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Solvability of the boundary value problem for the equation of transition processes in semiconductors with a fractional time derivative

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Abstract. Necessary and sufficient conditions are established for the unique solvability of the initial boundary value problem for the equation describing the transition processes in semiconductors. The method of studying is the reducing to the Cauchy problem for a degenerate evolution equation of fractional order in a Banach space. Using the functional calculus in the Banach algebra of bounded linear operators a form of the considered problem solution is performed.

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Introduction

Let $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$, $\Delta = \sum\limits_{k=1}^{3} \frac{\partial^2}{\partial x_k^2}$ is the Laplace operator. For $\alpha > 0, \lambda, \beta, \theta \in \mathbb{R}$ consider the initial boundary value problem

$$D_t^\alpha(\lambda - \Delta)w(x,t) = \beta w(x,t) + f(x,t), \quad (x,t) \in \Omega \times \overline{\mathbb{R}}_+, \tag{1}$$

$$(1-\theta)w(x) + \theta\frac{\partial}{\partial n}w(x) = 0, \quad (x,t) \in \partial\Omega \times \overline{\mathbb{R}}_+, \tag{2}$$

$$\frac{\partial^k w}{\partial t^k}(x,0) = w_k(x), \ x \in \Omega, \ k = 0,1,\dots,m-1, \tag{3}$$

Here D_t^α is a fractional Caputo derivative, m is a smallest integer not exceeding or equal to α. It is worth noting that the fractional derivatives play an increasingly important role in mathematical modeling, partially for describing various physical processes [3–6].

In the case of $\alpha = 1$ equation (1) describes the transition processes in semiconductors [2]. Function $w(x,t)$ has a physical sense of the electric field potential. The unique solvability of problem (1)–(3) with $\alpha = 1$ was studied in [2]. A mixed–type optimal control problem for the corresponding distributed control system was researched in [7].

In this paper by means of solution operators theory for fractional differential equations in Banach spaces the conditions of problem (1)–(3) unique solvability in the fractional case are found and the form of solution is performed in the present work.

1 Cauchy problem for abstract fractional order equation

Let $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$, $\overline{\mathbb{R}}_+ = \{0\} \cup \mathbb{R}_+$, for $\delta > 0$ $g_\delta(t) = t^{\delta-1}/\Gamma(\delta),\ t > 0$,

$$J_t^\delta h(t) = (g_\delta * h)(t) = \int_0^t g_\delta(t-s)h(s)ds = \frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}h(s)ds.$$

Let $\alpha > 0$, m is the smallest integer that is greater than or equal to α, D_t^m is a usual derivative of order m for $m \in \mathbb{N}$, D_t^α is Caputo derivative [1], i. e.

$$D_t^\alpha h(t) = D_t^m J_t^{m-\alpha}\left(h(t) - \sum_{k=0}^{m-1} h^{(k)}(0) g_{k+1}(t)\right) = J_t^{m-\alpha} D_t^m h(t),$$

when the expression on the right side is defined.

Let $\mathfrak{U}$ and $\mathfrak{V}$ be Banach spaces, $L \in \mathcal{L}(\mathfrak{U};\mathfrak{V})$ (linear and continuous operator), $M \in \mathcal{C}l(\mathfrak{U};\mathfrak{V})$ (linear, closed and densely defined operator), D_M is a domain of the operator M, $f : [0,T] \to \mathfrak{V}$ is a given function. Consider the Cauchy problem

$$u^{(k)}(0) = u_k,\ k = 0, 1, \dots, m-1, \tag{4}$$

for the fractional differential equation

$$D_t^\alpha Lu(t) = Mu(t) + f(t). \tag{5}$$

Various initial–boundary value problems for partial differential equations or systems of equations not solved with respect to the time–fractional derivatives can be reduced to the Cauchy problem (4), (5). Such equations arise in mathematical modeling of various processes in natural and technical sciences [2, 8–10]. Partially it concerns problem (1)–(3).

The theory of fractional differential equations has been intensively developed in the last decades [1, 3–6], but a few articles concern the fractional differential equations of the form (1), not solved with respect to the fractional derivative. See [9–11].

Define L–resolvent set of operator M $\rho^L(M) = \left\{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathfrak{V};\mathfrak{U})\right\}$. Operator M is called (L,σ)–bounded if the complement to the set $\rho^L(M)$ is bounded in $\mathbb{C}$. Define $R_\mu^L(M) = (\mu L - M)^{-1}L$, $L_\mu^L(M) = L(\mu L - M)^{-1}$,

$$P = \frac{1}{2\pi i}\int_\gamma R_\mu^L(M)d\mu \in \mathcal{L}(\mathfrak{U};\mathfrak{U}), \quad Q = \frac{1}{2\pi i}\int_\gamma L_\mu^L(M)d\mu \in \mathcal{L}(\mathfrak{V};\mathfrak{V}), \tag{6}$$

where the integrals are taken along a circle γ with a radius a, enclosing the complement to $\rho^L(M)$ in the complex plane $\mathbb{C}$. It is easy to check that operators P and Q are projectors [8]. Denote $\mathfrak{U}^0 = \ker P$, $\mathfrak{V}^0 = \ker Q$, $\mathfrak{U}^1 = \mathrm{im} P$, $\mathfrak{V}^1 = \mathrm{im} Q$. Let $L_k(M_k)$ be the restrictions of operator $L(M)$ to the subspace $\mathfrak{U}^k(D_{M_k} = D_M \cap \mathfrak{U}^k)$, $k = 0, 1$.

