][m](a)$, $H_{m,\alpha}^{[q]}(z) = \prod_{k=0}^{p-1}(z - z_k)^{s_k}$, $z_i \neq z_j$, $\forall i \neq j$. Then $a_n = I_m^{[a]} \cdot ((B^{[a]})^T)^{-1} \cdot (\beta_n^{[a]})^T$, $\forall n \in \mathbb{N}$, where $\beta_i^{[a]} = \left(\dfrac{\tau_{ij}}{z_k^i}\right)_{k=\overline{0,p-1},\ j=\overline{0,s_k-1}}$, $\tau_{ij} = \begin{cases} i^j, & if\ i^2+j^2 \neq 0 \\ 1, & if\ i=j=0 \end{cases}$, $i \in \mathbb{N}$, $B^{[a]} = (\beta_i^{[a]})_{i=0}^{m-1}$.*

The dimension and the unique minimal generating vector of the sequence $a \in Rol^*[\mathbb{C}][m]$ can be determined by using the following minimization method:

Theorem 7. *If $a \in Rol^*[\mathbb{C}][m]$ is a sequence with at least one non-zero element, then $dim[\mathbb{C}](a) = R$ and $q = (q_0, q_1, \ldots, q_{R-1}) \in G^*[\mathbb{C}][R](a)$, where*

$$R = rank(A_m^{[a]}),\ A_n^{[a]} = (a_{i+j})_{i,j=\overline{0,n-1}},\ f_n^{[a]} = (a_k)_{k=\overline{n,2n-1}}, \forall n \geq 1$$

and the vector $x = (q_{R-1}, q_{R-2}, \ldots, q_0)$ represents the unique solution of the system $A_R^{[a]} x^T = (f_R^{[a]})^T$.

2.2 Subsequences of Homogeneous Linear Recurrences

Next, we will extend these properties with the following new results related to homogeneous linear recurrences. These results will be very important in the process of probabilistic characterization of the game duration and determination of the win probabilities of players.

The following two theorems analyze subsequences of degenerate and non-degenerate homogeneous linear recurrences.

Theorem 8. *If $a \in Rol[\mathbb{C}][m]$, then $b = (a_{cn+t})_{n=0}^{\infty} \in Rol[\mathbb{C}][m]$, $\forall c, t \in \mathbb{N}$, with a generating vector that does not depend on t.*

Proof. Let $a \in Rol[\mathbb{C}][m]$ with generating vector $u \in G[\mathbb{C}][m](a)$. We consider all distinct roots $z_0,\ z_1,\ \ldots,\ z_{p-1}$ (of corresponding multiplicity $s_0,\ s_1,\ \ldots,\ s_{p-1}$) of the characteristic polynomial $H_m^{[u]}(z)$. Let $b = (a_{cn+t})_{n=0}^{\infty}$, where c and t are two fixed nonnegative integers.

We consider the decomposition $x^{[a]} = I_m^{[a]}((B^{[a]})^T)^{-1} = (A_{k,j})_{k=\overline{0,p-1},\ j=\overline{0,s_k-1}}$. Using Theorem 6, we have

$$a_n = x^{[a]}(\beta_n^{[a]})^T = \sum_{k=0}^{p-1}\sum_{j=0}^{s_k-1} A_{k,j}\frac{n^j}{z_k^n},\ n = \overline{0,\infty},$$

that implies

$$b_n = a_{cn+t} = \sum_{k=0}^{p-1}\sum_{j=0}^{s_k-1} A_{k,j}\frac{(cn+t)^j}{z_k^{cn+t}} = \sum_{k=0}^{p-1}\sum_{j=0}^{s_k-1} \alpha_{kjt}h_{kjtc}(n),$$

where $\alpha_{kjt} = \dfrac{A_{k,j}}{z_k^t}$ and $h_{kjtc}(n) = \dfrac{(cn+t)^j}{(z_k^c)^n}$, $k = \overline{0, p-1}$, $j = \overline{0, s_k-1}$, $n \in \mathbb{N}$.

Since $h_{jtc} = ((cn+t)^j)_{n=0}^{\infty}$ is a sequence of polynomials of degree j, applying Theorem 4, we have $h_{jtc} \in Rol[\mathbb{C}][j+1]$ with characteristic polynomial $(1-z)^{j+1} \in H[\mathbb{C}](h_{jtc})$, $j = \overline{0, s_k-1}$. Also, because $g_{kc} = \left(\dfrac{1}{(z_k^c)^n}\right)_{n=\overline{0,\infty}}$ is a geometric progression with common ratio $\dfrac{1}{z_k^c}$, we have $g_{kc} \in Rol[\mathbb{C}][1]$ with generating vector $\left(\dfrac{1}{z_k^c}\right) \in G[\mathbb{C}](g_{kc})$, $k = \overline{0, p-1}$. From these relations, applying Theorem 5, we obtain $h_{kjtc} = (h_{kjtc}(n))_{n=\overline{0,\infty}} = h_{jtc} \cdot g_{kc} \in Rol[\mathbb{C}][j+1]$ with characteristic polynomial $\left(1 - \dfrac{z}{z_k^c}\right)^{j+1} \in H[\mathbb{C}](h_{kjtc})$, $k = \overline{0, p-1}$, $j = \overline{0, s_k-1}$.

Next, using Theorem 3, for $k = \overline{0, p-1}$ we have

$$f_{ktc} = \sum_{j=0}^{s_k-1} \alpha_{kjt}h_{kjtc} \in Rol[\mathbb{C}][s_k]$$

with characteristic polynomial

$$L.C.M.\left(\left\{\left(1 - \frac{z}{z_k^c}\right)^{j+1} \mid j = \overline{0, s_k-1}\right\}\right) = \left(1 - \frac{z}{z_k^c}\right)^{s_k} \in H[\mathbb{C}](f_{ktc}).$$

Since $b_n = \sum\limits_{k=0}^{p-1} f_{ktc}(n)$, where $f_{ktc} = (f_{ktc}(n))_{n=0}^{\infty}$, applying Theorem 3, we obtain $b \in Rol[\mathbb{C}][m]$ with characteristic polynomial

$$L.C.M.\left(\left\{\left(1 - \frac{z}{z_k^c}\right)^{s_k} \mid k = \overline{0, p-1}\right\}\right) = \prod_{k=0}^{p-1}\left(1 - \frac{z}{z_k^c}\right)^{s_k} \in H[\mathbb{C}][m](b).$$

It is easy to observe that this characteristic polynomial does not depend on t. So, also the corresponding generating vector does not depend on t. In conclusion, $\forall c, t \in \mathbb{N}$, we have $b = (a_{cn+t})_{n=0}^{\infty} \in Rol[\mathbb{C}][m]$ with a generating vector that does not depend on t. □

Theorem 9. *If $a \in Rol^*[\mathbb{C}][m]$, then $b = (a_{cn+t})_{n=0}^{\infty} \in Rol^*[\mathbb{C}][m]$, $\forall c, t \in \mathbb{N}$, with a generating vector that does not depend on t.*

Proof. Let $a \in Rol^*[\mathbb{C}][m]$ with generating vector $u = (u_k)_{k=0}^{m-1} \in G^*[\mathbb{C}][m](a)$. Let s be the degree of the characteristic polynomial $H_m^{[u]}(z)$.

