Regular, Intra-regular and Duo $\Gamma$-Semirings

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Abstract. In this paper we give several characterizations of a regular $\Gamma$-semiring, intra-regular $\Gamma$-semiring and a duo $\Gamma$-semiring by using ideals, interior-ideals, quasi-ideals and bi-ideals of a $\Gamma$-semiring.


Keywords and phrases: Quasi-ideal, bi-ideal, interior-ideal, regular $\Gamma$-semiring, intra-regular $\Gamma$-semiring, duo $\Gamma$-semiring.

1 Introduction


The notion of a bi-ideal was first introduced for semigroups by Good and Hughes in [4]. The concept of a bi-ideal for a ring was given by Lajos [9]. Also in [10,11] Lajos discussed some characterizations of bi-ideals in semigroups. Shabir, Ali, Batool in [14] gave some properties of bi-ideals in a semiring.

The concept of a regular ring was introduced by J. von Neumann in [12] and he gave the definition of a regular ring as follows: a ring $R$ is regular if for any $b \in R$ there exists $x \in R$ such that $b = bx b$. Analogously the concept of a regular semigroup was introduced by Green in [5] and a regular semiring was introduced by Zelznikov [17]. This concept of regularity was extended to a $\Gamma$-semiring by Rao [13] and was studied by Dutta and Sardar in [3].

In this paper efforts are made to prove various characterizations of a regular $\Gamma$-semiring, intra-regular $\Gamma$-semiring and a duo $\Gamma$-semiring by using ideals, interior-ideals, quasi-ideals and bi-ideals of a $\Gamma$-semiring.

2 Preliminaries

First we recall some definitions of the basic concepts of $\Gamma$-semirings that we need in sequel. For this we follow Dutta and Sardar [3].
Definition 1. Let $S$ and $\Gamma$ be two additive commutative semigroups. $S$ is called a $\Gamma$-semiring if there exists a mapping $S \times \Gamma \times S \to S$ denoted by $a\alpha b$ for all $a, b \in S$ and $\alpha \in \Gamma$ satisfying the following conditions:

(i) $a\alpha (b + c) = (a\alpha b) + (a\alpha c)$,

(ii) $(b + c)\alpha a = (b\alpha a) + (c\alpha a)$,

(iii) $a(\alpha + \beta)c = (a\alpha c) + (a\beta c)$,

(iv) $a\alpha (b\beta c) = (a\alpha b)\beta c$; for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$.

Definition 2. An element $0 \in S$ is said to be an absorbing zero if $0\alpha a = 0 = a\alpha 0$, and $a + 0 = 0 + a = a$ for all $a \in S$ and $\alpha \in \Gamma$.

Definition 3. A non-empty subset $T$ of a $\Gamma$-semiring $S$ is said to be a sub-$\Gamma$-semiring of $S$ if $(T, +)$ is a subsemigroup of $(S, +)$ and $a\alpha b \in T$ for all $a, b \in T$ and $\alpha \in \Gamma$.

Definition 4. A non-empty subset $T$ of a $\Gamma$-semiring $S$ is called a left (respectively right) ideal of $S$ if $T$ is a subsemigroup of $(S, +)$ and $x\alpha a \in T$ (respectively $a\alpha x \in T$) for all $a \in T$, $x \in S$ and $\alpha \in \Gamma$.

Definition 5. If $T$ is both left and right ideal of a $\Gamma$-semiring $S$, then $T$ is known as an ideal of $S$.

A quasi-ideal $Q$ in a $\Gamma$-semiring $S$ is defined as follows.

Definition 6. A subsemigroup $Q$ of $(S, +)$ is a quasi-ideal of $S$ if $(S \Gamma Q) \cap (Q \Gamma S) \subseteq Q$.

Example. Consider a $\Gamma$-semiring $S = M_{2 \times 2}(N_0)$, where $N_0$ denotes the set of natural numbers with zero and $\Gamma = S$. Define $A\alpha B =$ usual matrix product of $A, \alpha$ and $B$; for all $A, \alpha, B \in S$. Then $Q = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid a \in N_0 \right\}$ is a quasi-ideal of a $\Gamma$-semiring $S$.

Definition 7. A non-empty subset $B$ of a $\Gamma$-semiring $S$ is a bi-ideal of a $\Gamma$-semiring $S$ if $B$ is a sub-$\Gamma$-semiring of $S$ and $B \Gamma S \subseteq B$.

Example. Let $N$ be the set of natural numbers and $\Gamma = 2N$. Then $N$ and $\Gamma$ both are additive commutative semigroups. An image of a mapping $N \times \Gamma \times N \to N$ is denoted by $a\alpha b$ and defined as $a\alpha b =$ product of $a, \alpha, b$, for all $a, b \in S$ and $\alpha \in \Gamma$. Then $N$ forms a $\Gamma$-semiring. $B = 3N$ is a bi-ideal of $N$.

Now we define a generalized bi-ideal and an interior-ideal of a $\Gamma$-semiring $S$.

Definition 8. A non-empty subset $B$ of a $\Gamma$-semiring $S$ is a generalized bi-ideal of a $\Gamma$-semiring $S$ if $B \Gamma S \subseteq B$.

Example. Let $N$ be the set of natural numbers and $\Gamma = 2N$. Then $N$ and $\Gamma$ both are additive commutative semigroups. An image of a mapping $N \times \Gamma \times N \to N$ is denoted by $a\alpha b$ and defined as $a\alpha b =$ product of $a, \alpha, b$, for all $a, b \in S$ and $\alpha \in \Gamma$. Then $N$ forms a $\Gamma$-semiring. $B = 3N$ is a bi-ideal of $N$.

Definition 9. A non-empty subset $I$ of a $\Gamma$-semiring $S$ is an interior-ideal of a $\Gamma$-semiring $S$ if $I$ is a subsemigroup of $S$ and $S \Gamma I \Gamma S \subseteq I$. 


Proposition 1. For each non-empty subset $X$ of a $\Gamma$-semiring $S$ the following statements hold.

(i) $S \Gamma X$ is a left ideal of $S$.
(ii) $X \Gamma S$ is a right ideal of $S$.
(iii) $S \Gamma X \Gamma S$ is an ideal of $S$.

Proposition 2. If $S$ is a $\Gamma$-semiring and $a \in S$, then the following statements hold.

