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Some Homomorphic Properties of Multigroups

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Abstract. Multigroup is an algebraic structure of multiset that generalized crisp group theory. In this paper, we study the concept of homomorphism and its properties in multigroups context. Some related results are established.

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1 Introduction

The idea of multigroup was proposed in [5] as an algebraic structure of multiset that generalized the concept of group. The notion is consistent with other non-classical groups in [4]. Although other researchers in [2, 3, 6, 7, 10, 11] earlier used the term multigroup as an extension of group theory (with each of them having a divergent view), the notion of multigroup in [5] is quite acceptable because it is in consonant with other non-classical groups and defined over multiset (see [9] for multisets details).

Some new results on multigroups following [5] were presented in [1]. In this paper, we study the notion of homomorphism in multigroups context, present some of its properties and obtain some results.

2 Preliminaries

Definition 1 (see [8])**.** Let $X = \{x_1, x_2, ..., x_n, ...\}$ be a set. A multiset A over X is a cardinal-valued function, that is, $C_A : X \to \mathbb{N}$ such that $x \in Dom(A)$ implies $A(x)$ is a cardinal and $A(x) = C_A(x) > 0$, where $C_A(x)$, denotes the number of times an object x occur in A. Whenever $C_A(x) = 0$, implies $x \notin Dom(A)$. The set X is called the ground or generic set of the class of all multisets (for short, msets) containing objects from X.

A multiset $A = [a, a, b, b, c, c, c]$ can be represented as $A = [a, b, c]_{2,2,3}$. Different forms of representing multiset exist other than this. See [8, 9, 12] for details.

We denote the set of all multisets by $MS(X)$.

Definition 2 (see [9])**.** Let A and B be two multisets over X, A is called a submultiset of B written as $A \subseteq B$ if $C_A(x) \leq C_B(x)$ $\forall x \in X$. Also, if $A \subseteq B$ and $A \neq B$, then A is called a proper submultiset of B and denoted as $A \subset B$. A multiset is called the parent in relation to its submultiset.

Definition 3. Two multisets A and B over X are comparable to each other if $A \subseteq B$ or $B \subseteq A$.

Definition 4 (see [12]). Let A and B be two multisets over X. Then the intersection and union of A and B, denoted by $A \cap B$ and $A \cup B$ respectively, are defined by the rules that for any object $x \in X$,

(i) $C_{A\cap B}(x) = C_A(x) \wedge C_B(x)$,

(ii) $C_{A\cup B}(x) = C_A(x) \vee C_B(x)$,

where $\wedge$ and $\vee$ denote minimum and maximum.

Definition 5. Let $\{A_i\}_{i\in I}$ be a family of multisets over X. Then

(i) $C_{\bigcap_{i\in I} A_i}(x) = \bigwedge_{i\in I} C_{A_i}(x) \forall x \in X$,

(ii) $C_{\bigcup_{i\in I} A_i}(x) = \bigvee_{i\in I} C_{A_i}(x) \forall x \in X$.

Definition 6 (see [5]). Let X be a group. A multiset G is called a multigroup of X if it satisfies the following conditions:

(i) $C_G(xy) \geq C_G(x) \wedge C_G(y) \forall x, y \in X$,

(ii) $C_G(x^{-1}) \geq C_G(x) \forall x \in X$,

where C_G denotes the count function of G from X into a natural number $\mathbb{N}$.

By implication, a multiset G is called a multigroup of a group X if

$$C_G(xy^{-1}) \geq C_G(x) \wedge C_G(y), \forall x, y \in X.$$

It follows immediately from the definition that $C_G(e) \geq C_G(x)$ $\forall x \in X$, where e is the identity element of X. A multigroup G of X is complete if $G_* = X$, where $G_* = \{x \in X \mid C_A(x) > 0\}$. Also, the set G^* is defined by

$$G^* = \{x \in X \mid C_A(x) = C_A(e)\},$$

where e is the identity of X. We denote the set of all multigroups of X by $MG(X)$.

Example 1. The following are examples of multigroups.

(i) Let $Z_4 = \{0, 1, 2, 3\}$ be a group with respect to addition. Then $G = [0, 1, 2, 3]_{4,3,4,3}$ is a multigroup of Z_4.

(ii) The zeros of $f(x) = x^8 - 2x^4 + 1$ form a multigroup of a group $X = \{1, -1, i, -i\}$.

(iii) Let $X = \{\rho_0, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5\}$ be a permutation group on a set $S = \{1, 2, 3\}$ such that

$$\rho_0 = (1),\ \rho_1 = (123),\ \rho_2 = (132),\ \rho_3 = (23), \rho_4 = (13),\ \rho_5 = (12).$$

Then $A = [\rho_0, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5]_{7,4,4,3,3,3}$ is a multigroup of X.