We consider the subsequence $\alpha = (\alpha_n)_{n=0}^{\infty}$, where $\alpha_n = a_{n+m-s}$, $\forall n \geq 0$. It is easy to observe that $\alpha \in Rol[\mathbb{C}][s]$ with generating vector $u^{(s)} \in G[\mathbb{C}][s](\alpha)$, where $u^{(s)} = (u_k)_{k=0}^{s-1}$. Applying Theorem 8, we have $\beta = (\beta_n)_{n=0}^{\infty} \in Rol[\mathbb{C}][s]$, $\forall c, t \in \mathbb{N}$, with a generating vector $v^{(s)} = (v_k)_{k=0}^{s-1} \in G[\mathbb{C}][s](\beta)$ that does not depend on t, where $\beta_n = \alpha_{cn+t} = a_{cn+t+m-s}$, $\forall n \geq 0$. From this relation, we obtain that $b = (b_n)_{n=0}^{\infty} \in Rol^*[\mathbb{C}][m]$ with a generating vector $v = (v_k)_{k=0}^{m-1} \in G^*[\mathbb{C}][m](b)$ that does not depend on t, where $b_n = a_{cn+t}$, $\forall n \geq 0$, $\forall c, t \in \mathbb{N}$ and $v_k = 0$, $k = \overline{s, m-1}$. □

2.3 Five-Dimensional Homogeneous Linear Recurrences

The following theorem analyzes homogeneous linear recurrences on the set of squared matrices with squared matrices as components:

Theorem 10. *If* $a \subseteq (\mathbb{C}^r)^t$ *and* $a \in Rol^*[\mathcal{M}_t(\mathcal{M}_r(K))][m]$*, then* $a \in Rol^*[K][mtr]$.

Proof. Let $a \subseteq (\mathbb{C}^r)^t$, $a \in Rol^*[\mathcal{M}_t(\mathcal{M}_r(K))][m]$ and $q \in G^*[\mathcal{M}_t(\mathcal{M}_r(K))][m]$. We have $a_n = \sum\limits_{k=0}^{m-1} q^{(k)} a_{n-1-k}$, $\forall n \geq m$, where $a = (a_n)_{n=0}^{\infty}$ and $q = (q^{(k)})_{k=0}^{m-1}$.

We consider the set $\Lambda(n) = \{0, 1, \ldots, n-1\}$, $\forall n \in \mathbb{N}$. Let $a_n = (a_{ni})_{i \in \Lambda(t)}$, where $a_{ni} = (a_{nij})_{j \in \Lambda(r)}$, $i \in \Lambda(t)$, $\forall n \in \mathbb{N}$. We obtain the recurrence relation

$$a_{ni} = \sum_{k=0}^{m-1} \sum_{s=0}^{t-1} q_{is}^{(k)} a_{n-1-k,s}, \ i \in \Lambda(t), \ \forall n \geq m,$$

where $q^{(k)} = (q_{is}^{(k)})_{i,s \in \Lambda(t)}$, $k \in \Lambda(m)$. This formula implies the recurrence relation

$$a_{nij} = \sum_{k=0}^{m-1} \sum_{s=0}^{t-1} \sum_{\ell=0}^{r-1} q_{isj\ell}^{(k)} a_{n-1-k,s,\ell}, \ i \in \Lambda(t), \ j \in \Lambda(r),$$

where $q_{is}^{(k)} = (q_{isj\ell}^{(k)})_{j,\ell \in \Lambda(r)}$, $i, s \in \Lambda(t)$.

Let $a^{(i,j)} = (a_{nij})_{n=0}^{\infty}$ and $q_{isj\ell} = (q_{isj\ell}^{(k)})_{k=0}^{m-1}$, $i, s \in \Lambda(t)$, $j, \ell \in \Lambda(r)$. We will determine the generating function of the sequence $a^{(i,j)}$, $i \in \Lambda(t)$, $j \in \Lambda(r)$.

$$G^{[a^{(i,j)}]}(z) = \sum_{n=0}^{\infty} a_{nij} z^n = \sum_{n=m}^{\infty} z^n \sum_{k=0}^{m-1} \sum_{s=0}^{t-1} \sum_{\ell=0}^{r-1} q_{isj\ell}^{(k)} a_{n-1-k,s,\ell} + \sum_{n=0}^{m-1} a_{nij} z^n =$$

$$= G_m^{[a^{(i,j)}]}(z) + \sum_{k=0}^{m-1} \sum_{s=0}^{t-1} \sum_{\ell=0}^{r-1} q_{isj\ell}^{(k)} z^{k+1} \sum_{n=m}^{\infty} a_{n-1-k,s,\ell} z^{n-1-k} =$$

$$= G_m^{[a^{(i,j)}]}(z) + \sum_{k=0}^{m-1} z^{k+1} \sum_{s=0}^{t-1} \sum_{\ell=0}^{r-1} q_{isj\ell}^{(k)} \sum_{n=m-1-k}^{\infty} a_{ns\ell} z^n =$$

$$= G_m^{[a^{(i,j)}]}(z) + \sum_{k=0}^{m-1} z^{k+1} \sum_{s=0}^{t-1}\sum_{\ell=0}^{r-1} q_{isj\ell}^{(k)} \left(G^{[a^{(s,\ell)}]}(z) - G_{m-1-k}^{[a^{(s,\ell)}]}(z) \right) =$$

$$= \left(G_m^{[a^{(i,j)}]}(z) - \sum_{k=0}^{m-1} z^{k+1} \sum_{s=0}^{t-1}\sum_{\ell=0}^{r-1} q_{isj\ell}^{(k)} G_{m-1-k}^{[a^{(s,\ell)}]}(z) \right) +$$

$$+ z \sum_{s=0}^{t-1}\sum_{\ell=0}^{r-1} G^{[a^{(s,\ell)}]}(z) \sum_{k=0}^{m-1} q_{isj\ell}^{(k)} z^k =$$

$$= F_{ij}(z) + z \sum_{s=0}^{t-1}\sum_{\ell=0}^{r-1} G_m^{[q_{isj\ell}]} G^{[a^{(s,\ell)}]}(z),$$

where $F_{ij}(z) = G_m^{[a^{(i,j)}]}(z) - \sum_{k=0}^{m-1} z^{k+1} \sum_{s=0}^{t-1}\sum_{\ell=0}^{r-1} q_{isj\ell}^{(k)} G_{m-1-k}^{[a^{(s,\ell)}]}(z)$.