(i) $S \Gamma a$ is a left ideal of $S$.
(ii) $a \Gamma S$ is a right ideal of $S$.
(iii) $S \Gamma a \Gamma S$ is an ideal of $S$.

Now onwards $S$ denotes a $\Gamma$-semiring with absorbing zero unless otherwise stated.

3 Regular $\Gamma$-Semiring

An element $a$ of a $\Gamma$-semiring $S$ is said to be regular if $a \in a \Gamma S \Gamma a$.
If all elements of a $\Gamma$-semiring $S$ are regular, then $S$ is known as a regular $\Gamma$-semiring.
The following theorem was proved in [8] by the authors.

Theorem 1. In $S$ the following statements are equivalent.

(1) $S$ is regular.
(2) For every left ideal $L$ and a right ideal $R$ of $S$, $R \cap L = R \Gamma L = R$.
(3) For every left ideal $L$ and a right ideal $R$ of $S$,
   (i) $R^2 = R \Gamma R = R$,
   (ii) $L^2 = L \Gamma L = L$,
   (iii) $R \cap L = R \Gamma L$ is a quasi-ideal of $S$.
(4) The set of all quasi-ideals of $S$ is a regular $\Gamma$-semigroup.
(5) Every quasi-ideal of $S$ is of the form $Q \Gamma S \Gamma Q = Q$.

Theorem 2. The following statements are equivalent in $S$.

(1) $S$ is regular.
(2) For any bi-ideal $B$ of $S$, $B \Gamma S \Gamma B = B$.
(3) For any quasi-ideal $Q$ of $S$, $Q \Gamma S \Gamma Q = Q$.

Proof. (1) $\Rightarrow$ (2) Let $B$ be a bi-ideal of $S$ and $b \in B$. As $S$ is regular, $b \in b \Gamma S \Gamma b \subseteq B \Gamma S \Gamma B$. Therefore $B \subseteq B \Gamma S \Gamma B$. Hence $B = B \Gamma S \Gamma B$.
(2) $\Rightarrow$ (3) As every quasi-ideal is a bi-ideal, implication (2) $\Rightarrow$ (3) holds.
(3) $\Rightarrow$ (1) Let $R$ be a right ideal and $L$ be a left ideal of $S$. Then $R \cap L$ is a quasi-ideal of $S$. Hence by assumption $R \cap L = R \cap L \subseteq (R \cap L) \Gamma S \Gamma (R \cap L) \subseteq (R \cap L) \Gamma L \subseteq R \Gamma L$.
Therefore $R \cap L = R \Gamma L$. Thus $S$ is a regular $\Gamma$-semiring by Theorem 1.

Theorem 3. In $S$ the following statements are equivalent.

(1) $S$ is regular.
(2) For every bi-ideal $B$ and an ideal $I$ of $S$, $B \cap I = B \Gamma I \Gamma B$.
(3) For every quasi-ideal $Q$ and an ideal $I$ of $S$, $Q \cap I = Q \Gamma I \Gamma Q$. 

Proof. (1) ⇒ (2) Let $B$ be a bi-ideal and $I$ be an ideal of $S$. Now $B \Gamma \Gamma B \subseteq B \Gamma I \Gamma B \subseteq B$. Therefore $B \Gamma \Gamma B \subseteq B \cap I$. For the reverse inclusion, let $a \in B \cap I$. As $S$ is regular, $a \in \alpha \Gamma S \Gamma a$. Then $\alpha \Gamma S \Gamma a \subseteq (\alpha \Gamma S \Gamma a) \Gamma \Gamma (\alpha \Gamma S \Gamma a) \subseteq B \Gamma \Gamma B \cap (S \Gamma I \Gamma S) \Gamma B \subseteq B \Gamma \Gamma B$. Therefore $a \in Q \Gamma \Gamma B$. Hence we have $B \cap I \subseteq B \Gamma \Gamma B$. Thus we get $B \Gamma \Gamma B = B \cap I$.

(2) ⇒ (3) Implication follows as every quasi-ideal of $S$ is a bi-ideal.

(3) ⇒ (1) Let $R$ be a right ideal and $L$ be a left ideal of $S$. Then by assumption we have, $R = R \cap S = R \Gamma S \cap R \cap R S \subseteq R \Gamma S \cap L$ and $L \cap S = L \cap S \cap L \subseteq L \cap L$. Also $R \cap L = R \Gamma L$ is a quasi-ideal of $S$. Hence by Theorem 1, $S$ is a regular $\Gamma$-semiring. □

Proof of the following theorem is straightforward.

**Theorem 4.** In $S$ the following statements are equivalent.

1. $S$ is regular.
2. For every bi-ideal $B$ and a left ideal $L$ of $S$, $B \cap L \subseteq B \Gamma L$.
3. For every quasi-ideal $Q$ and a left ideal $L$ of $S$, $Q \cap L \subseteq Q \Gamma L$.
4. For every bi-ideal $B$ and a right ideal $R$ of $S$, $B \cap R \subseteq B \Gamma S$.
5. For every right ideal $R$ and a quasi-ideal $Q$ of $S$, $R \cap Q \subseteq R \Gamma Q$.
6. For every left ideal $L$, every right ideal $R$ and every bi-ideal $B$ of $S$, $L \cap R \cap B \subseteq L \Gamma B \cap R \Gamma L$.
7. For every left ideal, every right ideal $R$ and every quasi-ideal $Q$ of $S$, $L \cap R \cap Q \subseteq R \Gamma Q \Gamma L$.

**Theorem 5.** In $S$ the following conditions are equivalent.

1. $S$ is regular.
2. $I \cap Q = Q \Gamma I \Gamma Q$, for an ideal $I$ and a quasi-ideal $Q$ of $S$.
3. $I \cap Q = Q \Gamma I \Gamma Q$, for an interior ideal $I$ and a quasi-ideal $Q$ of $S$.

