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A New Characterization of Curves in Euclidean 4-Space $\mathbb{E}^4$

Günay Öztürk, Selin Gürpınar, Kadri Arslan

Abstract. In the present study, we characterize a regular curve whose position vector can be written as a linear combination of its Serret-Frenet vectors in Euclidean 4-space $\mathbb{E}^4$. We investigate such curves in terms of their curvature functions. Further, we obtain some results of T-constant, N-constant and constant ratio curves in $\mathbb{E}^4$.

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1 Introduction

Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve in Euclidean 4-space $\mathbb{E}^4$. Let us denote $T(s) = x'(s)$ and call as a unit tangent vector of x at s. We denote the first Serret-Frenet curvature of x by $\kappa_1(s) = \|x''(s)\|$. If $\kappa_1(s) \neq 0$, then the unit principal normal vector $N_1(s)$ of the curve x at s is given by $N_1'(s) + \kappa_1(s)T(s) = \kappa_2(s)N_2(s)$, where κ_2 is the second Serret-Frenet curvature of x. If $\kappa_2(s) \neq 0$, then the unit second principal normal vector $N_2(s)$ of the curve x at s is given by $N_2'(s) + \kappa_2(s)N_1(s) = \kappa_3(s)N_3(s)$, where κ_3 is the third Serret-Frenet curvature of x. Then we have the Serret-Frenet formulae (see [12]):

$$\begin{aligned} T'(s) &= \kappa_1(s)N_1(s), \\ N_1'(s) &= -\kappa_1(s)T(s) + \kappa_2(s)N_2(s), \\ N_2'(s) &= -\kappa_2(s)N_1(s) + \kappa_3(s)N_3(s), \\ N_3'(s) &= -\kappa_3(s)N_2(s). \end{aligned} \tag{1}$$

If the Serret-Frenet curvatures $\kappa_1(s), \kappa_2(s)$ and $\kappa_3(s)$ of x are constant functions then x is called a screw line or a helix [11]. Since these curves are the traces of 1-parameter family of the groups of Euclidean transformations, F. Klein and S. Lie called them *W-curves* [20]. If the tangent vector T of the curve x makes a constant angle with a unit vector U of $\mathbb{E}^4$ then this curve is called a general helix (or inclined curve) in $\mathbb{E}^4$ [22]. It is known that a regular curve in $\mathbb{E}^n$ is said to have constant curvature ratios if the ratios of the consecutive curvatures are constant [21]. The Frenet curves with constant curvature ratios are called ccr-curves [22]. We remark that a regular curve in $\mathbb{E}^4$ is a ccr-curve if $H_1(s) = \frac{\kappa_1}{\kappa_2}(s)$ and $H_2(s) = \frac{\kappa_3}{\kappa_2}(s)$ are constant functions.

Rectifying curves in Euclidean 3-space $\mathbb{E}^3$ are introduced by B. Y. Chen in [4] as space curves whose position vector (denoted also by x) lies in its rectifying plane, spanned by the tangent and the binormal normal vector fields $T(s)$ and $N_2(s)$ of the curve. In the same paper, B. Y. Chen gave a simple characterization of rectifying curves. In particular, it is shown in [8] that there exists a simple relation between rectifying curves and centrodes, which play an important role in mechanics kinematics as well as in differential geometry in defining the curves of constant procession. It is also provided that a twisted curve is congruent to a non-constant linear function of s [4]. Further, in the Minkowski 3-space $\mathbb{E}^3_1$, the rectifying curves are investigated in [10, 15, 16]. In [16] a characterization of the spacelike, the timelike and the null rectifying curves in the Minkowski 3-space in terms of centrodes is given.

For a unit speed regular curve $x : I \subset \mathbb{R} \to \mathbb{E}^4$, the hyperplanes at each point of $x(s)$ which are spanned by $\{T, N_1, N_3\}$, $\{T, N_2, N_3\}$ are known as the *first osculating hyperplane* and the *second osculating hyperplane,* respectively. If the position vector x lies on its first (resp. second) *osculating hyperplane* then $x(s)$ is called *osculating curve of first (resp. second)* kind. In [17], Ilarslan and Nesovic considered the rectifying curve in Euclidean 4-space $\mathbb{E}^4$. They characterized the rectifying curves given by the equation

$$x(s) = \lambda(s)T(s) + \mu(s)N_2(s) + \upsilon(s)N_3(s), \tag{2}$$

for some differentiable functions $\lambda(s)$, $\mu(s)$ and $\upsilon(s)$. Actually, these curves are *osculating curves of* second kind. Further, in the Minkowski 4-space $\mathbb{E}^4_1$, the rectifying curves are investigated in [1, 18, 19]. Recently, quaternionic rectifying curves in the semi-Euclidean space $\mathbb{E}^4_2$ have been considered in [9].