So, for $i \in \Lambda(t)$ and $j \in \Lambda(r)$, we have

$$G^{[a^{(i,j)}]}(z) - z \sum_{s\in\Lambda(t),\ \ell\in\Lambda(r)} G_m^{[q_{isj\ell}]} G^{[a^{(s,\ell)}]}(z) = F_{ij}(z).$$

If we denote $x_{ij} = G^{[a^{(i,j)}]}(z)$, $i \in \Lambda(t)$, $j \in \Lambda(r)$, we obtain the following system of tr linear equations with tr unknown variables:

$$x_{ij}(z) - z \sum_{s\in\Lambda(t),\ \ell\in\Lambda(r)} G_m^{[q_{isj\ell}]} x_{s,\ell}(z) = F_{ij}(z),\ i \in \Lambda(t),\ j \in \Lambda(r).$$

In matrix form, this system can be written as follows:

$$W(z)x(z) = F(z),$$

where

$$x(z) = (x_{ij}(z))_{(i,j)\in\Lambda(t)\times\Lambda(r)},$$

$$F(z) = (F_{ij}(z))_{(i,j)\in\Lambda(t)\times\Lambda(r)},$$

$$Q = ((q_{(i,j),(s,\ell)}^{(k)})_{(i,j),(s,\ell)\in\Lambda(t)\times\Lambda(r)})_{k=0}^{m-1},$$

$$q_{(i,j),(s,\ell)}^{(k)} = q_{isj\ell}^{(k)},\ i,s \in \Lambda_t,\ j,\ell \in \Lambda_r,$$

$$W(z) = I - zG_m^{[Q]}(z).$$

So, we have $x(z) = W^{-1}(z)F(z)$, $\forall z \in D\backslash F$, where D is the domain of convergence of $G^{[a]}(z)$ and F is the set of roots of the polynomial $|W(z)|$. From this relation, we can conclude that $x_{ij}(z)$ are rational fractions, $\forall i \in \Lambda(t)$, $\forall j \in \Lambda(r)$. Using Theorem 2, we have that $a^{(i,j)} \in Rol^*[K][mtr]$, $\forall i \in \Lambda(t)$, $\forall j \in \Lambda(r)$, which implies also $a \in Rol^*[K][mtr]$. □

3 Game Duration

In this section we will determine the distribution law of the game duration T. We will prove that this distribution is a homogeneous linear recurrence.

3.1 Determining the Distribution of the Game Duration

Initially, we consider the sets $X_j = \{x_j\}$ and $\overline{X}_j = V \backslash X_j$, $j = \overline{1, m}$. Also, we consider the notations $\pi_j = p^*(x_j)$, $\pi_{ij}^{(\ell)} = p^{(\ell)}(x_i, x_j)$ and $\omega_j^{(\ell)} = \prod_{k=3}^{j} \pi_{k-1,\, k}^{(\ell\, \oplus (k-3))}$, for each $i, j = \overline{1, m}$ and $\ell = \overline{0, r-1}$, where $c \oplus d = (c + d) \mod r$, $\forall c, d \in \mathbb{Z}$.

If for each $\ell = \overline{0, r-1}$ there exists an index $j_\ell \in \{2, \ldots, m\}$ such that $\pi_{j_\ell - 1,\, j_\ell}^{(\ell \oplus (j_\ell - 2))} = 0$, then the evolution of the stochastic system is not finite, i.e. the game duration is unlimited. In other words, in this case we have $a_n = 0$, $\forall n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = 1$. In conclusion, also the n-order moments of the game duration are infinite. Next, we investigate the rest of cases, when the game duration is finite.

We consider $\forall n \in \mathbb{Z}$. Let be $S(V) = \{A \mid A \subseteq V\}$. Denote by $P_\Phi^{(\ell)}(n)$ the probability that $T = n$, $v(j) \in \Phi_j$, $j = \overline{0, t-1}$ and the player $\mathcal{P}_\ell$ acts first, supposing that the initial state of the system is known, for all $\Phi = (\Phi_j)_{j=0}^{t-1} \in (S(V))^t$, $t \in \mathbb{N}$ and $\ell = \overline{0, r-1}$. We introduce the following functions on $\mathbb{Z}$, $k = \overline{0, m}$, $\ell = \overline{0, r-1}$:

$$\begin{aligned} \alpha_k^{(\ell)}(n) &= P^{(\ell)}_{(X_1, X_2, \ldots, X_{k-1}, \overline{X}_k)}(n), \\ \beta_k^{(\ell)}(n) &= P^{(\ell)}_{(X_1, X_2, \ldots, X_k)}(n), \\ \gamma_k^{(\ell)}(n) &= P^{(\ell)}_{(X_2, X_3, \ldots, X_k)}(n). \end{aligned} \tag{1}$$

Also, we consider the sets

$$T_s = \{s+1\} \cup \{t \in \{2, 3, \ldots, s\} \mid x_{t-1+j} = x_j,\ j = \overline{1, s+1-t}\},\ s = \overline{1, m}.$$

The minimal elements from these sets are

$$t_s = \min_{k \in T_s} k,\ s = \overline{1, m}. \tag{2}$$

The value t_s represents the auto superposition level of the sequence $(x_1, x_2, \ldots, x_s)$, i.e. t_s is the position in the sequence $(x_1, x_2, \ldots, x_s)$ starting with which, if we overlap the same sequence, the superposed elements are equal.

Initially, we study the case $m \geq 2$. We have

$$\beta_k^{(\ell)}(n) = P^{(\ell)}_{(X_1, X_2, \ldots, X_k)}(n) = P_n^{(\ell)} - \sum_{j=1}^{k} \alpha_j^{(\ell)}(n),\ k = \overline{0, m}, \ell = \overline{0, r-1}, \tag{3}$$

where $P_n^{(\ell)} = P^{(\ell)}_{(\ )}(n)$, $\ell = \overline{0, r-1}$.

Directly from definition we obtain

$$\gamma_1^{(\ell)}(n) = P^{(\ell)}(n),\ \ell = \overline{0, r-1},\ \forall n \in \mathbb{Z}. \tag{4}$$

Let be $s \geq 2$. For $t_s \leq s$ and $\ell = \overline{0, r-1}$ we have

$$\gamma_s^{(\ell)}(n) = P^{(\ell)}_{(X_2, X_3, \ldots, X_s)}(n) = \pi^{(\ell)}_{2,3}\pi^{(\ell\ \oplus 1)}_{3,4} \ldots \pi^{(\ell\ \oplus (t_s-3))}_{t_s-1,\ t_s} P^{(\ell\ \oplus (t_s-2))}_{(X_{t_s}, \ldots X_s)}(n - t_s + 2) =$$

$$= \omega^{(\ell)}_{t_s} P^{(\ell\ \oplus (t_s-2))}_{(X_1, \ldots, X_{s+1-t_s})}(n - t_s + 2) = \omega^{(\ell)}_{t_s} \beta^{(\ell\ \oplus (t_s-2))}_{s+1-t_s}(n - t_s + 2) =$$

$$= \omega^{(\ell)}_{t_s} \left(P^{(\ell\ \oplus (t_s-2))}(n - t_s + 2) - \sum_{j=1}^{s+1-t_s} \alpha_j^{(\ell\ \oplus (t_s-2))}(n - t_s + 2) \right) \tag{5}$$

and in the case $t_s = s+1$, for $\ell = \overline{0, r-1}$, we obtain

$$\gamma_s^{(\ell)}(n) = P^{(\ell)}_{(X_2, X_3, \ldots, X_s)}(n) = \omega_s^{(\ell)} \sum_{y \in V} p^{(\ell\ \oplus (s-2))}(x_s, y) P^{(\ell\ \oplus (s-1))}_{(\{y\})}(n - s + 1) =$$

$$= \sum_{y \in V} \omega_s^{(\ell)} p^{(\ell\ \oplus (s-2))}(x_s, y) P^{(\ell\ \oplus (s-1))}_{(\{y\})}(n - t_s + 2). \tag{6}$$