Proof. (1) ⇒ (2) Let $Q$ be a quasi-ideal and $I$ be an ideal of $S$. Now $Q \Gamma I \Gamma Q \subseteq Q \Gamma S \Gamma Q \subseteq Q \Gamma S$ by Proposition 1. Similarly we get $Q \Gamma I \Gamma Q \subseteq S \Gamma Q$. Therefore $Q \Gamma I \Gamma Q \subseteq (S \Gamma Q) \cap (Q \Gamma S) \subseteq Q$, since $Q$ is a quasi-ideal. Also $Q \Gamma I \Gamma Q \subseteq I$ as $I$ is an ideal. Therefore $Q \Gamma I \Gamma Q \subseteq Q \cap I$. For the reverse inclusion, let $a \in Q \cap I$. As $S$ is regular, $a \in \alpha \Gamma S \Gamma a$. We have $a \in (\alpha \Gamma S \Gamma a) \Gamma \Gamma (\alpha \Gamma S \Gamma a) \subseteq (Q \Gamma S \Gamma Q) \Gamma \Gamma (S \Gamma I \Gamma S) \Gamma Q \subseteq Q \Gamma I \Gamma Q$. Hence $Q \cap I \subseteq Q \Gamma I \Gamma Q$. Therefore $Q \Gamma I \Gamma Q = Q \cap I$.

(2) ⇒ (1) Let $Q$ be a quasi-ideal of $S$. By (2), $Q \Gamma S \Gamma Q = Q \cap S$. Hence $Q \Gamma S \Gamma Q = Q$. Therefore $S$ is regular by Theorem 2.

(1) ⇒ (3) Let $Q$ be a quasi-ideal and $I$ be an interior ideal of $S$. Now $Q \Gamma I \Gamma Q \subseteq Q \Gamma S \Gamma Q \subseteq Q \Gamma S$ by Proposition 1. Similarly we get $Q \Gamma I \Gamma Q \subseteq S \Gamma Q$. Therefore $Q \Gamma I \Gamma Q \subseteq (S \Gamma Q) \cap (Q \Gamma S) \subseteq Q$. Also $Q \Gamma I \Gamma Q \subseteq I$ as $I$ is an interior ideal. Therefore $Q \Gamma I \Gamma Q \subseteq Q \cap I$. For the reverse inclusion, let $a \in Q \cap I$. As $S$ is regular, $a \in \alpha \Gamma S \Gamma a$. Therefore $a \in (\alpha \Gamma S \Gamma a) \Gamma \Gamma (\alpha \Gamma S \Gamma a) \subseteq (Q \Gamma S \Gamma Q) \Gamma \Gamma (S \Gamma I \Gamma S) \Gamma Q \subseteq Q \Gamma I \Gamma Q$. Therefore $Q \cap I \subseteq Q \Gamma I \Gamma Q$. Hence $Q \Gamma I \Gamma Q = Q \cap I$.

(3) ⇒ (1) Let $Q$ be a quasi-ideal of $S$. By (3), $Q \Gamma S \Gamma Q = Q \cap S$. Hence $Q \Gamma S \Gamma Q = Q$. Hence by Theorem 2, $S$ is regular. □
Theorem 6. In $S$ the following statements are equivalent.

1. $S$ is regular.
2. $Q \cap L \subseteq QFGL$, for a quasi-ideal $Q$ and a left ideal $L$ of $S$.
3. $Q \cap R \subseteq RGQ$, for a quasi-ideal $Q$ and a right ideal $R$ of $S$.

Theorem 7. $S$ is regular if and only if $R \cap Q \cap L \subseteq RFGQL$, for a right ideal $R$, quasi-ideal $Q$ and a left ideal $L$ of $S$.

Proof. Suppose that $S$ is a regular $\Gamma$-semiring. Let $R$ be a right ideal, $Q$ be a quasi-ideal and $L$ be a left ideal of $S$. Let $a \in R \cap Q \cap L$. As $S$ is regular, $a \in a\Gamma S\Gamma a$. Therefore $a \in (a\Gamma S\Gamma a)\Gamma S\Gamma a \subseteq (R\Gamma S)\Gamma S\Gamma (S\Gamma L) \subseteq R\Gamma QGL$. Hence $R \cap Q \cap L \subseteq RFGQL$. Conversely, let $R$ be a right ideal and $L$ be a left ideal of $S$. By assumption $R \cap S \cap L \subseteq R\Gamma S\Gamma L$. Therefore $R \cap L \subseteq R\Gamma L$. Thus we have $R \cap L = R\Gamma L$. Hence $S$ is regular by Theorem 1. 

4 Intra-regular $\Gamma$-semiring

Now we give the definition of an intra-regular $\Gamma$-semiring.

Definition 10. A $\Gamma$-semiring $S$ is said to be an intra-regular $\Gamma$-semiring if for any $x \in S$, $x \in S\Gamma x \Gamma x S$.

Theorem 8. $S$ is intra-regular if and only if each right ideal $R$ and left ideal $L$ of $S$ satisfy $R \cap L \subseteq LR\Gamma$.

Proof. Suppose that $S$ is an intra-regular $\Gamma$-semiring and $R$ and $L$ be a right ideal and a left ideal of $S$ respectively. Let $a \in R \cap L$. As $S$ is intra-regular, $a \in S\Gamma a \Gamma a \Gamma S$. Now $S\Gamma a \Gamma a \Gamma S = (S\Gamma a)\Gamma (a\Gamma S) \subseteq (S\Gamma L)\Gamma (R\Gamma S) \subseteq LR\Gamma$. Therefore $R \cap L \subseteq LR\Gamma$. Conversely, for $a \in S$, $(a)_l = N_0 a + S\Gamma a$, $(a)_r = N_0 a + a\Gamma S$. By assumption $(a)_l \cap (a)_l \subseteq (a)_l \Gamma (a)_r$. Then $(a)_r \cap (a)_l \subseteq (a)_l \Gamma (a)_r = (N_0 a + S\Gamma a) \Gamma (N_0 a + a\Gamma S)$. Also by assumption we have $(a)_r \subseteq S\Gamma a + S\Gamma a \Gamma S$ and $(a)_l \subseteq a\Gamma S + S\Gamma a \Gamma S$. Hence we have $(a)_r \subseteq S\Gamma a + S\Gamma a \Gamma S \subseteq S\Gamma a \Gamma a \Gamma S$. Therefore we get $a \in S\Gamma a \Gamma a \Gamma S$. Thus any $a \in S$ is an intra-regular element of $S$. Therefore $S$ is an intra-regular $\Gamma$-semiring.

Theorem 9. In $S$ the following statements are equivalent.