For a regular curve $x(s)$, the position vector x can be decomposed into its tangential and normal components at each point, i.e., $x = x^T + x^N$. A curve $x(s)$ with $\kappa_1(s) > 0$ is said to be of *constant ratio* if the ratio $\|x^T\| : \|x^N\|$ is constant on $x(I)$ where $\|x^T\|$ and $\|x^N\|$ denote the length of x^T and x^N, respectively [2].

Clearly a curve x in $\mathbb{E}^n$ is of constant ratio if and only if $x^T = 0$ or $\|x^T\| : \|x\|$ is constant [2]. The distance function $\rho = \|x\|$ satisfies $\|grad\rho\| = c$ for some constant c if and only if we have $\|x^T\| = c\,\|x\|$. In particular, if $\|grad\rho\| = c$ then $c \in [0, 1]$. In [4], B. Y. Chen gave a classification of constant ratio curves in Euclidean space. A curve in $\mathbb{E}^n$ is called T*-constant* (resp. N*-constant*) if the tangential component x^T (resp. the normal component x^N) of its position vector x is of constant length [3, 6]. Recently the present authors have studied curves with constant ratio in Euclidean 3-space $\mathbb{E}^3$ in [13]. For more details see also [5, 7].

In the present study, we give a generalization of rectifying curves in Euclidean 4-space $\mathbb{E}^4$. First of all, we consider a regular curve in Euclidean 4-space $\mathbb{E}^4$ as a curve whose position vector satisfies the parametric equation

$$x(s) = m_0(s)T(s) + m_1(s)N_1(s) + m_2(s)N_2(s) + m_3(s)N_3(s), \tag{3}$$

for some differentiable functions $m_i(s)$, $0 \le i \le 3$. Next, we characterize osculating curves of first and second kind in terms of their curvature functions $\kappa_1(s), \kappa_2(s)$ and

$\kappa_3(s)$. We give necessary and sufficient conditions for the curves given with the parametrization (3) to become W-curves. Furthermore, we obtain some results for these types of curves to become ccr-curves. Finally, we consider T-constant and N-constant curves in $\mathbb{E}^4$. Moreover, we obtain some explicit equations of constant-ratio curves in $\mathbb{E}^4$.

2 Characterization of Curves in $\mathbb{E}^4$

In the present section, we consider unit speed curves with Serret-Frenet curvatures $\kappa_1(s) > 0$, $\kappa_2(s)$, and $\kappa_3(s)$. By definition of the position vector of the curve (also defined by x), it satisfies the vectorial equation (3) for some differentiable functions $m_i(s)$, $0 \le i \le 3$. By taking the derivative of (3) with respect to arclength parameter s and using the Serret-Frenet equations (1), we obtain

$$\begin{aligned} x'(s) \;=\; & (m_0'(s) - \kappa_1(s)m_1(s))T(s) \\ & +(m_1'(s) + \kappa_1(s)m_0(s) - \kappa_2(s)m_2(s))N_1(s) \\ & +(m_2'(s) + \kappa_2(s)m_1(s) - \kappa_3(s)m_3(s))N_2(s) \\ & +(m_3'(s) + \kappa_3(s)m_2(s))N_3(s). \end{aligned} \tag{4}$$

It follows that

$$\begin{aligned} m_0' - \kappa_1 m_1 &= 1, \\ m_1' + \kappa_1 m_0 - \kappa_2 m_2 &= 0, \\ m_2' + \kappa_2 m_1 - \kappa_3 m_3 &= 0, \\ m_3' + \kappa_3 m_2 &= 0. \end{aligned} \tag{5}$$

The following result explicitly determines the W-curves in $\mathbb{E}^4$.