Next, we determine the values $\alpha_k^{(\ell)}(n)$, $k = \overline{1, m}$, $\ell = \overline{0, r-1}$. We have

$$\alpha_1^{(\ell)}(n) = P^{(\ell)}_{(\overline{X}_1)}(n) = \sum_{x \in V \setminus \{x_1\}} P^{(\ell)}_{(\{x\})}(n) =$$

$$= \sum_{x \in V \setminus \{x_1\}} \sum_{y \in V} P^{(\ell)}_{(\{x\},\ \{y\})}(n) = \sum_{x \in V \setminus \{x_1\}} \sum_{y \in V} p^{(\ell)}(x, y) P^{(\ell\ \oplus 1)}_{\{y\})}(n-1) =$$

$$= \sum_{y \in V} P^{(\ell\ \oplus 1)}_{\{y\})}(n-1) \sum_{x \in V \setminus \{x_1\}} p^{(\ell)}(x, y) = \sum_{y \in V} \psi_1^{(\ell)}(y) P^{(\ell\ \oplus 1)}_{(\{y\})}(n-1), \tag{7}$$

where

$$\psi_1^{(\ell)}(y) = \sum_{x \in V \setminus \{x_1\}} p^{(\ell)}(x, y),\ \forall y \in V. \tag{8}$$

For $k = 2$ we obtain

$$\alpha_2^{(\ell)}(n) = P^{(\ell)}_{(X_1,\ \overline{X}_2)}(n) = \sum_{y \neq x_2} P^{(\ell)}_{(X_1,\ \{y\})}(n) = \sum_{y \neq x_2} p^{(\ell)}(x_1, y) P^{(\ell\ \oplus 1)}_{(\{y\})}(n-1) \tag{9}$$

and for $k \geq 3$ we have

$$\alpha_k^{(\ell)}(n) = P^{(\ell)}_{(X_1, X_2, \ldots, X_{k-1}, \overline{X}_k)}(n) = \pi^{(\ell)}_{1,2} P^{(\ell\ \oplus 1)}_{(X_2, X_3, \ldots, X_{k-1}, \overline{X}_k)}(n-1) =$$

$$= \pi^{(\ell)}_{1,2} \left(P^{(\ell\ \oplus 1)}_{(X_2, X_3, \ldots, X_{k-1})}(n-1) - P^{(\ell\ \oplus 1)}_{(X_2, X_3, \ldots, X_k)}(n-1) \right) =$$

$$= \pi^{(\ell)}_{1,2} \left(\gamma^{(\ell\ \oplus 1)}_{k-1}(n-1) - \gamma^{(\ell\ \oplus 1)}_{k}(n-1) \right). \tag{10}$$

From the following equality

$$P^{(\ell)}(n) = \sum_{k=1}^{m} \alpha_k^{(\ell)}(n) = \alpha_1^{(\ell)}(n) + \alpha_2^{(\ell)}(n) + \sum_{k=3}^{m} \alpha_k^{(\ell)}(n),\ \forall n \geq m, \tag{11}$$

using the relations (3), (9) and (10), we obtain the formula

$$P_{X_1}^{(\ell)}(n) = \beta_1^{(\ell)}(n) = P^{(\ell)}(n) - \alpha_1^{(\ell)}(n) = \alpha_2^{(\ell)}(n) + \sum_{k=3}^{m} \alpha_k^{(\ell)}(n) =$$

$$= \sum_{y \neq x_2} p^{(\ell)}(x_1, y) P_{(\{y\})}^{(\ell\, \oplus 1)}(n-1) + \sum_{k=3}^{m} \pi_{1,2}^{(\ell)} \left(\gamma_{k-1}^{(\ell\, \oplus 1)}(n-1) - \gamma_k^{(\ell\, \oplus 1)}(n-1) \right) =$$

$$= \sum_{y \neq x_2} p^{(\ell)}(x_1, y) P_{(\{y\})}^{(\ell\, \oplus 1)}(n-1) + \pi_{1,2}^{(\ell)} \left(\gamma_2^{(\ell\, \oplus 1)}(n-1) - \gamma_m^{(\ell\, \oplus 1)}(n-1) \right) =$$

$$= \sum_{y \in V} p^{(\ell)}(x_1, y) P_{(\{y\})}^{(\ell\, \oplus 1)}(n-1) - \pi_{1,2}^{(\ell)} \gamma_m^{(\ell\, \oplus 1)}(n-1), \ \forall n \geq m. \tag{12}$$

For $x \neq x_1$, we have

$$P_{(\{x\})}^{(\ell)}(n) = \sum_{y \in V} p^{(\ell)}(x, y) P_{(\{y\})}^{(\ell\, \oplus 1)}(n-1). \tag{13}$$

According to the relations (4) – (10), using the mathematical induction, we can prove that there exist real coefficients $u_{jk\ell}^{(i)}(y)$ and $v_{jk\ell}^{(i)}(y)$, $j = \overline{1, m}$, $k = \overline{0, j-1}$, $y \in V$, $\ell = \overline{0, r-1}$, $i = \overline{0, r-1}$ such that, for all $n \in \mathbb{Z}$, the following relations hold:

$$\begin{cases} \alpha_j^{(\ell)}(n) = \sum\limits_{i=0}^{r-1} \sum\limits_{k=0}^{j-1} \sum\limits_{y \in V} u_{jk\ell}^{(i)}(y) \ P_{(\{y\})}^{(i)}(n-1-k), \\ \gamma_j^{(\ell)}(n-1) = \sum\limits_{i=0}^{r-1} \sum\limits_{k=0}^{j-1} \sum\limits_{y \in V} v_{jk\ell}^{(i)}(y) \ P_{(\{y\})}^{(i)}(n-1-k). \end{cases} \tag{14}$$

For $n < m-1$ these relations are obvious and are true for all reals $u_{jk\ell}^{(i)}(y)$ and $v_{jk\ell}^{(i)}(y)$. We should prove these relations for $n \geq m$, using mathematical induction method on parameter j.