1. $S$ is intra-regular.
2. For bi-ideals $B_1$ and $B_2$ of $S$, $B_1 \cap B_2 \subseteq S\Gamma B_1 \Gamma B_2 \Gamma S$.
3. For every bi-ideal $B$ and a quasi-ideal $Q$ of $S$, $B \cap Q \subseteq (S\Gamma Q \Gamma B \Gamma S) \cap (S\Gamma B \Gamma Q \Gamma S)$.
4. For every quasi-ideals $Q_1$ and $Q_2$ of $S$, $Q_1 \cap Q_2 \subseteq S\Gamma Q_1 \Gamma Q_2 \Gamma S$.

Proof. (1) $\Rightarrow$ (2) Suppose that $S$ is intra-regular. Let $B_1$ and $B_2$ be bi-ideals of $S$. Let $a \in B_1 \cap B_2$. As $S$ is intra-regular, $a \in S\Gamma a \Gamma a \Gamma S$. $a \in S\Gamma a \Gamma a \Gamma S \subseteq S\Gamma B_1 \Gamma B_2 \Gamma S$. Therefore $B_1 \cap B_2 \subseteq S\Gamma B_1 \Gamma B_2 \Gamma S$.

(2) $\Rightarrow$ (3), (3) $\Rightarrow$ (4) Implications follow as every quasi-ideal is a bi-ideal.
(4) \implies (1) Let \( L \) be a left ideal and \( R \) be a right ideal of \( S \). Then \( R \) and \( L \) both are quasi-ideals of \( S \). By (4), \( R \cap L \subseteq S \Gamma \Delta_\Gamma R \Gamma S = (S \Gamma L) \Gamma (R \Gamma S) \subseteq L \Gamma R \). Therefore we get \( R \cap L \subseteq L \Gamma R \). Thus by Theorem 8, \( S \) is an intra-regular \( \Gamma \)-semiring. Thus we have proved (1) \implies (2) \implies (3) \implies (4) \implies (1).

\[\square\]

**Theorem 10.** In \( S \) the following statements are equivalent.

(1) \( S \) is intra-regular.

(2) For a left ideal \( L \) and a bi-ideal \( B \) of \( S \), \( L \cap B \subseteq L \Gamma B \Gamma S \).

(3) For a left ideal \( L \) and a quasi-ideal \( Q \) of \( S \), \( L \cap Q \subseteq L \Gamma Q \Gamma S \).

(4) For a right ideal \( R \) and a bi-ideal \( B \) of \( S \), \( R \cap B \subseteq S \Gamma B \Gamma R \).

(5) For a right ideal \( R \) and a quasi-ideal \( Q \) of \( S \), \( R \cap Q \subseteq S \Gamma Q \Gamma R \).

**Proof.** (1) \implies (2) Suppose that \( S \) is intra-regular. Let \( L \) be a left ideal and \( B \) be a bi-ideal of \( S \). Let \( a \in B \cap L \). As \( S \) is intra-regular, \( a \in S \Gamma a \Gamma S \). Hence \( B \cap L \subseteq L \Gamma B \Gamma S \).

(2) \implies (3), (4) \implies (5) As every quasi-ideal is a bi-ideal, implications follow.

(3) \implies (1) Let \( L \) be a left ideal and \( R \) be a right ideal of \( S \). Then \( R \) is a quasi-ideal of \( S \). By (3), \( R \cap L \subseteq L \Gamma R \Gamma S \subseteq L \Gamma R \). Therefore we get \( R \cap L \subseteq L \Gamma R \). Thus by Theorem 8, \( S \) is an intra-regular \( \Gamma \)-semiring.

(1) \implies (4) Suppose that \( S \) is intra-regular. Let \( R \) be a right ideal and \( B \) be a bi-ideal of \( S \). Let \( a \in B \cap R \). As \( S \) is intra-regular, \( a \in S \Gamma a \Gamma S \). Hence \( B \cap R \subseteq S \Gamma B \Gamma R \). This shows that \( B \cap R \subseteq S \Gamma B \Gamma R \).

(5) \implies (1) Let \( L \) be a left ideal and \( R \) be a right ideal of \( S \). By (5), \( R \cap L \subseteq S \Gamma L \Gamma R \subseteq L \Gamma R \), since \( L \) is a quasi-ideal of \( S \). Therefore we get \( R \cap L \subseteq L \Gamma R \). This shows that \( S \) is an intra-regular \( \Gamma \)-semiring by Theorem 8.

\[\square\]

**Theorem 11.** In \( S \) the following statements are equivalent.

(1) \( S \) is intra-regular.

(2) \( K \cap B \cap R \subseteq K \Gamma B \Gamma R \), for a bi-ideal \( B \), a right ideal \( R \) and an interior ideal \( K \) of \( S \).

(3) \( I \cap B \cap R \subseteq I \Gamma B \Gamma R \), for a bi-ideal \( B \), a right ideal \( R \) and an ideal \( I \) of \( S \).

(4) \( K \cap Q \cap R \subseteq K \Gamma Q \Gamma R \), for a quasi-ideal \( Q \), a right ideal \( R \) and an interior ideal \( K \) of \( S \).

(5) \( I \cap Q \cap R \subseteq I \Gamma Q \Gamma R \), for a quasi-ideal \( Q \), a right ideal \( R \) and an ideal \( I \) of \( S \).

**Proof.** (1) \implies (2) Suppose that \( S \) is intra-regular. Let \( R \) be a right ideal, \( K \) be an interior ideal and \( B \) be a bi-ideal of \( S \). Let \( a \in K \cap B \cap R \). As \( S \) is intra-regular, \( a \in S \Gamma a \Gamma S \). Therefore \( a \in S \Gamma a \Gamma S \subseteq (S \Gamma K \Gamma S) \Gamma B \Gamma (R \Gamma S \Gamma S) \subseteq K \Gamma B \Gamma R \). Thus we have \( K \cap B \cap R \subseteq K \Gamma B \Gamma R \).

(2) \implies (3), (4) \implies (5) As every ideal is an interior ideal, implications follow.

(2) \implies (4), (3) \implies (5) Clearly implications follow, since quasi-ideal is a bi-ideal.

(5) \implies (1) Let \( L \) be a left ideal and \( R \) be a right ideal of \( S \). As \( L \) is a quasi-ideal of \( S \), by (5) we have \( S \cap L \cap R \subseteq S \Gamma L \Gamma R \subseteq L \Gamma R \). Therefore we have \( R \cap L \subseteq L \Gamma R \). Hence by Theorem 8, \( S \) is an intra-regular \( \Gamma \)-semiring.

\[\square\]
Theorem 12. In $S$ the following statements are equivalent.