Definition 7 (see [5])**.** Let $A, B \in MG(X)$. Then the product of A and B denoted as $A \circ B$, is governed by

$$C_{A \circ B}(x) = \bigvee_{x=yz} (C_A(y) \wedge C_B(z)), \forall y, z \in X.$$

Definition 8 (see [5])**.** For any multigroup $A \in MG(X)$, there exists its inverse, A^{-1} defined by

$$C_{A^{-1}}(x) = C_A(x^{-1}) \ \forall x \in X.$$

For example, let $X = \{0, 1, 2, 3\}$ be a group of $(Z_4, +)$. Let $A = [0, 1, 2, 3]_{4,3,2,3}$ be a multigroup of X, then $A^{-1} = [0, 3, 2, 1]_{4,3,2,3}$. From Definition 6 (ii), $C_A(x^{-1}) \geq C_A(x) \forall x \in X$ and also, $C_A(x) = C_A((x^{-1})^{-1}) \geq C_A(x^{-1})$. Hence, $C_A(x) = C_A(x^{-1})$. Since $C_{A^{-1}}(x) = C_A(x^{-1})$, we have $C_A(x) = C_{A^{-1}}(x)$. Therefore, $A = A^{-1}$ for every $A \in MG(X)$.

Proposition 1 (see [5])**.** *Let $A \in MS(X)$. Then $A \in MG(X)$ if and only if A satisfies the following conditions;*

(i) $A \circ A \subseteq A$,

(ii) $A^{-1} \subseteq A$ *or* $A \subseteq A^{-1}$ *or* $A^{-1} = A$,

(iii) $A \circ A^{-1} \subseteq A$.

Proposition 2 (see [5])**.** *Let $A, B \in MG(X)$, then the following hold.*

(i) $A \circ A = A$,

(ii) $A \circ B = B \circ A$,

(iii) $(A \circ B)^{-1} = B^{-1} \circ A^{-1}$,

(iv) $(A \circ B) \circ C = A \circ (B \circ C)$.

Proposition 3 (see [5])**.** *Let $A, B \in MG(X)$. Then $A \circ B \in MG(X)$ if and only if $A \circ B = B \circ A$.*

Definition 9. Let $\{A_i\}_{i \in I}, I = 1, ..., n$ be an arbitrary family of multigroups of X. Then $\{A_i\}_{i \in I} \in X$ is said to have descending or ascending chain if either $A_1 \subseteq A_2 \subseteq ... \subseteq A_n$ or $A_1 \supseteq A_2 \supseteq ... \supseteq A_n$, respectively.

3 Main Results

Throughout this section, we assume that multigroups are completely defined over the underlying groups.

Definition 10. Let X and Y be groups and let $f : X \to Y$ be a homomorphism. Let A and B be multisets over X and Y respectively. Then

(i) the image of A under f, denoted by $f(A)$, is a multigroup of Y defined by

$$C_{f(A)}(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} C_A(x), & f^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

for each $y \in Y$.

(ii) the inverse image of B under f, denoted by $f^{-1}(B)$, is a multigroup of X defined by

$$C_{f^{-1}(B)}(x) = C_B(f(x)) \ \forall x \in X.$$

Definition 11. Let X and Y be groups and let $A \in MG(X)$ and $B \in MG(Y)$, respectively.

(i) A homomorphism f from X to Y is called a weak homomorphism from A to B if $f(A) \subseteq B$. If f is a weak homomorphism of A into B, then we say that A is weakly homomorphic to B denoted by $A \sim B$.

(ii) An isomorphism f from X to Y is called a weak isomorphism from A to B if $f(A) \subseteq B$. If f is a weak isomorphism of A into B, then we say that A is weakly isomorphic to B denoted by $A \simeq B$.

(iii) A homomorphism f from X to Y is called a homomorphism from A to B if $f(A) = B$. If f is a homomorphism of A onto B, then A is homomorphic to B denoted by $A \approx B$.

(iv) An isomorphism f from X to Y is called an isomorphism from A to B if $f(A) = B$. If f is an isomorphism of A onto B, then A is isomorphic to B denoted by $A \cong B$.

Definition 12. Let $f : X \to Y$ be a homomorphism. Suppose A and B are multigroups of X and Y, respectively and A is homomorphic to B. Then the kernel of f from A to B is defined by

$$ker f = \{x \in X \mid C_A(x) = C_B(e'), f(e) = e'\},$$

where e and e' are the identities of X and Y, respectively.