For $j = 1$ we have

$$\alpha_1^{(\ell)}(n) = \sum_{y \in V} \psi_1^{(\ell)}(y) P_{(\{y\})}^{(\ell\, \oplus 1)}(n-1) =$$

$$= \sum_{i=0}^{r-1} \sum_{k=0}^{0} \sum_{y \in V} u_{1k\ell}^{(i)}(y) P_{(\{y\})}^{(i)}(n-1-k) = \sum_{i=0}^{r-1} \sum_{y \in V} u_{10\ell}^{(i)}(y) P_{(\{y\})}^{(i)}(n-1),$$

where

$$u_{10\ell}^{(i)}(y) = \begin{cases} \psi_1^{(\ell)}(y), & if\ i = \ell\, \oplus 1 \\ 0, & if\ i \neq \ell\, \oplus 1 \end{cases} \tag{15}$$

and

$$\gamma_1^{(\ell)}(n-1) = P^{(\ell)}(n-1) = \sum_{y \in V} P_{(\{y\})}^{(\ell)}(n-1) = \sum_{i=0}^{r-1} \sum_{y \in V} v_{10\ell}^{(i)}(y) P_{(\{y\})}^{(i)}(n-1),$$

where

$$v^{(i)}_{10\ell}(y) = \begin{cases} 1, & if\ i = \ell \\ 0, & if\ i \neq \ell. \end{cases} \tag{16}$$

For $j = 2$ we obtain

$$\alpha^{(\ell)}_2(n) = \sum_{y\neq x_2} p^{(\ell)}(x_1, y)P^{(\ell\ \oplus 1)}_{(\{y\})}(n-1) = \sum_{i=0}^{r-1}\sum_{k=0}^{1}\sum_{y\in V} u^{(i)}_{2k\ell}(y)P^{(i)}_{(\{y\})}(n-1-k),$$

where

$$u^{(i)}_{20\ell}(y) = \begin{cases} 0, & if\ y = x_2 \\ 0, & if\ y \neq x_2\ and\ i \neq \ell\ \oplus 1 \\ p^{(\ell)}(x_1, y), & if\ y \neq x_2\ and\ i = \ell\ \oplus 1 \end{cases} \tag{17}$$

and

$$u^{(i)}_{21\ell}(y) = 0,\ \forall y \in V. \tag{18}$$

Also, we have

$$\gamma^{(\ell)}_2(n-1) = P^{(\ell)}_{(X_2)}(n-1) = \sum_{i=0}^{r-1}\sum_{k=0}^{1}\sum_{y\in V} v^{(i)}_{2k\ell}(y)P^{(i)}_{(\{y\})}(n-1-k),$$

where

$$v^{(i)}_{20\ell}(y) = \begin{cases} 0, & if\ y \neq x_2 \\ 0, & if\ y = x_2\ and\ i \neq \ell \\ 1, & if\ y = x_2\ and\ i = \ell \end{cases} \tag{19}$$

and

$$v^{(i)}_{21\ell}(y) = 0,\ \forall y \in V. \tag{20}$$

So, the relations are true for $\forall j \in \{1, 2\}$. Let these relations be true for $j = \overline{1, s-1}$, $s \geq 3$, $\forall n < \tau$ and $\forall y \in V$. We have

$$\alpha^{(\ell)}_s(\tau) = \pi^{(\ell)}_{1,2}\left(\gamma^{(\ell\ \oplus 1)}_{s-1}(\tau-1) - \gamma^{(\ell\ \oplus 1)}_s(\tau-1)\right) =$$

$$= \pi^{(\ell)}_{1,2}\left(\sum_{i=0}^{r-1}\sum_{k=0}^{s-2}\sum_{y\in V} v^{(i)}_{s-1,k,\ell}(y)P^{(i)}_{(\{y\})}(\tau-1-k) - \right.$$

$$\left. - \sum_{i=0}^{r-1}\sum_{k=0}^{s-1}\sum_{y\in V} v^{(i)}_{s,k,\ell}(y)P^{(i)}_{(\{y\})}(\tau-1-k)\right) =$$

$$= \sum_{i=0}^{r-1}\sum_{k=0}^{s-1}\sum_{y\in V} u^{(i)}_{sk\ell}(y)P^{(i)}_{(\{y\})}(\tau-1-k),$$

where

$$u^{(i)}_{sk\ell}(y) = \begin{cases} \pi^{(\ell)}_{1,2}(v^{(i)}_{s-1,k,\ell}(y) - v^{(i)}_{sk\ell}(y)), & if\ 0 \leq k \leq s-2 \\ -\pi^{(\ell)}_{1,2}v^{(i)}_{s,s-1,\ell}(y), & if\ k = s-1. \end{cases} \tag{21}$$

For $t_s \le s$ we obtain

$$\gamma_s^{(\ell)}(\tau-1) = \omega_{t_s}^{(\ell)} \left(\sum_{y\in V} P_{(\{y\})}^{(\ell\ \oplus (t_s-2))}(\tau - t_s + 1) - \sum_{j=1}^{s+1-t_s} \alpha_j^{(\ell\ \oplus (t_s-2))}(\tau - t_s + 1) \right) =$$

$$= \omega_{t_s}^{(\ell)} \left(\sum_{y\in V} P_{(\{y\})}^{(\ell\ \oplus (t_s-2))}(\tau - t_s + 1) - \right.$$

$$\left. - \sum_{j=1}^{s+1-t_s} \sum_{i=0}^{r-1} \sum_{k=0}^{j-1} \sum_{y\in V} u_{j,k,\ell\ \oplus (t_s-2)}^{(i)}(y) P_{(\{y\})}^{(i)}(\tau - t_s - k) \right) =$$

$$= \omega_{t_s}^{(\ell)} \left(\sum_{y\in V} P_{(\{y\})}^{(\ell\ \oplus (t_s-2))}(\tau - 1 - (t_s - 2)) - \right.$$

$$\left. - \sum_{i=0}^{r-1} \sum_{k=t_s-1}^{s-1} \sum_{y\in V} P_{(\{y\})}^{(i)}(\tau - 1 - k) \sum_{j=k-t_s+2}^{s+1-t_s} u_{j,k-t_s+1,\ell\ \oplus (t_s-2)}^{(i)}(y) \right) =$$

$$= \sum_{i=0}^{r-1} \sum_{k=0}^{s-1} \sum_{y\in V} v_{sk\ell}^{(i)}(y) P_{(\{y\})}^{(i)}(\tau - 1 - k),$$

where

$$v_{sk\ell}^{(i)}(y) = \begin{cases} 0, & if\ 0 \le k \le t_s - 3 \\ 0, & if\ k = t_s - 2\ and\ i \ne \ell\ \oplus (t_s - 2) \\ \omega_{t_s}^{(\ell)}, & if\ k = t_s - 2,\ and\ i = \ell\ \oplus (t_s - 2) \\ -\omega_{t_s}^{(\ell)} \sum\limits_{j=k-t_s+2}^{s+1-t_s} u_{j,\ k-t_s+1,\ \ell\ \oplus (t_s-2)}^{(i)}(y), & if\ t_s - 1 \le k \le s - 1 \end{cases} \qquad (22)$$

and for $t_s = s + 1$ we have

$$\gamma_s^{(\ell)}(\tau-1) = \sum_{y\in V} \omega_s^{(\ell)} p^{(\ell\ \oplus (s-2))}(x_s, y) P_{(\{y\})}^{(\ell\ \oplus (s-1))}(\tau - 1 - (t_s - 2)) =$$

$$= \sum_{i=0}^{r-1} \sum_{k=0}^{j-1} \sum_{y\in V} v_{sk\ell}^{(i)}(y) P_{(\{y\})}^{(i)}(\tau - 1 - k),$$

where

$$v_{sk\ell}^{(i)}(y) = \begin{cases} 0, & if\ 0 \le k \le s - 2 \\ 0, & if\ k = s - 1\ and\ i \ne \ell\ \oplus (s - 1) \\ \omega_s^{(\ell)} p^{(\ell\ \oplus (s-2))}(x_s, y), & if\ k = s - 1\ and\ i = \ell\ \oplus (s - 1). \end{cases} \qquad (23)$$