(1) $S$ is intra-regular.
(2) $I \cap B \cap L \subseteq L \Gamma B \Gamma I$, for a bi-ideal $B$, a left ideal $L$ and an interior ideal $I$ of $S$.
(3) $I \cap B \cap L \subseteq L \Gamma B \Gamma I$, for a bi-ideal $B$, a left ideal $L$ and an ideal $I$ of $S$.
(4) $I \cap Q \cap L \subseteq L \Gamma Q \Gamma I$, for a quasi-ideal $Q$, a left ideal $L$ and an interior ideal $I$ of $S$.
(5) $I \cap Q \cap L \subseteq L \Gamma Q \Gamma I$, for a quasi-ideal $Q$, a left ideal $L$ and an ideal $I$ of $S$.

Proof. (1) $\Rightarrow$ (2) Suppose that $S$ is intra-regular. Let $L$ be a left ideal, $I$ be an interior ideal and $B$ be a bi-ideal of $S$. Let $a \in I \cap B \cap L$. As $S$ is intra-regular, $a \in S \alpha \Gamma a \Gamma S$. Let $a \in S \alpha \Gamma a \Gamma S \subseteq (S \Gamma S \Gamma L) \Gamma B \Gamma (S \Gamma I S) \subseteq L \Gamma B \Gamma I$. Thus we have $I \cap B \cap L \subseteq L \Gamma B \Gamma I$.

(2) $\Rightarrow$ (3), (4) $\Rightarrow$ (5) Clearly implications follow, since an ideal is an interior ideal.

(2) $\Rightarrow$ (4), (4) $\Rightarrow$ (5) As every quasi-ideal is a bi-ideal, implications follow.

(5) $\Rightarrow$ (1) Let $L$ be a left ideal and $R$ be a right ideal of $S$. As right ideal $R$ is a quasi-ideal, and $S$ itself is an ideal of $S$, $S \cap R \cap L \subseteq L \Gamma R \Gamma S$ by (5). Therefore $L \Gamma R \Gamma S \subseteq L \Gamma R$. Thus we get $R \cap L \subseteq L \Gamma R$. Therefore $S$ is an intra-regular $\Gamma$-semiring by Theorem 8. \hfill $\Box$

5 Regular and Intra-regular $\Gamma$-semiring

Theorem 13. For $S$ the following statements are equivalent.

(1) $S$ is regular and intra-regular.
(2) Each right ideal $R$ and left ideal $L$ of $S$ satisfy $R \cap L = R \Gamma L \subseteq L \Gamma R$.
(3) Each bi-ideal $B$ of $S$ satisfies $B = B^2 = B \Gamma B$.
(4) Each quasi-ideal $Q$ of $S$ satisfies $Q = Q^2 = Q \Gamma Q$.

Proof. (1) $\Leftrightarrow$ (2) Proof follows from Theorems 1 and 8.

(1) $\Rightarrow$ (3) Suppose that $S$ is regular and intra-regular. Let $B$ be a bi-ideal of $S$. Then $B^2 = B \Gamma B \subseteq B$. For the reverse inclusion, let $a \in B$. As $S$ is regular and intra-regular, we have $a \in a \Gamma a \Gamma a \subseteq a \Gamma S \Gamma a \subseteq a \Gamma S \Gamma (S \alpha \Gamma a \Gamma S) \Gamma a \subseteq (B \Gamma S \Gamma B) \Gamma (B \Gamma S \Gamma B) \subseteq B \Gamma B$. Therefore $B \subseteq B \Gamma B$. Thus we get $B = B \Gamma B = B^2$.

(3) $\Rightarrow$ (4) As every quasi-ideal is a bi-ideal, implication follows.

(4) $\Rightarrow$ (1) Let $L$ be a left ideal and $R$ be a right ideal of $S$. Then $R \cap L$ is a quasi-ideal of $S$. By (4), $R \cap L = (R \cap L)^2 = (R \cap L) \Gamma (R \cap L) \subseteq L \Gamma R$. This shows that $S$ is an intra-regular $\Gamma$-semiring by Theorem 8. Similarly $R \cap L = (R \cap L)^2 = (R \cap L) \Gamma (R \cap L) \subseteq R \Gamma L$. Hence we get $R \cap L = R \Gamma L$. Therefore $S$ is a regular $\Gamma$-semiring by Theorem 1. \hfill $\Box$

Theorem 14. In $S$ the following statements are equivalent.

(1) $S$ is regular and intra-regular.
(2) For bi-ideals $B_1$ and $B_2$ of $S$, $B_1 \cap B_2 \subseteq (B_1 \Gamma B_2) \cap (B_2 \Gamma B_1)$.
(3) For every bi-ideal $B$ and a quasi-ideal $Q$ of $S$, $B \cap Q \subseteq (Q \Gamma B) \cap (B \Gamma Q)$.
(4) For quasi-ideals $Q_1$ and $Q_2$ of $S$, $Q_1 \cap Q_2 \subseteq (Q_1 \Gamma Q_2) \cap (Q_2 \Gamma Q_1)$.
(5) For every quasi-ideal $Q$ and a generalized bi-ideal $G$ of $S$, $G \cap Q \subseteq (GTQ) \cap (QFG)$.

(6) For every left ideal $L$ and a bi-ideal $B$ of $S$, $B \cap L \subseteq (BGL) \cap (LGB)$.

(7) For every left ideal $L$ and a quasi-ideal $Q$ of $S$, $Q \cap L \subseteq (QFL) \cap (LQG)$.

(8) For every right ideal $R$ and a bi-ideal $B$ of $S$, $B \cap R \subseteq (BFR) \cap (RFB)$.

(9) For every quasi-ideal $Q$ and a right ideal $R$ of $S$, $R \cap Q \subseteq (RQG) \cap (QGR)$.

(10) For every left ideal $L$ and a right ideal $R$ of $S$, $R \cap L \subseteq (RFL) \cap (LGR)$.