Proposition 4. *Let $f : X \to Y$ be a homomorphism. For $A, B \in MG(X)$, if $A \subseteq B$, then $f(A) \subseteq f(B)$.*

Proof. Straightforward. □

Proposition 5. *Let X, Y be groups and f be a homomorphism of X into Y. For $A, B \in MG(Y)$, if $A \subseteq B$, then $f^{-1}(A) \subseteq f^{-1}(B)$.*

Proof. Straightforward. □

Definition 13. Let f be a homomorphism of a group X into a group Y, and $A \in MG(X)$. If for all $x, y \in X$, $f(x) = f(y)$ implies $C_A(x) = C_A(y)$, then A is f-invariant.

Lemma 1. *Let $f : X \to Y$ be groups homomorphism and $A \in MG(X)$. If $\forall x, y \in X$, $f(x) = f(y)$, then A is f-invariant.*

Proof. Suppose $f(x) = f(y)$ $\forall x, y \in X$. Then $C_{f(A)}(f(x)) = C_{f(A)}(f(y))$ implies $C_A(x) = C_A(y)$. Hence, A is f-invariant. □

Lemma 2. *If $f : X \to Y$ is a homomorphism and $A \in MG(X)$, then*

(i) $f(A^{-1}) = (f(A))^{-1}$,

(ii) $f^{-1}(f(A^{-1})) = f((f(A))^{-1})$.

Proof. (i) Let $y \in Y$. Then we get

$$\begin{aligned} C_{f(A^{-1})}(y) &= C_{A^{-1}}(f^{-1}(y)) = C_A(f^{-1}(y)) \\ &= C_{f(A)}(y) = C_{(f(A))^{-1}}(y) \forall y \in Y. \end{aligned}$$

Hence, $f(A^{-1}) = (f(A))^{-1}$.

(ii) Similar to (i). □

Proposition 6. *Let X and Y be groups such that $f : X \to Y$ is an isomorphic mapping. If $A \in MG(X)$ and $B \in MG(Y)$, respectively, then*

(i) $(f^{-1}(B))^{-1} = f^{-1}(B^{-1})$,

(ii) $f^{-1}(f(A)) = f^{-1}(f(f^{-1}(B)))$.

Proof. Recall that, if f is an isomorphism, then $f(x) = y$ $\forall x \in X$, $\forall y \in Y$. Consequently, $f(A) = B$.
(i)

$$\begin{aligned} C_{(f^{-1}(B))^{-1}}(x) &= C_{f^{-1}(B)}(x^{-1}) = C_{f^{-1}(B)}(x) \\ &= C_B(f(x)) = C_{B^{-1}}((f(x))^{-1}) \\ &= C_{B^{-1}}(f(x)) = C_{f^{-1}(B^{-1})}(x). \end{aligned}$$

Hence, $(f^{-1}(B))^{-1} = f^{-1}(B^{-1})$.

(ii) Similar to (i). □

Proposition 7. *Let $f : X \to Y$ be a homomorphism of groups. If $\{A_i\}_{i\in I} \in MG(X)$ and $\{B_i\}_{i\in I} \in MG(Y)$, respectively, then*

(i) $f(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} f(A_i)$,

(ii) $f(\bigcap_{i\in I} A_i) = \bigcap_{i\in I} f(A_i)$,

(iii) $f^{-1}(\bigcap_{i\in I} B_i) = \bigcap_{i\in I} f^{-1}(B_i)$,

(iv) $f^{-1}(\bigcup_{i\in I} B_i) = \bigcup_{i\in I} f^{-1}(B_i)$.

Proof. (i) Let $x \in X$ and $y \in Y$. Since f is a homomorphism, so $f(x) = y$. Then we have,

$$\begin{aligned} C_{f(\bigcup_{i\in I} A_i)}(y) &= C_{\bigcup_{i\in I} A_i}(f^{-1}(y)) \\ &= \bigvee_{i\in I} C_{A_i}(f^{-1}(y)) \\ &= \bigvee_{i\in I} C_{f(A_i)}(y) \\ &= C_{\bigcup_{i\in I} f(A_i)}(y), \forall y \in Y. \end{aligned}$$

Hence, $f(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} f(A_i)$.

The proofs of (ii)-(iv) are similar to (i). □

Theorem 1. *Let X be a group and $f : X \to X$ be an automorphism. If $A \in MG(X)$, then $f(A) = A \Leftrightarrow f^{-1}(A) = A$, consequently, $f(A) = f^{-1}(A)$.*

Proof. Let $x \in X$, and suppose $f(A) = A$, we get

$$\begin{aligned} C_{f(A)}(x) &= C_A(f^{-1}(x)) = C_A(x) \\ &= C_A(f(x)) = C_{f^{-1}(A)}(x) \end{aligned}$$

implies that $f^{-1}(A) = A$.