So, we proved the truth of the relations (14), obtaining the formulas (15) – (23) for determining coefficients of the decompositions. Substituting the decompositions (14) in the relations (12) and (13), we have

$$P^{(\ell)}_{(X_1)}(n) = \sum_{y\in V} p^{(\ell)}(x_1,y)P^{(\ell\,\oplus 1)}_{(\{y\})}(n-1) - \pi^{(\ell)}_{1,2}\gamma_m^{(\ell\,\oplus 1)}(n-1) =$$

$$= \sum_{y\in V} p^{(\ell)}(x_1,y)P^{(\ell\,\oplus 1)}_{(\{y\})}(n-1) - \pi^{(\ell)}_{1,2}\sum_{i=0}^{r-1}\sum_{k=0}^{m-1}\sum_{y\in V} v^{(i)}_{m,\;k,\;\ell\,\oplus 1}(y)P^{(i)}_{(\{y\})}(n-1-k) =$$

$$= \sum_{i=0}^{r-1}\sum_{k=0}^{m-1}\sum_{y\in V} w^{(i)}_{k,\ell}(x_1,y)P^{(i)}_{(\{y\})}(n-1-k)$$

and, for all $x \neq x_1$,

$$P^{(\ell)}_{(\{x\})}(n) = \sum_{y\in V} p^{(\ell)}(x,y)P^{(\ell\,\oplus 1)}_{(\{y\})}(n-1) = \sum_{i=0}^{r-1}\sum_{k=0}^{m-1}\sum_{y\in V} w^{(i)}_{k,\ell}(x,y)P^{(i)}_{(\{y\})}(n-1-k),$$

where

$$w^{(i)}_{k\ell}(x,y) = \begin{cases} p^{(\ell)}(x_1,y) - \pi^{(\ell)}_{1,2}v^{(\ell\,\oplus 1)}_{m,0,\ell\,\oplus 1}(y), & if\ x=x_1,\ k=0,\ i=\ell\,\oplus 1\\ -\pi^{(\ell)}_{1,2}v^{(i)}_{m,0,\ell\,\oplus 1}(y), & if\ x=x_1,\ k=0,\ i\neq\ell\,\oplus 1\\ -\pi^{(\ell)}_{1,2}v^{(i)}_{m,k,\ell\,\oplus 1}(y), & if\ x=x_1,\ 1\leq k\leq m-1\\ p^{(\ell)}(x,y), & if\ x\neq x_1,\ k=0,\ i=\ell\,\oplus 1\\ 0, & if\ x\neq x_1,\ k=0,\ i\neq\ell\,\oplus 1\\ 0, & if\ x\neq x_1,\ 1\leq k\leq m-1 \end{cases} \qquad (24)$$

Thus, we obtained the recurrence relation

$$P^{(\ell)}_{(\{x\})}(n) = \sum_{i=0}^{r-1}\sum_{k=0}^{m-1}\sum_{y\in V} w^{(i)}_{k\ell}(x,y)P^{(i)}_{(\{y\})}(n-1-k),\ \forall x\in V,\ \forall n\geq m,\ \ell=\overline{0,r-1}.$$

So, we have

$$P_{(\{x\})}(n) = \sum_{k=0}^{m-1}\sum_{y\in V} W_k(x,y)P_{(\{y\})}(n-1-k),\ \forall x\in V,\ \forall n\geq m,$$

where $W_k(x,y) = (w^{(i)}_{k\ell}(x,y))_{\ell,\ i=\overline{0,\ r-1}}$, $P_{(\{x\})}(n) = (P^{(\ell)}_{(\{x\})}(n))_{\ell\ =\overline{0,\ r-1}}$, $\forall x,y\in V$, $k=\overline{0,m-1}$. This recurrence relation can be written in the form

$$h_n = \sum_{k=0}^{m-1} W_k h_{n-1-k},\ \forall n\geq m,$$

where $W_k = (W_k(x,y))_{x,y\in V}$ and $h_n = ((P_{(\{x\})}(n))_{x\in V})^T$, $k = \overline{1,m}$, $\forall n \in \mathbb{Z}$. From this relation, we obtain that $h = (h_n)_{n=0}^{\infty} \in Rol^*[\mathcal{M}_\omega(\mathcal{M}_r(\mathbb{R}))][m]$ with generating vector $W = (W_k)_{k=0}^{m-1} \in G^*[\mathcal{M}_\omega(\mathcal{M}_r(\mathbb{R}))][m](h)$. Using Theorem 10, we have $h \in Rol^*[\mathbb{R}][mr\omega]$, which implies that also

$$\left(P^{(\ell)}_{(\{x\})}(n)\right)_{n=0}^{\infty} \in Rol^*[\mathbb{R}][mr\omega],\ \forall x \in V,\ \ell = \overline{0, r-1},$$

with the same generating vector. Since

$$a^{(\ell)}(n) = \sum_{x\in V} p^*(x) P^{(\ell)}_{(\{x\})}(n),\ \forall n \in \mathbb{N},$$

we have $a^{(\ell)} = (a^{(\ell)}(n))_{n=0}^{\infty} \in Rol^*[\mathbb{R}][mr\omega]$, $\ell = \overline{0, r-1}$, with the same generating vector. Because the game is started by player $\mathcal{P}^{(0)}$, then the distribution a of the game duration T coincides with $a^{(0)}$, i.e. $a = (a_n)_{n=0}^{\infty} \in Rol^*[\mathbb{R}][mr\omega]$ with the same generating vector.

Next, we will use only the relation $a \in Rol^*[\mathbb{C}][mr\omega]$, the minimal generating vector being determined using the minimization method based on the matrix rank, given by Theorem 7. So, according to this method, we have that the minimal generating vector $q = (q_0, q_1, \ldots, q_{R-1}) \in G^*[\mathbb{C}][R](a)$ is obtained from the unique solution $x = (q_{R-1}, q_{R-2}, \ldots, q_0)$ of the system

$$A_R^{[a]} x^T = (f_R^{[a]})^T, \tag{25}$$

where

$$f_R^{[a]} = (a_R, a_{R+1}, \ldots, a_{2R-1}),\ A_n^{[a]} = (a_{i+j})_{i,j=\overline{0,n-1}},\ \forall n \in \mathbb{N}^* \tag{26}$$

and R is the rank of the matrix $A^{[a]}_{mr\omega}$.