Theorem 1. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a regular curve given with the parametrization (3). If x is a W-curve of $\mathbb{E}^4$ then the position vector x is given by the curvature functions*

$$\begin{aligned} m_0(s) &= \kappa_1 \left(\frac{-c_1 e^{-\lambda s} + c_2 e^{\lambda s}}{\lambda} + \frac{-c_3 e^{-\mu s} + c_4 e^{\mu s}}{\mu} \right) + c_0, \\ m_1(s) &= c_1 e^{-\lambda s} + c_2 e^{\lambda s} + c_3 e^{-\mu s} + c_4 e^{\mu s} - \frac{1}{\kappa_1}, \\ m_2(s) &= \frac{1}{\kappa_2} \left(\left(\frac{\lambda^2 + \kappa_1^2}{\lambda} \right) \left(-c_1 e^{-\lambda s} + c_2 e^{\lambda s} \right) + \left(\frac{\mu^2 + \kappa_1^2}{\mu} \right) \left(-c_3 e^{-\mu s} + c_4 e^{\mu s} \right) \right) \\ & \quad + \frac{\kappa_1}{\kappa_2} c_0 \\ m_3(s) &= -\kappa_3 \int m_2(s) ds \end{aligned} \tag{6}$$

where c_i $(0 \le i \le 4)$ are integral constants and

$$\lambda = \frac{\sqrt{-2a - 2\sqrt{a^2 - 4b}}}{2},$$

$$
\begin{aligned}
\mu &= \frac{\sqrt{-2a+2\sqrt{a^2-4b}}}{2}, \\
a &= \kappa_1^2+\kappa_2^2+\kappa_3^2, \\
b &= \kappa_1^2\kappa_3^2,
\end{aligned} \tag{7}
$$

are real constants.

Proof. Let x be a regular W-curve in $\mathbb{E}^4$, then by the use of the equations (5) we get

$$
\begin{aligned}
m_0' &= \kappa_1 m_1 + 1, \\
m_1'' &= \kappa_2 m_2' - \kappa_1\left(\kappa_1 m_1 + 1\right), \\
m_2'' &= -\kappa_3^2 m_2 - \kappa_2 m_1',
\end{aligned} \tag{8}
$$

In particular, one can show that the system of equations (8) has a non-trivial solution (6). Thus, the theorem is proved. □

2.1 Osculating curve of first kind in $\mathbb{E}^4$

Definition 1. Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a regular curve in $\mathbb{E}^4$ given with the arclength parameter s. If the position vector x lies in the hyperplane spanned by $\{T, N_1, N_3\}$ then x is called an osculating curve of first kind in $\mathbb{E}^4$.

Assume that $x : I \subset \mathbb{R} \to \mathbb{E}^4$ is an osculating curve of first kind in $\mathbb{E}^4$ given with the arclength parameter s. By definition the curvature function m_2 vanishes identically. So, from (5) we get

$$
\begin{aligned}
m_0' - \kappa_1 m_1 &= 1, \\
m_1' + \kappa_1 m_0 &= 0, \\
\kappa_2 m_1 - \kappa_3 m_3 &= 0, \\
m_3 &= c,
\end{aligned} \tag{9}
$$

and therefore

$$
\begin{aligned}
m_0 &= \frac{-cH_2'}{\kappa_1}, \\
m_1 &= cH_2, \\
m_3 &= c,
\end{aligned} \tag{10}
$$

where $H_2(s) = \frac{\kappa_3}{\kappa_2}(s)$ and $c \in \mathbb{R}$ is a real constant. So, the position vector of x is given by

$$
x(s) = c\left\{\frac{-H_2'}{\kappa_1}T(s) + H_2 N_1(s) + N_3(s)\right\}. \tag{11}
$$

By the use of (9) with (10) we obtain the following result.

Lemma 1. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed in $\mathbb{E}^4$. Then, x is congruent to an osculating curve of first kind if and only if*

$$\left(\frac{cH_2'}{\kappa_1}\right)' + c\kappa_1 H_2 + 1 = 0 \tag{12}$$

holds, where $H_2(s) = \frac{\kappa_3}{\kappa_2}(s)$ and $c \in \mathbb{R}$.

As a consequence of (12), we obtain the following result.

Theorem 2. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a regular curve congruent to an osculating curve of first kind. If x is a ccr-curve then*

$$H_2 = \frac{-1}{c\kappa_1},$$

vhere $c = m_3$ is a real constant.