Conversely, let $f^{-1}(A) = A$, we have

$$\begin{aligned} C_{f^{-1}(A)}(x) &= C_A(f(x)) = C_A(x) \\ &= C_A(f^{-1}(x)) = C_{f(A)}(x). \end{aligned}$$

Hence, $f(A) = A$.
Therefore, $f(A) = A \Leftrightarrow f^{-1}(A) = A$. □

Theorem 2. *Let $f : X \to Y$ be a homomorphism. If $A \in MG(X)$, then $f^{-1}(f(A)) = A$, whenever f is injective.*

Proof. Suppose f is injective, then $f(x) = y \; \forall x \in X$ and $\forall y \in Y$. Now

$$\begin{aligned} C_{f^{-1}(f(A))}(x) &= C_{f(A)}(f(x)) = C_{f(A)}(y) \\ &= C_A(f^{-1}(y)) = C_A(x). \end{aligned}$$

Hence, $f^{-1}(f(A)) = A$. □

Corollary 1. *Let $f : X \to Y$ be a homomorphism. If $B \in MG(Y)$, then $f(f^{-1}(B)) = B$, whenever f is surjective.*

Proof. Similar to Theorem 2. □

Remark. Let $f : X \to Y$ be a homomorphism, $A \in MG(X)$ and $B \in MG(Y)$, respectively. If $kerf = \{e\}$ that is, $kerf \subseteq A^*$, then $f^{-1}(f(A)) = A$ since f is one-to-one.

Proposition 8. *Let X, Y and Z be groups and $f : X \to Y, f : Y \to Z$ be homomorphisms. If $\{A_i\}_{i\in I} \in MG(X)$ and $\{B_i\}_{i\in I} \in MG(Y)$ for each $i \in I$, then*

(i) $f(A_i) \subseteq B_i \Rightarrow A_i \subseteq f^{-1}(B_i)$,

(ii) $g[f(A_i)] = [gf](A_i)$,

(iii) $f^{-1}[g^{-1}(B_i)] = [gf]^{-1}(B_i)$.

Proof. The proof of (i) is trivial.

(ii) Since f and g are homomorphisms, then $f(x) = y$ and $g(y) = z$ $\forall x \in X, \forall y \in Y$ and $\forall z \in Z$ respectively. Now

$$\begin{aligned} C_{g[f(A_i)]}(z) &= C_{f(A_i)}(g^{-1}(z)) = C_{f(A_i)}(y) \\ &= C_{A_i}(f^{-1}(y)) = C_{A_i}(x), \end{aligned}$$

and

$$\begin{aligned} C_{[gf](A_i)}(z) &= C_{g(f(A_i))}(z) = C_{f(A_i)}(g^{-1}(z)) \\ &= C_{f(A_i)}(y) = C_{A_i}(f^{-1}(y)) \\ &= C_{A_i}(x)\forall x \in X. \end{aligned}$$

Hence, $g[f(A_i)] = [gf](A_i)$.

(iii) Similar to (ii). □

Theorem 3. *Let X and Y be groups and $f : X \to Y$ be an isomorphism. Then the following statements hold.*

(i) $A \in MG(X)$ *if and only if* $f(A) \in MG(Y)$.

(ii) $B \in MG(Y)$ *if and only if* $f^{-1}(B) \in MG(X)$.

Proof. (i) Suppose $A \in MG(X)$. Let $x, y \in Y$, then $\exists f(a) = x$ and $f(b) = y$ since f is an isomorphism for all $a, b \in X$. We know that

$$C_B(x) = C_A(f^{-1}(x)) = \bigvee_{a\in f^{-1}(x)} C_A(a)$$

and

$$C_B(y) = C_A(f^{-1}(y)) = \bigvee_{b \in f^{-1}(y)} C_A(b).$$

Clearly, $a \in f^{-1}(x) \neq \emptyset$ and $b \in f^{-1}(y) \neq \emptyset$. For $a \in f^{-1}(x)$ and $b \in f^{-1}(y) \Rightarrow x = f(a)$ and $y = f(b)$. Thus $f(ab^{-1}) = f(a)f(b^{-1}) = f(a)(f(b))^{-1} = xy^{-1}$. Let $c = ab^{-1} \Rightarrow c \in f^{-1}(xy^{-1})$. Now,

$$\begin{aligned} C_B(xy^{-1}) &= \bigvee_{c \in f^{-1}(xy^{-1})} C_A(c) \\ &= C_A(ab^{-1}) \\ &\geq C_A(a) \wedge C_A(b) \\ &= C_{f^{-1}(B)}(a) \wedge C_{f^{-1}(B)}(b) \\ &= C_B(f(a)) \wedge C_B(f(b)) \\ &= C_B(x) \wedge C_B(y) \forall x, y \in Y. \end{aligned}$$

Hence, $f(A) \in MG(Y)$.