Proof. (1) $\Rightarrow$ (2) Suppose that $S$ is regular and intra-regular. Let $B_1$ and $B_2$ be bi-ideals of $S$. Let $a \in B_1 \cap B_2$. As $S$ is regular and intra-regular, $a \in a\Gamma S\Gamma a$ and $a \in S\Gamma a\Gamma S$. Hence $a \in a\Gamma S\Gamma a \subseteq (a\Gamma S\Gamma S\Gamma a) \subseteq (B_1\Gamma S\Gamma B_1) \Gamma (B_2\Gamma S\Gamma B_2) \subseteq B_1\Gamma B_2$.

Similarly we can show that $a \in B_2\Gamma B_1$. Therefore $a \in B_1 \cap B_2$ implies $a \in B_1 \Gamma B_2$ and $a \in B_2 \Gamma B_1$. This gives $B_1 \cap B_2 \subseteq (B_1 \Gamma B_2) \cap (B_2 \Gamma B_1)$.

(2) $\Rightarrow$ (1) Let $L$ be a left ideal and $R$ be a right ideal of $S$. Then $R$ and $L$ both are quasi-ideals of $S$. By (4), $R \cap L \subseteq (R\Gamma L) \cap (L\Gamma R)$. Hence $R \cap L \subseteq L\Gamma R$ implies $S$ is an intra-regular $\Gamma$-semiring by Theorem 8. Also $R \cap L \subseteq R\Gamma L$. Therefore we get $R \cap L = R\Gamma L$. Hence by Theorem 1, $S$ is a regular $\Gamma$-semiring.

(1) $\Rightarrow$ (5) Suppose that $S$ is regular and intra-regular. Let $G$ be a generalized bi-ideal and $Q$ be quasi-ideal of $S$. Let $a \in G \cap Q$. As $S$ is regular and intra-regular, $a \in a\Gamma S\Gamma a$ and $a \in S\Gamma a\Gamma S$. Therefore $a \in a\Gamma S\Gamma a \subseteq a\Gamma S\Gamma (a\Gamma S\Gamma a) \subseteq (a\Gamma S\Gamma S\Gamma a) \subseteq (G\Gamma S\Gamma G) \subseteq G\Gamma (S\Gamma S\Gamma G) \subseteq G\Gamma Q$. Therefore $a \in G\cap Q$. Similarly we can show that $a \in Q\cap G$. Therefore $a \in G\cap Q$ implies $a \in G\Gamma Q$ and $a \in Q\Gamma G$, which gives $G \cap Q \subseteq (GTQ) \cap (QFG)$.

(5) $\Rightarrow$ (1) Let $L$ be a left ideal and $R$ be a right ideal of $S$ respectively. As $R$ is a generalized bi-ideal and $L$ is a quasi-ideal of $S$, proof follows from (4) $\Rightarrow$ (1).

(1) $\Rightarrow$ (6) Suppose that $S$ is regular and intra-regular. Let $B$ be a bi-ideal and $L$ be a left ideal of $S$. Let $a \in B \cap L$. As $S$ is regular and intra-regular, $a \in a\Gamma S\Gamma a$ and $a \in S\Gamma a\Gamma S$. $a \in a\Gamma S\Gamma a \subseteq a\Gamma S\Gamma (a\Gamma S\Gamma a) \subseteq (a\Gamma S\Gamma S\Gamma a) \subseteq (B\Gamma S\Gamma B) \subseteq B\Gamma L$. Therefore we get $a \in B\Gamma L$. Similarly we can show that $a \in L\Gamma B$. Therefore $a \in B \cap L$ implies $a \in B\Gamma L$ and $a \in L\Gamma B$. Hence $B \cap L \subseteq (B\Gamma L) \cap (L\Gamma B)$.

(6) $\Rightarrow$ (7) As every quasi-ideal is a bi-ideal, implication follows.

(7) $\Rightarrow$ (1) Let $L$ be a left ideal and $R$ be a right ideal of $S$ respectively. As $R$ is a quasi-ideal of $S$, proof follows from (4) $\Rightarrow$ (1).

(1) $\Rightarrow$ (8) Suppose that $S$ is regular and intra-regular. Let $R$ be right ideal and $B$ be a bi-ideals of $S$. Let $a \in B \cap R$. As $S$ is regular and intra-regular, $a \in a\Gamma S\Gamma a$ and $a \in S\Gamma a\Gamma S$. Therefore $a \in a\Gamma S\Gamma a \subseteq a\Gamma S\Gamma (a\Gamma S\Gamma a) \subseteq (a\Gamma S\Gamma S\Gamma a) \subseteq (B\Gamma S\Gamma B) \subseteq B\Gamma R$. Therefore we get $a \in B\Gamma R$. Similarly we can show that $a \in R\Gamma B$. Therefore $a \in B \cap R$ implies $a \in B\Gamma R$ and $a \in R\Gamma B$, which gives $B \cap R \subseteq (B\Gamma R) \cap (R\Gamma B)$.

(8) $\Rightarrow$ (9), (9) $\Rightarrow$ (10) Implications follow as every left ideal is a quasi-ideal.
(10) \Rightarrow (1) Let \( L \) be a left ideal and \( R \) be a right ideal of \( S \). Proof follows from (4) \Rightarrow (1).

**Theorem 15.** In \( S \) the following statements are equivalent.

(1) \( S \) is regular and intra-regular.

(2) For bi-ideals \( B_1 \) and \( B_2 \) of \( S \), \( B_1 \cap B_2 \subseteq (B_1 \Gamma B_2 \Gamma B_1) \cap (B_2 \Gamma B_1 \Gamma B_2) \).

(3) For a quasi-ideal \( Q \) and a bi-ideal \( B \) of \( S \), \( Q \cap B \subseteq (B \Gamma Q \Gamma B) \cap (Q \Gamma B \Gamma Q) \).

(4) For quasi-ideals \( Q_1 \) and \( Q_2 \) of \( S \), \( Q_1 \cap Q_2 \subseteq (Q_1 \Gamma Q_2 \Gamma Q_1) \cap (Q_2 \Gamma Q_1 \Gamma Q_2) \).

(5) For a bi-ideal \( B \) and a left ideal \( L \) of \( S \), \( B \cap L \subseteq B \Gamma L \Gamma B \).

(6) For a quasi-ideal \( Q \) and a left ideal \( L \) of \( S \), \( Q \cap L \subseteq Q \Gamma L \Gamma Q \).

(7) For a bi-ideal \( B \) and a right ideal \( R \) of \( S \), \( B \cap R \subseteq B \Gamma R \Gamma B \).