For this, we need to have only the values a_k, $k = \overline{0, 2mr\omega - 1}$. These values are determined from the formula

$$a_k = a_k^{(0)},\ k = \overline{0, 2mr\omega - 1}, \tag{27}$$

using the relations (3) – (13) and the initial conditions

$$a_n = a_n^{(\ell)} = P^{(\ell)}(n) = P^{(\ell)}_{(\{x\})}(n) = 0,\ \forall x \in V,\ \ell = \overline{0, r-1},\ n = \overline{0, m-2},$$

$$\alpha_k^{(\ell)}(n) = 0,\ k = \overline{1,m},\ n = \overline{0, m-1},\ \ell = \overline{0, r-1},$$

$$P^{(\ell)}(m-1) = \pi_{1,2}^{(\ell)} w_m^{(\ell+1)},\ a_{m-1}^{(\ell)} = \pi_1 P^{(\ell)}(m-1),\ \ell = \overline{0, r-1},$$

$$P^{(\ell)}_{(\{x_1\})}(m-1) = P^{(\ell)}(m-1),\ \ell = \overline{0, r-1},$$

$$P^{(\ell)}_{(\{x\})}(m-1) = 0,\ \forall x \in V \backslash \{x_1\},\ \ell = \overline{0, r-1}. \tag{28}$$

For the case $m = 1$ we have other formulas for determining the values of conditional probabilities $P^{(\ell)}_{(\{x\})}(n)$, $\ell = \overline{0, r-1}$, $\forall x \in V$, $\forall n \in \mathbb{N}$. It is easy to observe that these values can be obtained using the following formulas:

$$P^{(\ell)}_{(X_1)}(0) = 1,\ P^{(\ell)}_{(X_1)}(n) = 0,\ \forall n \in \mathbb{N}^*,\ \ell = \overline{0, r-1},$$

$$P^{(\ell)}_{(\{x\})}(0) = 0,\ P^{(\ell)}_{(\{x\})}(n) = \sum_{y \in V} p^{(\ell)}(x,y) P^{(\ell\ \oplus 1)}_{(\{y\})}(n-1),\ \forall n \in \mathbb{N}^*,\ \forall x \in V \backslash \{x_1\}. \quad (29)$$

3.2 Describing the developed algorithm

In the previous subsection we theoretically grounded the following algorithm for determining the main probabilistic characteristics (the distribution $(\mathbb{P}(T = n))_{n=0}^{\infty}$, the expectation $\mathbb{E}(T)$, the variance $\mathbb{V}(T)$, the mean square deviation $\sigma(T)$ and the k-order moments $\nu_k(T)$, $k = 1, 2, \ldots$) of the game duration T.

Algorithm 1.

Input: $X = (x_1, x_2, \ldots, x_m) \in V^m$, π_j, $\pi^{(\ell)}_{i,j}$, $i, j = \overline{1, m}$, $\ell = \overline{0, r-1}$;
Output: $\mathbb{E}(T),\ \mathbb{V}(T),\ \sigma(T),\ \nu_k(T),\ k = \overline{1, t},\ t \geq 2.$

1. Determine the values a_k, $k = \overline{0, 2mr\omega - 1}$, using the formula (27), the relations (3) – (13) and the initial conditions (28) – (29);

2. Find the minimal generating vector $q = (q_0, q_1, \ldots, q_{R-1}) \in G^*[\mathbb{R}][R](a)$ by solving the system (25), taking into account the relation (26);

3. Consider the distribution $a = (a_n)_{n=0}^{\infty} = (\mathbb{P}(T = n))_{n=0}^{\infty}$ of the game duration T as a homogeneous linear recurrence with the initial state $I^{[a]}_R = (a_n)_{n=0}^{R-1}$ and the minimal generating vector $q = (q_k)_{k=0}^{R-1}$, determined at the steps 1 and 2;

4. Determine the expectation $\mathbb{E}(T)$, the variance $\mathbb{V}(T)$, the mean square deviation $\sigma(T)$ and the k-order moments $\nu_k(T)$, $k = \overline{1, t}$, of the game duration T by using the corresponding algorithm from [2].

4 Win Probabilities

Another problem that is interesting for us is the determination of the win probabilities of the players. For solving this problem, we will consider the given game in more general case.

We consider a finite game Γ with r players $\mathcal{P}_\ell$, $\ell = \overline{0, r-1}$, who apply their own stochastic strategy S_ℓ, $0 \leq \ell < r$, in a given cyclic order $(S_0, S_1, \ldots, S_{r-1}, S_0, S_1, \ldots)$. Let T be the duration of the game Γ. The player $\mathcal{P}_{T mod\ r}$, who applies the last strategy, is considered the winner of the game.

Suppose that the distribution $d = (d_n)_{n=0}^{\infty} = (\mathbb{P}(T = n))_{n=0}^{\infty}$ of the game duration T is a homogeneous linear recurrence, i.e. there exist $m \in \mathbb{N}^*$ and the generating vector $q = (q_k)_{k=0}^{m-1} \in \mathbb{C}^m$, such that $d_n = \sum_{k=0}^{m-1} q_k d_{n-1-k}$, $\forall n \geq m$. We have $d \in Rol^*[\mathbb{C}][m]$ and $q \in G^*[\mathbb{C}][m](d)$. Next, we will show how to determine the win probability ω_ℓ for each player $\mathcal{P}_\ell$, $\ell = \overline{0, r-1}$.

If we consider the subsequence $d^{(\ell)} = (d_{rn+\ell})_{n=0}^{\infty}$ of the sequence d, then we have $\omega_\ell = G^{[d^{(\ell)}]}(1)$, $\ell = \overline{0, r-1}$. Using Theorem 8, we obtain $d^{(\ell)} \in Rol^*[\mathbb{C}][m]$, $\ell = \overline{0, r-1}$, with a common generating vector.

The minimal generating vector of these sequences can be determined using the minimization method based on matrix rank, given by Theorem 7, and the initial states of these sequences can be obtained using the initial state and the generating vector of the duration distribution d. Finally, the win probability ω_ℓ is obtained applying the formula, given by Theorem 1, for $z := 1$ and $a := d^{(\ell)}$, $\ell = \overline{0, r-1}$.

5 Optimal Cooperative Strategies of the Players

Next, we consider that the distributions p^* and $p^{(\ell)}$, $\ell = \overline{0, r-1}$, are not fixed. So, we have the game $\Gamma(p^*, p^{(0)}, p^{(1)}, \ldots, p^{(r-1)})$ with final sequence of states X, initial distribution of the states p^* and strategies of players $p^{(\ell)}$, $\ell = \overline{0, r-1}$, for every parameters p^* and $p^{(\ell)}$, $\ell = \overline{0, r-1}$. The problem is to determine the optimal distribution $\overline{p}^* = p^*$ and optimal strategies $\overline{p}^{(\ell)} = p^{(\ell)}$, $\ell = \overline{0, r-1}$, that minimize the expectation of the game duration $T(p^*, p^{(0)}, p^{(1)}, \ldots, p^{(r-1)})$ for the game $\Gamma(p^*, p^{(0)}, p^{(1)}, \ldots, p^{(r-1)})$.