Theorem 1. *[8] Let operator M be (L,σ)–bounded. Then*
(i) $M_1 \in \mathcal{L}(\mathfrak{U}^1;\mathfrak{V}^1)$, $M_0 \in \mathcal{C}l(\mathfrak{U}^0;\mathfrak{V}^0)$, $L_k \in \mathcal{L}(\mathfrak{U}^k;\mathfrak{V}^k)$, $k=0,1$;
(ii) *there exist operators* $M_0^{-1} \in \mathcal{L}(\mathfrak{V}^0;\mathfrak{U}^0)$, $L_1^{-1} \in \mathcal{L}(\mathfrak{V}^1;\mathfrak{U}^1)$.

Let us denote $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, $H = M_0^{-1}L_0$. For $p \in \mathbb{N}_0$ operator M is called (L,p)–bounded if it is (L,σ)–bounded, $H^p \neq \mathbb{O}$, $H^{p+1} = \mathbb{O}$.

The solution of Cauchy problem (4), (5) is a function $u \in C^{m-1}(\overline{\mathbb{R}}_+;\mathfrak{U}) \cap C(\overline{\mathbb{R}}_+;D_M)$ such that $Lu \in C^{m-1}(\overline{\mathbb{R}}_+;\mathfrak{V})$, $g_{m-\alpha} * \left(Lu - \sum\limits_{k=0}^{m-1}(Lu)^{(k)}(0)g_{k+1}\right) \in C^m(\overline{\mathbb{R}}_+;\mathfrak{V})$, equalities (4) and (5) are valid for all $t \in \overline{\mathbb{R}}_+$.

The unique solvability of problem (4), (5) was investigated in [10]. Formulate the theorem on the existence and uniqueness of problem (4), (5) solution.

Theorem 2. *[10] Let operator M be (L,p)–bounded, $\gamma = \{\mu \in \mathbb{C} : |\mu| = r > a\}$,*

$$U(t) = \frac{1}{2\pi i}\int\limits_{\gamma} R_\mu^L(M)E_{\alpha,\beta}(\mu t^\alpha)d\mu,\ t \in \overline{\mathbb{R}}_+,$$

where $E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}$ is Mittag–Leffler function. Then for all $u_k \in \mathfrak{U}^1$, $k = 0,1,\dots m-1$, there exists a unique solution of problem (4), (5), and it has the form

$$u(t) = \sum_{k=0}^{m-1} J_t^k U(t)u_k. \tag{7}$$

If for some $l \in \{0,1,\dots m-1\}$ $u_l \notin \mathfrak{U}^1$, then problem (4), (5) has no solutions.

2 Solvability of the equation of transition processes in semiconductors

Let us return to problem (1)–(3) and reduce it to Cauchy problem (4), (5). Define the formal differential operator

$$B_\theta = (1-\theta) + \theta\frac{\partial}{\partial n}, \quad \theta \in \mathbb{R}.$$

Operator $A \in \mathcal{C}l(L_2(\Omega))$ is defined as acting on its domain

$$D_A = H_\theta^2(\Omega) = \left\{u \in H^2(\Omega) : B_\theta u(x) = 0,\ x \in \partial\Omega\right\}$$

by $Au = \Delta u$. Denote by $\{\varphi_k :\ k \in \mathbb{N}\}$ the orthonormal in the sense of the scalar product $\langle\cdot,\cdot\rangle$ in $L_2(\Omega)$ eigenfunctions of operator A, numbered in the non–increasing order with respect to the corresponding eigenvalues $\{\lambda_k :\ k \in \mathbb{N}\}$, counting their multiplicities. Note that the spectrum of operator A is real, discrete and condensed to $-\infty$ [12].

Let $\mathfrak{U} = \left\{u \in H^2(\Omega) : B_\theta u(x) = 0, x \in \partial\Omega\right\}$ (the Sobolev space), $\mathfrak{V} = L_2(\Omega)$ (the Lebesgue space), $L = \lambda - A$, $M = \beta I \in \mathcal{L}(\mathfrak{U};\mathfrak{V})$.