(8) For a quasi-ideal \( Q \) and a right ideal \( R \) of \( S \), \( Q \cap R \subseteq Q \Gamma R \Gamma Q \).

(9) For a quasi-ideal \( Q \) and a generalized bi-ideal \( G \) of \( S \), \( Q \cap G \subseteq (Q \Gamma G \Gamma Q) \cap (G \Gamma Q \Gamma G) \).

**Proof.** (1) \Rightarrow (2) Suppose that \( S \) is regular and intra-regular. Let \( B_1 \) and \( B_2 \) be bi-ideals of \( S \). Let \( a \in B_1 \cap B_2 \). As \( S \) is regular and intra-regular, \( a \in \alpha \Gamma STa \) and \( a \in STa \Gamma \alpha S \). Hence \( a \in \alpha \Gamma STa \subseteq \alpha \Gamma STa(\alpha \Gamma STa) \subseteq (\alpha \Gamma STa) \Gamma (\alpha \Gamma STa) \subseteq (B_1 \Gamma STB_1) \Gamma (B_2 \Gamma STB_2) \Gamma (B_1 \Gamma STB_1) \subseteq B_2 \Gamma B_2 \Gamma B_1 \). Therefore \( B_1 \cap B_2 \subseteq B_1 \Gamma B_2 \Gamma B_1 \). In the same manner we can show that \( B_1 \cap B_2 \subseteq B_2 \Gamma B_2 \Gamma B_1 \). Thus we get \( B_1 \cap B_2 \subseteq (B_1 \Gamma B_2 \Gamma B_1) \cap (B_2 \Gamma B_2 \Gamma B_1) \).

(2) \Rightarrow (3), (3) \Rightarrow (4) Implications follow as every quasi-ideal is a bi-ideal.

(4) \Rightarrow (1) Let \( L \) be a left ideal and \( R \) be a right ideal of \( S \). Then \( R \cap L \) is a quasi-ideal of \( S \). By (4), \((R \cap L) \cap (R \cap L) \subseteq ((R \cap L) \Gamma (R \cap L) \Gamma (R \cap L)) \subseteq LR \Gamma R \subseteq LR \). Hence \( R \cap L \subseteq LR \Gamma R \). This shows that \( S \) is an intra-regular \( \Gamma \)-semiring by Theorem 8. Also \( R \cap L \subseteq ((R \cap L) \Gamma (R \cap L) \Gamma (R \cap L)) \) implies \( R \cap L \subseteq LR \). Therefore \( R \cap L = LR \Gamma R \). Thus \( S \) is a regular \( \Gamma \)-semiring by Theorem 1.

(1) \Rightarrow (5) Suppose that \( S \) is regular and intra-regular. Let \( B \) be a bi-ideal and \( L \) be a left ideal of \( S \). Let \( a \in B \cap L \). As \( S \) is regular and intra-regular, \( a \in \alpha \Gamma STa \) and \( a \in STa \Gamma \alpha S \). Therefore \( a \in \alpha \Gamma STa \subseteq \alpha \Gamma STa(\alpha \Gamma STa) \subseteq (\alpha \Gamma STa) \Gamma (\alpha \Gamma STa) \subseteq (B \Gamma S \Gamma B) \Gamma (S \Gamma L) \Gamma (B \Gamma S \Gamma B) \subseteq B \Gamma L \Gamma B \). Hence we have \( B \cap L \subseteq B \Gamma L \Gamma B \).

(5) \Rightarrow (6) As every quasi-ideal is a bi-ideal, implication follows.

(6) \Rightarrow (7) Proof is similar to (4) \Rightarrow (1).

(1) \Rightarrow (7) \Rightarrow (8) \Rightarrow (1) can be proved similarly to (1) \Rightarrow (5) \Rightarrow (6) \Rightarrow (1).

Proof of (1) \Rightarrow (9) is similar to (1) \Rightarrow (2) and proof of (9) \Rightarrow (1) is parallel to (1) \Rightarrow (4) \Rightarrow (1).

Thus we have shown that (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (1), (1) \Rightarrow (5) \Rightarrow (6) \Rightarrow (1) and (1) \Rightarrow (7) \Rightarrow (8) \Rightarrow (1) and (1) \Rightarrow (9) \Rightarrow (1).

**Theorem 16.** In \( S \) the following statements are equivalent.

(1) \( S \) is regular and intra-regular.

(2) \( B \cap R \cap L \subseteq B \Gamma R \Gamma L \), for a bi-ideal \( B \), right ideal \( R \) and a left ideal \( L \) of \( S \).

(3) \( Q \cap R \cap L \subseteq Q \Gamma R \Gamma L \), for a quasi-ideal \( Q \), right ideal \( R \) and left ideal \( R \) of \( S \).
Proof. (1) ⇒ (2) Suppose that $S$ is regular and intra-regular. Let $B$ be a bi-ideal, $R$ be a right ideal and $L$ be a left ideal of $S$. Let $a \in B \cap R \cap L$. As $S$ is regular and intra-regular, $a \in a\Gamma S \subseteq a\Gamma S \subseteq (a\Gamma S) \subseteq (B\Gamma S) \subseteq B\Gamma S$. Therefore $B \cap R \cap L \subseteq B\Gamma S$.

(2) ⇒ (3) As every quasi-ideal is a bi-ideal, implication follows.

(3) ⇒ (1) Let $L$ be a left ideal and $R$ be a right ideal of $S$. As $R$ is a quasi-ideal and $S$ itself a left ideal of $S$, by (3) we have $R \cap S \subseteq L \cap R \subseteq L \cap R$. Therefore $L \cap R \subseteq L \cap R$. Thus we get $L \cap R = L \cap R$. Hence $S$ is a regular $\Gamma$-semiring by Theorem 1. Similarly $L$ is a quasi-ideal and $S$ itself a left ideal of $S$ gives $L \cap R \subseteq L \cap R$ by (3). Thus $L \cap R \subseteq L \cap R$. This shows that $S$ is an intra-regular $\Gamma$-semiring by Theorem 8.

6 Duo $\Gamma$-semiring

Now we define the notion of a duo $\Gamma$-semiring as follows.

**Definition 11.** A $\Gamma$-semiring $S$ is said to be a left (right) duo $\Gamma$-semiring if every left (right) ideal of $S$ is a right (left) ideal.