Conversely, let $a, b \in X$ and suppose $f(A) \in MG(Y)$. Then

$$\begin{aligned} C_A(ab^{-1}) &= C_{f^{-1}(B)}(ab^{-1}) \\ &= C_B(f(ab^{-1})) \\ &= C_B(f(a)f(b^{-1})) \\ &= C_B(f(a)(f(b))^{-1}) \\ &\geq C_B(f(a)) \wedge C_B(f(b)) \\ &= C_{f^{-1}(B)}(a) \wedge C_{f^{-1}(B)}(b) \\ &= C_A(a) \wedge C_A(b) \end{aligned}$$

$\forall a, b \in X$. Hence, $A \in MG(X)$.

(ii) Similar to (i). □

Corollary 2. *Let X and Y be groups and $f : X \to Y$ be an isomorphism. Then the following statements hold.*

(i) $A^{-1} \in MG(X)$ if and only if $f(A^{-1}) \in MG(Y)$,

(ii) $B^{-1} \in MG(Y)$ if and only if $f^{-1}(B^{-1}) \in MG(X)$.

Proof. By combining Definition 8 and Theorem 3, the result follows. □

Corollary 3. *Let X and Y be groups and $f : X \to Y$ be homomorphism. If $\bigcap_{i \in I} A_i \in MG(X)$ and $\bigcap_{i \in I} B_i \in MG(Y)$, then*

(i) $f(\bigcap_{i \in I} A_i) \in MG(Y)$,

(ii) $f^{-1}(\bigcap_{i \in I} B_i) \in MG(X)$.

Proof. Straightforward from Theorem 3. □

Corollary 4. *Let $f : X \to Y$ be groups homomorphism. If $\bigcup_{i\in I} A_i \in MG(X)$ and $\bigcup_{i\in I} B_i \in MG(Y)$, whenever $\{A_i\}_{i\in I}$ and $\{B_i\}_{i\in I}$ have sup/inf assuming chain, then*

(i) $f(\bigcup_{i\in I} A_i) \in MG(Y)$,

(ii) $f^{-1}(\bigcup_{i\in I} B_i) \in MG(X)$.

Proof. Straightforward from Theorem 3. □

Theorem 4. *Let $f : X \to Y$ be an isomorphism. If $A \in MG(X)$ and $B \in MG(Y)$, then*

(i) $f(A) \circ B \in MG(Y)$ *if and only if* $f(A) \circ B = B \circ f(A)$,

(ii) $f^{-1}(B) \circ A \in MG(X)$ *if and only if* $f^{-1}(B) \circ A = A \circ f^{-1}(B)$.

Proof. (i) By Theorem 3, it follows that $f(A) \in MG(Y)$. So, $f(A), B \in MG(Y)$. Suppose $f(A) \circ B \in MG(Y)$. Then

$$\begin{aligned} C_{f(A)\circ B}(y) &= C_{(f(A))^{-1}\circ B^{-1}}(y) \\ &= C_{(B\circ f(A))^{-1}}(y) \\ &= C_{B\circ f(A)}(y) \forall y \in Y. \end{aligned}$$

Conversely, suppose $f(A) \circ B = B \circ f(A)$. Then

$$\begin{aligned} C_{(f(A)\circ B)^{-1}}(y) &= C_{(B\circ f(A))^{-1}}(y) \\ &= C_{(f(A))^{-1}\circ B^{-1}}(y) \\ &= C_{f(A)\circ B}(y) \forall y \in Y, \end{aligned}$$

and

$$\begin{aligned} C_{(f(A)\circ B)\circ(f(A)\circ B)}(y) &= C_{f(A)\circ(B\circ f(A))\circ B}(y) \\ &= C_{f(A)\circ(f(A)\circ B)\circ B}(y) \\ &= C_{(f(A)\circ f(A))\circ(B\circ B)}(y) \\ &= C_{f(A)\circ B}(x) \forall y \in Y. \end{aligned}$$

Hence, $f(A) \circ B \in MG(Y)$ by Propositions 1, 2 and 3.

(ii) Combining Propositions 1, 2 and 3, Definition 10, Theorem 3 and (i), the proof follows. □

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