Theorem 3. *Let $\beta \neq 0$ or the spectrum $\sigma(A)$ do not contain λ. Then operator M is $(L, 0)$–bounded.*

Proof. In conditions of the theorem consider the operator

$$\mu L - M = \sum_{k=1}^{\infty} (\mu(\lambda - \lambda_k) - \beta) \langle \cdot, \varphi_k \rangle \varphi_k.$$

Show that for

$$|\mu| > \sup_{\lambda \neq \lambda_k} \left| \frac{\beta}{\lambda - \lambda_k} \right|$$

the operator

$$(\mu L - M)^{-1} = \sum_{k=1}^{\infty} \frac{\langle \cdot, \varphi_k \rangle \varphi_k}{\mu(\lambda - \lambda_k) - \beta} : L_2(\Omega) \to \mathfrak{U}$$

exists and is continuous. For $f \in L_2(\Omega)$

$$||(\mu L - M)^{-1} f||^2_{H^2(\Omega)} = \sum_{k=1}^{\infty} \frac{(1 + \lambda_k^2) | \langle f, \varphi_k \rangle |^2}{|\mu(\lambda - \lambda_k) - \beta|^2} =$$

$$= \sum_{\lambda_k = \lambda} \frac{(1 + \lambda_k^2) | \langle f, \varphi_k \rangle |^2}{\beta^2} + \sum_{\lambda_k \neq \lambda} \frac{(1 + \lambda_k^2) | \langle f, \varphi_k \rangle |^2}{|\lambda - \lambda_k|^2 \left| \mu - \frac{\beta}{\lambda - \lambda_k} \right|^2} \leq C ||f||^2_{L_2(\Omega)}$$

because of finitness of the first sum in the last line. Indeed,

$$\lim_{k \to \infty} \frac{1 + \lambda_k^2}{|\lambda - \lambda_k|^2} = 1,$$

so the corresponding sequence is bounded. Furthermore, the inequalities

$$\left| \mu - \frac{\beta}{\lambda - \lambda_k} \right| \geq |\mu| - \left| \frac{\beta}{\lambda - \lambda_k} \right| \geq d > 0$$

are true. Thus, the operator M is (L, σ)–bounded with a constant

$$a = \sup_{\lambda \neq \lambda_k} \left| \frac{\beta}{\lambda - \lambda_k} \right|.$$

Construct the projector

$$P = \frac{1}{2\pi i} \int\limits_{|\mu| = a+1} \sum_{\lambda_k \neq \lambda} \frac{\langle \cdot, \varphi_k \rangle \varphi_k}{\mu - \frac{\beta}{\lambda - \lambda_k}} d\mu = \sum_{\lambda_k \neq \lambda} \langle \cdot, \varphi_k \rangle \varphi_k \in \mathcal{L}(\mathfrak{U}).$$

It is obvious that the projector Q has the same form but is defined in $L_2(\Omega)$. Consequently $\mathfrak{U}^0 = \mathfrak{V}^0 = \operatorname{span} \{\varphi_k : \lambda_k = \lambda\}$, $\mathfrak{U}^1$ and $\mathfrak{V}^1$ are the closures of span $\{\varphi_k : \lambda_k \neq \lambda\}$ in the norm of spaces $\mathfrak{U}$ and $\mathfrak{V}$ respectively. Note that $\ker L = \ker P$, hence

$$0 = Lu = (\lambda I - A)u = \sum_{k=1}^{\infty}(\lambda - \lambda_k)\langle u, \varphi_k\rangle = \sum_{\lambda\neq\lambda_k}(\lambda - \lambda_k)\langle u, \varphi_k\rangle,$$

then $u = \sum_{\lambda_k=\lambda} c_k\varphi_k$ for some $c_k \in \mathbb{R}$, therefore $u \in \mathfrak{U}^0$. Inversely, if $u = \sum_{\lambda_k=\lambda} c_k\varphi_k$, then $(\lambda I - A)u = \sum_{\lambda_k=\lambda} c_k(\lambda - \lambda_k)\varphi_k = 0$ and $u \in \ker L$. Therefore $H = \mathbb{O}$ and the operator M is $(L, 0)$–bounded. □

Theorem 4. *Let $\beta \neq 0$ or the spectrum $\sigma(A)$ do not contain λ, for all $k \in \mathbb{N}$ such that $\lambda_k = \lambda$ the equalities $\langle u_l, \varphi_k\rangle = 0$, $l = 0, 1, \ldots, m-1$, are true. Then there exists a unique solution of problem (1)–(3), and it has the form*

$$u(x,t) = \sum_{\lambda_k\neq\lambda}\sum_{l=0}^{m-1} t^l E_{\alpha,\beta+l}\left(\frac{\beta t^\alpha}{\lambda - \lambda_k}\right)\langle u_l, \varphi_k\rangle\,\varphi_k(x).$$

Proof. Reduce the problem (1)–(3) to problem (4), (5). By Theorems 2, 3 obtain the required assertion. The solution is calculated using the formula (7) and the residue theorem in the same way as the projector is calculated in the previous theorem. Note that the properties of Mittag–Leffler functions imply the equality

$$u(x,t) = \sum_{\lambda_k\neq\lambda}\sum_{l=0}^{m-1} J_t^l E_{\alpha,\beta}\left(\frac{\beta t^\alpha}{\lambda - \lambda_k}\right)\langle u_l, \varphi_k\rangle\,\varphi_k(x) =$$

$$= \sum_{\lambda_k\neq\lambda}\sum_{l=0}^{m-1} t^l E_{\alpha,\beta+l}\left(\frac{\beta t^\alpha}{\lambda - \lambda_k}\right)\langle u_l, \varphi_k\rangle\,\varphi_k(x).$$

□

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