Moreover, if two of the curvature functions are constant, we may consider the following cases.

Suppose that $\kappa_1(s) = \text{constant} > 0$, $\kappa_2(s) = \text{constant} \neq 0$, and $\kappa_3(s)$ is a non-constant function. By the use of (12), we obtain the differential equation

$$c\kappa_3''(s) + c\kappa_1^2\kappa_3(s) + \kappa_1\kappa_2 = 0, \tag{13}$$

which has a non-trivial solution

$$\kappa_3(s) = -\frac{\kappa_2}{c\kappa_1} + c_1 \cos(\kappa_1 s) + c_2 \sin(\kappa_1 s).$$

Similarly, assume that $\kappa_1(s) = \text{constant} > 0$, $\kappa_3(s) = \text{constant} \neq 0$, and $\kappa_2(s)$ is a non-constant function. Then the equation (12) implies the differential equation

$$\frac{c\kappa_3}{\kappa_1}\left(\frac{1}{\kappa_2(s)}\right)'' + \frac{c\kappa_1\kappa_3}{\kappa_2(s)} + 1 = 0. \tag{14}$$

Thus, the differential equation (14) has a non-trivial solution of the form

$$\kappa_2(s) = \frac{c\kappa_1\kappa_3}{c_1\kappa_3\cos(\kappa_1 s) - c_2\kappa_3 \sin(\kappa_1 s) - 1}.$$

Summing up these calculations, we obtain the following result.

Theorem 3. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve in $\mathbb{E}^4$. Then x is congruent to an osculating curve of first kind if*

i) $\kappa_1(s) = \text{constant} > 0$, $\kappa_2(s) = \text{constant} \neq 0$, and

$$\kappa_3(s) = -\frac{\kappa_2}{c\kappa_1} + c_1 \cos(\kappa_1 s) + c_2 \sin(\kappa_1 s),$$

ii) $\kappa_1(s) = \text{constant} > 0$, $\kappa_3(s) = \text{constant} \neq 0$, and

$$\kappa_2(s) = \frac{c\kappa_1\kappa_3}{c_1\kappa_3\cos(\kappa_1 s) - c_2\kappa_3 \sin(\kappa_1 s) - 1}$$

where $H_2(s) = \frac{\kappa_3}{\kappa_2}(s)$ and c, c_1 and $c_2 \in \mathbb{R}$.

2.2 Osculating curve of second kind in $\mathbb{E}^4$

Definition 2. Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a regular curve in $\mathbb{E}^4$ given with the arclength parameter s. If the position vector x lies in the hyperplane spanned by $\{T, N_2, N_3\}$ then x is called an osculating curve of second kind in $\mathbb{E}^4$.

In [17] K. Ilarslan and E. Nesovic considered the osculating curves of second kind in $\mathbb{E}^4$. Observe that they called them rectifying curves in $\mathbb{E}^4$. It means that the curvature function m_1 vanishes identically. So, from (5) we get

$$\begin{aligned} m_0' &= 1, \\ \kappa_2 m_2 - \kappa_1 m_0 &= 0, \\ m_2' - \kappa_3 m_3 &= 0, \\ m_3' + \kappa_3 m_2 &= 0, \end{aligned} \tag{15}$$

and therefore

$$\begin{aligned} m_0 &= s + b, \\ m_2 &= (s+b)H_1, \\ m_3 &= \frac{1}{\kappa_3}\left\{(s+b)H_1' + H_1\right\}, \end{aligned} \tag{16}$$

where $H_1(s) = \frac{\kappa_1}{\kappa_2}(s)$ is the first harmonic curvature of x and $b \in \mathbb{R}$. So, the position vector of x is given by

$$x(s) = (s+b)T(s) + (s+b)H_1 N_1(s) + \frac{(s+b)H_1' + H_1}{\kappa_3} N_3(s). \tag{17}$$

By the use of (9) with (10) we obtain the following result.

Theorem 4. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve in $\mathbb{E}^4$. Then, x is congruent to an osculating curve of second kind if and only if*

$$\left\{\frac{(s+b)H_1' + H_1}{\kappa_3}\right\}' + \kappa_3 (s+b) H_1 = 0 \tag{18}$$

holds, where $H_1(s) = \frac{\kappa_1}{\kappa_2}(s)$, $b \in \mathbb{R}$.