Similar with results obtained in [3], the following theorems hold:

Theorem 11. *The optimal initial distribution of the states is $\overline{p}^*$, where $\overline{p}^*(x_1) = 1$ and $\overline{p}^*(x) = 0$, $\forall x \in V \setminus \{x_1\}$.*

Theorem 12. *We consider the set of active final states $\overline{X} = \{x_1, x_2, \ldots, x_{m-1}\}$, the set of final transitions $\overline{Y} = \{(x_1, x_2), (x_2, x_3), \ldots, (x_{m-1}, x_m)\}$ and the set of branch states $\overline{Z} = \{y \in \overline{X} \setminus \{x_1\} \mid \exists x \in \overline{X}, \exists z \in \overline{X} \cup \{x_m\}, z \neq y : \ (x, y) \in \overline{Y}, (x, z) \in \overline{Y}\}$. The optimal strategies $\overline{p}^{(\ell)}$, $\ell = \overline{0, r-1}$, have the following properties:*

1. $\overline{p}^{(\ell)}(x, x_1) = 1$, *if* $(x, x_1) \in \overline{Y}$ *and* $(x, z) \notin \overline{Y}$, $\forall z \neq x_1$;
2. $\overline{p}^{(\ell)}(x, x_1) = 1$, $\forall x \notin \overline{X}$;
3. $\overline{p}^{(\ell)}(x, x_1) > 0$, $\forall x \in \overline{Z}$ *and* $\overline{p}^{(\ell)}(x, x_1) = 0$ *if* $(x, x_1) \notin \overline{Y}$, $x \in \overline{X} \setminus \overline{Z}$;
4. $\overline{p}^{(\ell)}(x, y) = 0$, *if* $(x, y) \notin \overline{Y}$ *and* $y \neq x_1$;
5. $\overline{p}^{(\ell)}(x, y) > 0$, $\forall (x, y) \in \overline{Y}$;
6. $\sum\limits_{(x,y) \in \overline{Y} \cup \{(x, x_1)\}} \overline{p}^{(\ell)}(x, y) = 1, \ \forall x \in \overline{X}$.

Theorem 13. *Let $p = (p^{(0)}, p^{(1)}, \dots, p^{(r-1)})$. If $\delta_{i,j}(p) \not\equiv 0$, $i, j = \overline{1,2}$, then the optimal transition matrix can be determined solving the following geometric programs with posynomial equality constraints:*

$$\mathbb{E}(T(p^*, p)) = d_1 d_2^{-1} \to \min, \tag{30}$$

subject to

$$\begin{cases} \sum\limits_{(x,y)\in\overline{Y}\cup\{(x,x_1)\}} p^{(\ell)}(x,y) = 1, \ \forall x \in \overline{X}, \ \ell = \overline{0, r-1} \\ d_{1,1}^{-1} d_1 + d_{1,1}^{-1} d_{1,2} = 1 \\ d_{2,1}^{-1} d_2 + d_{2,1}^{-1} d_{2,2} = 1 \\ d_{1,1}^{-1} \delta_{1,1}(p) = 1 \\ d_{1,2}^{-1} \delta_{1,2}(p) = 1 \\ d_{2,1}^{-1} \delta_{2,1}(p) = 1 \\ d_{2,2}^{-1} \delta_{2,2}(p) = 1 \\ d_1, d_2, d_{1,1}, d_{1,2}, d_{2,1}, d_{2,2} > 0 \\ p^{(\ell)}(x,y) > 0, \ \forall (x,y) \in \overline{Y}, \ \ell = \overline{0, r-1} \\ p^{(\ell)}(x,x_1) > 0, \ \forall x \in \overline{Z}, \ \ell = \overline{0, r-1} \end{cases} \tag{31}$$

and (30) subject to

$$\begin{cases} \sum\limits_{(x,y)\in\overline{Y}\cup\{(x,x_1)\}} p^{(\ell)}(x,y) = 1, \ \forall x \in \overline{X}, \ \ell = \overline{0, r-1} \\ d_{1,1}^{-1} d_1 + d_{1,1}^{-1} d_{1,2} = 1 \\ d_{2,1}^{-1} d_2 + d_{2,1}^{-1} d_{2,2} = 1 \\ d_{1,1}^{-1} \delta_{1,2}(p) = 1 \\ d_{1,2}^{-1} \delta_{1,1}(p) = 1 \\ d_{2,1}^{-1} \delta_{2,2}(p) = 1 \\ d_{2,2}^{-1} \delta_{2,1}(p) = 1 \\ d_1, d_2, d_{1,1}, d_{1,2}, d_{2,1}, d_{2,2} > 0 \\ p^{(\ell)}(x,y) > 0, \ \forall (x,y) \in \overline{Y}, \ \ell = \overline{0, r-1} \\ p^{(\ell)}(x,x_1) > 0, \ \forall x \in \overline{Z}, \ \ell = \overline{0, r-1} \end{cases} \tag{32}$$

according to the properties described by Theorems 11 and 12, where $\delta_{i,j}(p)$, $i, j = \overline{1,2}$, are the posynomials from the decomposition

$$\mathbb{E}(T(p^*, p)) = (\delta_{1,1}(p) - \delta_{1,2}(p))(\delta_{2,1}(p) - \delta_{2,2}(p))^{-1} \tag{33}$$

that follows from the algorithm developed in [2]. The signomial programs $(30) - (31)$ and $(30) - (32)$ can be handled as geometric programs using the way followed in [8]. If $\overline{p}^1$ is the optimal solution of the problem $(30) - (31)$ and $\overline{p}^2$ is the optimal solution of the problem $(30) - (32)$, then the optimal transition matrix is $\overline{p} \in \{\overline{p}^1, \overline{p}^2\}$ for which $\mathbb{E}(T(\overline{p^*}, \overline{p}))$ is minimal. If there exists at least one $\delta_{i^*,j^*}(p) \equiv 0$, then in (31) and (32) the corresponding posynomial equality constraints just disappear and the related substitution $d_{i^*,j^*} = 0$ is performed in (31) and substitution $d_{i^*,3-j^*} = 0$ is performed in (32).

So, Theorem 13 shows us how to determine the optimal cooperative strategies of the players using signomial and geometric programming approaches. These methods were described in details in [8].

6 Conclusions

In this paper stationary games defined on Markov processes with final sequence of states were studied and the duration and win probabilities of these games were analyzed. It was proved that the game duration is a discrete random variable with homogeneous linear recurrence distribution. Based on this fact, the generating function is applied for determining the win probabilities and the main probabilistic characteristics of the game duration. Also, using the signomial and geometric programming approaches, the optimal cooperative strategies that minimize the expectation of the game duration are determined.

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Alexandru Lazari
Moldova State University,
60 Mateevici str., Chişinău,
Republic of Moldova, MD−2009.
E-mail: *lazarialexandru@mail.md*

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