A $\Gamma$-semiring $S$ is said to be a duo $\Gamma$-semiring if every one-sided ideal of $S$ is a two-sided ideal.

That is a $\Gamma$-semiring $S$ is said to be a duo $\Gamma$-semiring if it is both left duo and right duo.

**Theorem 17.** If $S$ is regular, then $S$ is left duo if and only if for any two left ideals $A$ and $B$ of $S$, $A \cap B = A\Gamma B$.

**Proof.** Let $S$ be a regular $\Gamma$-semiring. Assume that $S$ is left duo. Let $A$ and $B$ be any two left ideals of $S$. As $S$ is left duo, $A$ is a right ideal of $S$. Then by Theorem 1, $A \cap B = A\Gamma B$. Conversely, suppose that the given condition holds. Let $L$ be a left ideal of $S$. Then by assumption $L \Gamma S \subseteq L \cap S \subseteq L \Gamma S$. This shows that $L$ is a right ideal of $S$. Therefore $S$ is a left duo $\Gamma$-semiring.

Proof of the following theorem is analogous to proof of Theorem 17.

**Theorem 18.** If $S$ is regular, then $S$ is right duo if and only if for any two right ideals $A$ and $B$ of $S$, $A \cap B = A\Gamma B$.

**Theorem 19.** If $S$ is regular, then $S$ is left duo if and only if every quasi-ideal of $S$ is a right ideal of $S$.

**Proof.** Let $S$ be a regular $\Gamma$-semiring. Suppose that $S$ is left duo. Let $Q$ be any quasi-ideal of $S$. Then there exists a right ideal $R$ and a left ideal $L$ of $S$ such that $Q = R \cap L$. Therefore $Q = R \cap L$ is a right ideal of $S$. Conversely, let $L$ be a left ideal of $S$. Then $L$ is a quasi-ideal of $S$. Hence by assumption $L$ is a right ideal of $S$. Therefore $S$ is a left duo $\Gamma$-semiring.
Proofs of the following theorems are similar to proof of Theorem 19.

**Theorem 20.** If $S$ is regular, then $S$ is right duo if and only if every quasi-ideal of $S$ is a left ideal of $S$.

**Theorem 21.** If $S$ is regular, then $S$ is duo if and only if every quasi-ideal of $S$ is an ideal of $S$.

**Theorem 22.** If $S$ is regular, then $S$ is duo if and only if every bi-ideal of $S$ is a ideal of $S$.

**Theorem 23.** In $S$ the following conditions are equivalent.
1. $S$ is regular duo.
2. $I \cap B = I \Gamma B \Gamma I$, for every ideal $I$ and a bi-ideal $B$ of $S$.
3. $I \cap Q = I \Gamma Q \Gamma I$, for every ideal $I$ and a quasi-ideal $Q$ of $S$.

**Proof.** (1) $\Rightarrow$ (2) Suppose that $S$ is a regular duo $\Gamma$-semiring. Let $I$ be an ideal and $B$ be a bi-ideal of $S$. Then by Theorem 22, $B$ is an ideal of $S$. Therefore $I \Gamma B \Gamma I \subseteq I$ and $I \Gamma B \Gamma I \subseteq B$, since $I$ and $B$ are ideals of $S$. Hence $I \Gamma B \Gamma I \subseteq I \cap B$. For the reverse inclusion, let $a \in I \cap B$. $S$ is regular implies $a \in a \Gamma S \Gamma a$. $a \in a \Gamma S \Gamma a \subseteq a \Gamma S \Gamma (\Gamma a \Gamma) \subseteq (I \Gamma S \Gamma) \Gamma B \Gamma (\Gamma S \Gamma) \subseteq I \Gamma B \Gamma I$. Therefore $I \cap B \subseteq I \Gamma B \Gamma I$. Hence $I \cap B = I \Gamma B \Gamma I$.

(2) $\Rightarrow$ (3) As every quasi-ideal of $S$ is a bi-ideal of $S$, implication follows.
(3) $\Rightarrow$ (1) Let $L$ be a left ideal and $R$ be a right ideal of $S$. Hence $S \cap L = S L \Gamma S$ and $S \cap R = S \Gamma R \Gamma S$ by (3). Therefore $L = S L \Gamma S$ and $R = S \Gamma R \Gamma S$. Now $L \Gamma S = S L \Gamma S \Gamma S \subseteq S \Gamma L \Gamma S = L$ and $S \Gamma R = S \Gamma S \Gamma R \Gamma S \subseteq S \Gamma R \Gamma S = R$. Hence $L \Gamma S \subseteq L$ and $S \Gamma R \subseteq R$. This shows that $L$ is a right ideal and $R$ is a left ideal of $S$. Therefore $S$ is a duo $\Gamma$-semiring by Definition 11. As $S$ is a duo $\Gamma$-semiring, $R \cap L = R \Gamma \Gamma L \Gamma R$ by (3). $R \cap L = R \Gamma L \Gamma R \subseteq R \Gamma L$. This shows that $R \cap L = R \Gamma L$. Hence by Theorem 1, $S$ is regular.

**Theorem 24.** If $S$ is a $\Gamma$-semiring then the following statements are equivalents.
1. $S$ is regular duo.
2. For every bi-ideals $A$ and $B$ of $S$, $A \cap B = A \Gamma B$.
3. For every bi-ideal $B$ and a quasi-ideal $Q$ of $S$, $B \cap Q = B \Gamma Q$.
4. For every bi-ideal $B$ and a right ideal $R$ of $S$, $B \cap R = B \Gamma R$.
5. For every quasi-ideal $Q$ and a bi-ideal $B$ of $S$, $Q \cap B = Q \Gamma B$.
6. For every quasi-ideals $Q_1$ and $Q_2$ of $S$, $Q_1 \cap Q_2 = Q_1 \Gamma Q_2$.
7. For every quasi-ideal $Q$ and a right ideal $R$ of $S$, $Q \cap R = Q \Gamma R$.
8. For every left ideal $L$ and a bi-ideal $B$ of $S$, $L \cap B = L \Gamma B$.
9. For every left ideal $L$ and a right ideal $R$ of $S$, $L \cap R = L \Gamma R$.

**Proof.** We can prove the equivalence of statements such as (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1), (1) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (1) and (1) $\Rightarrow$ (8) $\Rightarrow$ (9) $\Rightarrow$ (1). Proof of each implication is straightforward so omitted. \qed
References


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