In [17] K. Ilarslan and E. Nesovic gave the following result.

Theorem 5. *[17] There is no osculating curve of second kind with non-zero constant curvatures $\kappa_1(s), \kappa_2(s)$ and $\kappa_3(s)$.*

As a consequence of (18) we obtain the following result.

Theorem 6. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a regular curve congruent to an osculating curve of second kind. If x is a ccr-curve then*

$$\kappa_3(s) = \frac{\mp 1}{\sqrt{c - 2bs - s^2}}, \tag{19}$$

where $b, c \in \mathbb{R}$.

Proof. Let x be an osculating curve of second kind. If x is a ccr-curve then by definition, the curvature functions $H_1(s) = \frac{\kappa_1}{\kappa_2}$ and $H_2(s) = \frac{\kappa_3}{\kappa_2}$ are constant. So, by the use of (18) one can get

$$\kappa_3'(s) + (s+b)\kappa_3^3(s) = 0 \tag{20}$$

which has a nontrivial solution (19). □

As a consequence of differential equation (18) one can get the following solutions as in the previous section.

Corollary 1. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve in $\mathbb{E}^4$. Then x is congruent to an osculating curve of second kind if*

i) $\kappa_1(s) =$ constant > 0, $\kappa_2(s) =$ constant $\neq 0$, and $\kappa_3(s) = \frac{1}{\left|\sqrt{c_1 - s^2 - 2bs}\right|}$ (see [17]),

ii) $\kappa_2(s) =$ constant $\neq 0$, $\kappa_3(s) =$ constant $\neq 0$, and

$$\kappa_1(s) = \frac{1}{s+b}\left(c_2 \sin(\kappa_3 s) + c_1 \cos(\kappa_3 s)\right),$$

iii) $\kappa_1(s) =$ constant > 0, $\kappa_3(s) =$ constant $\neq 0$, and

$$\kappa_2(s) = \frac{(s+b)\,\kappa_1}{c_1 \cos(\kappa_1 s) - c_2 \sin(\kappa_1 s)},$$

where c_1, c_2 and $b \in \mathbb{R}$.

2.3 T-constant curves in $\mathbb{E}^4$

Definition 3. Let $x : I \subset \mathbb{R} \to \mathbb{E}^n$ be a unit speed curve in $\mathbb{E}^n$. If $\left\|x^T\right\|$ is constant then x is called a *T-constant curve.* For a T-constant curve x, either $\left\|x^T\right\| = 0$ or $\left\|x^T\right\| = \lambda$ for some non-zero smooth function λ (see [3, 6]). Further, a T-constant curve x is called of first kind if $\left\|x^T\right\| = 0$, otherwise of second kind.

As a consequence of (5), we get the following results.

Theorem 7. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve in $\mathbb{E}^4$ given with the parametrization (5). Then x is a T-constant curve of first kind if and only if*

$$H_2 R' + \left(\frac{\left(\frac{R'}{\kappa_2}\right)'}{\kappa_3} + \frac{R}{H_2}\right)' = 0. \tag{21}$$

where $H_2(s) = \frac{\kappa_3}{\kappa_2}(s)$ and $-m_1(s) = R(s) = \frac{1}{\kappa_1(s)}$ is the radius of the curvature of the curve x.

Proof. Let x be a T-constant curve of first kind, then from (5) we get

$$m_1 = -\frac{1}{\kappa_1},\ m_2 = \frac{m_1'}{\kappa_2},\ m_3 = \frac{m_2' + m_1\kappa_2}{\kappa_3}.$$

Further, substituting these values into $m_3' + \kappa_3 m_2 = 0$ we get the result. □

Remark 1. Any unit speed regular curve in $\mathbb{E}^4$ satisfying the equality (21) is a spherical curve lying on a sphere $S^3(r)$ of $\mathbb{E}^4$. Thus every T-constant curves of first kind are spherical.

The following theorem characterizes T-constant curve of second kind in $\mathbb{E}^4$.

Theorem 8. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve in $\mathbb{E}^4$ given with the parametrization (5).Then x is a T-constant curve of second kind if and only if*

$$H_2\left(\kappa_1 m_0 - R'\right) + \left(\frac{\left(H_1 m_0 - \frac{R'}{\kappa_2}\right)'}{\kappa_3} - \frac{R}{H_2}\right)' = 0, \tag{22}$$

where $m_0 \in \mathbb{R}$, $H_1(s) = \frac{\kappa_1}{\kappa_2}(s)$, $H_2(s) = \frac{\kappa_3}{\kappa_2}(s)$ and $-m_1(s) = R(s) = \frac{1}{\kappa_1(s)}$ is the radius of the curvature of the curve x.

Proof. Let x be a T-constant curve of second kind, then from (5) we get

$$m_1 = -\frac{1}{\kappa_1},\ m_2 = \frac{m_1' + \kappa_1 m_0}{\kappa_2},\ m_3 = \frac{m_2' + m_1\kappa_2}{\kappa_3}.$$

Further, substituting these values into $m_3' + \kappa_3 m_2 = 0$, we get the result. □

The following result explicitly determines the T-constant W-curves of second kind in $\mathbb{E}^4$.

Corollary 2. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a regular T-constant curve of second kind in $\mathbb{E}^4$. If x is a W-curve of $\mathbb{E}^4$, then the position vector x has the parametrization*

$$x(s) = \lambda T - RN_1 + H_1\lambda N_2 + (bs + c)\, N_3,$$

where $R = \frac{1}{\kappa_1}$, $H_1 = \frac{\kappa_1}{\kappa_2}$, c is integral constant, $b = -H_1\kappa_3\lambda$ and $\lambda \in \mathbb{R}$.

The following result provides a simple characterization of T-constant curve of second kind in $\mathbb{E}^4$.

Theorem 9. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a T-constant curve of second kind. Then the distance function $\rho = \|x\|$ satisfies*

$$\rho = \pm\sqrt{2\lambda s + c}. \tag{23}$$

for some real constants c and $\lambda = m_0$.

Proof. Differentiating the squared distance function $\rho^2 = \langle x(s), x(s)\rangle$ and using (3) we get $\rho\rho' = m_0$. If x is a T-constant curve of second kind then by definition, the curvature function $m_0(s)$ of x is constant. It is easy to show that this differential equation has a nontrivial solution (23). □

2.4 N-constant curves in $\mathbb{E}^4$

Definition 4. Let $x : I \subset \mathbb{R} \to \mathbb{E}^n$ be a unit speed curve in $\mathbb{E}^n$. If $\|x^N\|$ is constant then x is called an N-constant curve. For an N-constant curve x, either $\|x^N\| = 0$ or $\|x^N\| = \mu$ for some non-zero smooth function μ (see [3,6]). Further, an N-constant curve x is called of first kind if $\|x^N\| = 0$, otherwise of second kind.

So, for an N-constant curve x in $\mathbb{E}^4$

$$\|x^N(s)\|^2 = m_1^2(s) + m_2^2(s) + m_3^2(s) \tag{24}$$

becomes a constant function. Therefore, by differentiation

$$m_1 m_1' + m_2 m_2' + m_3 m_3' = 0. \tag{25}$$

For the N-constant curves of first kind we give the following result.

Proposition 1. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve in $\mathbb{E}^4$. Then x is an N-constant curve of first kind if and only if $x(I)$ is an open portion of a straight line through the origin.*

Proof. Suppose that x is an N-constant curve of first kind in $\mathbb{E}^4$, then by definition $\|x^N(s)\| = \mu = 0$. Further, differentiating $x(s) = m_0(s)T(s)$ and using the Frenet equation (1) we get $\kappa_1 = 0$. □

Further, for the N-constant curves of second kind, we obtain the following results.

Theorem 10. *Let $x(s) \in \mathbb{E}^4$ be a unit speed regular curve that fully lies in $\mathbb{E}^4$. If x is an N-constant curve of second kind, then the position vector x of the curve has the parametrization*

$$x(s) = (s+b)\,T(s) + (s+b)\,H_1 N_2(s) + \frac{(s+b)\,H_1' + H_1}{\kappa_3} N_3(s), \tag{26}$$

where $H_1(s) = \frac{\kappa_1}{\kappa_2}(s)$, $b \in \mathbb{R}$.

Proof. Suppose that x is an N-constant curve of second kind in $\mathbb{E}^4$, then from the equations in (5) and (25) we get $m_1 = 0$, $m_0(s) = s + b$, $m_2(s) = \frac{\kappa_1}{\kappa_2}(s)m_0$ and $m_3(s) = \frac{m_2'(s)}{\kappa_3(s)}$ for some constant function b. This completes the proof of the theorem. □

Corollary 3. *Every N-constant curve of second kind in $\mathbb{E}^4$ is an osculating curve of second kind.*

The following result provides a simple characterization of N-constant curve of second kind in $\mathbb{E}^4$.

Theorem 11. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be an N-constant curve of second kind. Then the distance function $\rho = \|x\|$ satisfies*

$$\rho = \mp\sqrt{s^2 + 2bs + d} \tag{27}$$

for some constant functions b, d.

Proof. Differentiating the squared distance function $\rho^2 = \langle x(s), x(s)\rangle$ and using (3) we get $\rho\rho' = m_0$. If x is an N-constant curve of second kind then from the previous theorem $m_0(s) = s + b$. It is easy to show that this differential equation has a nontrivial solution (27). □

Definition 5. Let $x : I \subset \mathbb{R} \to \mathbb{E}^n$ be a unit speed regular curve in $\mathbb{E}^n$. Then the position vector x can be decomposed into its tangential and normal components at each point:

$$x = x^T + x^N.$$

If the ratio $\left\|x^T\right\| : \left\|x^N\right\|$ is constant on $x(I)$ then x is said to be of *constant ratio, or equivalently* $\left\|x^T\right\| : \|x\| = c$ =constant [2].

For a unit speed regular curve x in $\mathbb{E}^n$, the gradient of the distance function $\rho = \|x(s)\|$ is given by

$$grad\rho = \frac{d\rho}{ds}x'(s) = \frac{< x(s), x'(s) >}{\|x(s)\|}T(s), \tag{28}$$

where T is the tangent vector field of x.

The following results characterize constant-ratio curves.

Theorem 12. *[7] Let $x : I \subset \mathbb{R} \to \mathbb{E}^n$ be a unit speed regular curve in $\mathbb{E}^n$. Then x is of constant ratio with $\left\|x^T\right\| : \|x\| = c$ if and only if $\|grad\rho\| = c$ which is constant.*

In particular, for a curve of constant ratio we have $\|grad\rho\| = c \leq 1$.

As a consequence of (28) we obtain the following result.

Corollary 4. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^n$ be a unit speed regular curve in $\mathbb{E}^n$. If x is of constant ratio then the distance function $\rho = \frac{m_0}{c}$, where $\|grad\rho\| = c$ and $m_0 =< x(s), x'(s) >$.*

Theorem 13. *[7] Let $x : I \subset \mathbb{R} \to \mathbb{E}^n$ be a unit speed regular curve in $\mathbb{E}^n$. Then $\|grad\rho\| = c$ holds for a constant c if and only if one of the following three cases occurs:*

(i) $\|grad\rho\| = 0 \Longleftrightarrow x(I)$ is contained in a hypersphere centered at the origin.
(ii) $\|grad\rho\| = 1 \Longleftrightarrow x(I)$ is an open portion of a line through the origin.
(iii) $\|grad\rho\| = c \Longleftrightarrow \rho = \|x(s)\| = cs$, for $c \in (0, 1)$.

The following result provides some simple characterization of T-constant and N-constant curves in $\mathbb{E}^4$. Observe that this result is also valid in 3-dimensional case (see [13]).

Corollary 5. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed regular curve in $\mathbb{E}^4$. Then up to a translation of the arc length function s, we have*

i) If x is a T-constant curve of first kind then $\|grad\rho\| = 0$,

ii) If x is an N-constant curve of first kind then $\|grad\rho\| = 1$,

iii) If x is a T-constant curve of second kind then $\rho^2 = m_0 s + b$,

iv) If x is an N-constant curve of second kind then $\rho^2 = (s+a)^2 + m_1$,

where m_0, m_1, a, b are real constants.

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Günay Öztürk
Kocaeli University,
Department of Mathematics,
Kocaeli, TURKEY.
E-mail: *ogunay@kocaeli.edu.tr*

Selin Gürpinar, Kadri Arslan
Uludağ University,
Department of Mathematics,
Bursa, TURKEY.
E-mail: *selin.grpnar@gmail.com, arslan@uludag.edu.tr*