BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 1(83), 2017, Pages 3–14
ISSN 1024–7696

Regular, Intra-regular and Duo Γ-Semirings

R. D. Jagatap, Y. S. Pawar

Abstract. In this paper we give several characterizations of a regular Γ-semiring, intra-regular Γ-semiring and a duo Γ-semiring by using ideals, interior-ideals, quasi-ideals and bi-ideals of a Γ-semiring.

Mathematics subject classification: 16Y60, 16Y99.
Keywords and phrases: Quasi-ideal, bi-ideal, interior-ideal, regular Γ-semiring, intra-regular Γ-semiring, duo Γ-semiring.

1 Introduction

The notion of a quasi-ideal was firstly introduced for semigroups in [15] and for rings in [16] by Steinfeld. Iseki in [6] discussed some characterizations of quasi-ideals for a semiring without zero. Using quasi-ideals, Shabir, Ali, Batool in [14] characterize a class of semirings. Chinram in [2] generalizes the concept of a quasi-ideal to a Γ-semigroup and discussed some of its properties. Also in [1] Chinram gave some different characterizations of quasi-ideals in a Γ-semiring while the concept of a Γ-semiring was coined by Rao in [13]. The authors studied quasi-ideals and minimal quasi-ideals in Γ-semirings in [7] and quasi-ideals in regular Γ-semirings in [8].

The notion of a bi-ideal was first introduced for semigroups by Good and Hughes in [4]. The concept of a bi-ideal for a ring was given by Lajos [9] . Also in [10,11] Lajos discussed some characterizations of bi-ideals in semigroups. Shabir, Ali, Batool in [14] gave some properties of bi-ideals in a semiring.

The concept of a regular ring was introduced by J. von Neumann in [12] and he gave the definition of a regular ring as follows: a ring R is regular if for any $b \in R$ there exists $x \in R$ such that $b = bxb$. Analogously the concept of a regular semigroup was introduced by Green in [5] and a regular semiring was introduced by Zelznikov [17]. This concept of regularity was extended to a Γ-semiring by Rao [13] and wos studied by Dutta and Sardar in [3].

In this paper efforts are made to prove various characterizations of a regular Γ-semiring, intra-regular Γ-semiring and a duo Γ-semiring by using ideals, interior-ideals, quasi-ideals and bi-ideals of a Γ-semiring.

2 Preliminaries

First we recall some definitions of the basic concepts of Γ-semirings that we need in sequel. For this we follow Dutta and Sardar [3].

Definition 1. Let S and Γ be two additive commutative semigroups. S is called a Γ-semiring if there exists a mapping $S \times \Gamma \times S \longrightarrow S$ denoted by $a\alpha b$ for all $a, b \in S$ and $\alpha \in \Gamma$ satisfying the following conditions:
(i) $a\alpha\,(b + c) = (a\alpha b) + (a\alpha c)$,
(ii) $(b + c)\,\alpha a = (b\alpha a) + (c\alpha a)$,
(iii) $a(\alpha + \beta)c = (a\alpha c) + (a\beta c)$,
(iv) $a\alpha\,(b\beta c) = (a\alpha b)\,\beta c$; for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$.

Definition 2. An element $0 \in S$ is said to be an absorbing zero if $0\alpha a = 0 = a\alpha 0$, and $a + 0 = 0 + a = a$ for all $a \in S$ and $\alpha \in \Gamma$.

Definition 3. A non-empty subset T of a Γ-semiring S is said to be a sub-Γ- semiring of S if (T,+) is a subsemigroup of (S,+) and $a\alpha b \in T$ for all $a, b \in T$and $\alpha \in \Gamma$.

Definition 4. A non-empty subset T of a Γ-semiring S is called a left (respectively right) ideal of S if T is a subsemigroup of (S,+) and $x\alpha a \in T$ (respectively $a\alpha x \in T$) for all $a \in T$, $x \in S$ and $\alpha \in \Gamma$.

Definition 5. If T is both left and right ideal of a Γ-semiring S, then T is known as an ideal of S.

A quasi-ideal Q in a Γ-semiring S is defined as follows.

Definition 6. A subsemigroup Q of $(S, +)$ is a quasi-ideal of S if $(S\Gamma Q) \cap (Q\Gamma S) \subseteq Q$.

Example. Consider a Γ-semiring $S = M_{2\times 2}(N_0)$, where N_0 denotes the set of natural numbers with zero and Γ= S. Define $A\alpha B$= usual matrix product of A, α and B; for all $A, \alpha\ , B\ \in S$. Then
$Q = \left\{ \left(\begin{array}{cc} a & 0 \\ 0 & 0 \end{array} \right) \mid a \in N_0 \right\}$ is a quasi-ideal of a Γ-semiring S.

Definition 7. A non-empty subset B of a Γ- semiring S is a bi-ideal of a Γ-semiring S if B is a sub-Γ-semiring of S and $B\Gamma S\Gamma B \subseteq B$.

Example. Let N be the set of natural numbers and $\Gamma = 2N$. Then N and Γ both are additive commutative semigroups. An image of a mapping $N \times \Gamma \times N \longrightarrow N$ is denoted by $a\alpha b$ and defined as $a\alpha b$ = product of a, α, b, for all $a, b \in S$ and $\alpha \in \Gamma$. Then N forms a Γ-semiring. $B = 3N$ is a bi-ideal of N.

Now we define a generalized bi-ideal and an interior-ideal of a Γ- semiring S.

Definition 8. A non-empty subset B of a Γ- semiring S is a generalized bi-ideal of a Γ- semiring S if $B\Gamma S\Gamma B \subseteq B$.

Definition 9. A non-empty subset I of a Γ- semiring S is an interior-ideal of a Γ-semiring S if I is a subsemigroup of S and $S\Gamma I\Gamma S \subseteq I$.

Proposition 1. *For each non-empty subset X of a Γ- semiring S the following statements hold.*
(i) $S\Gamma X$ is a left ideal of S.
(ii) $X\Gamma S$ is a right ideal of S.
(iii) $S\Gamma X\Gamma S$ is an ideal of S.

Proposition 2. *If S is a Γ- semiring S and $a \in S$, then the following statements hold.*
(i) $S\Gamma a$ is a left ideal of S.
(ii) $a\Gamma S$ is a right ideal of S.
(iii) $S\Gamma a\Gamma S$ is an ideal of S.

Now onwards S denotes a Γ-semiring with absorbing zero unless otherwise stated.

3 Regular Γ-Semiring

An element a of a Γ-semiring S is said to be regular if $a \in a\Gamma S\Gamma a$.
If all elements of a Γ-semiring S are regular, then S is known as a regular Γ-semiring. The following theorem was proved in [8] by the authors.

Theorem 1. *In S the following statements are equivalent.*
(1) S is regular.
(2) For every left ideal L and a right ideal R of S, $R\Gamma L = R \cap L$.
(3) For every left ideal L and a right ideal R of S,
(i) $R^2 = R\Gamma R = R$,
(ii) $L^2 = L\Gamma L = L$,
(iii) $R\Gamma L = R \cap L$ is a quasi-ideal of S.
(4) The set of all quasi-ideals of S is a regular Γ-semigroup.
(5) Every quasi-ideal of S is of the form $Q\Gamma S\Gamma Q = Q$.

Theorem 2. *The following statements are equivalent in S.*
(1) S is regular.
(2) For any bi-ideal B of S, $B\Gamma S\Gamma B = B$.
(3) For any quasi-ideal Q of S, $Q\Gamma S\Gamma Q = Q$.

Proof. (1) $\Rightarrow$ (2) Let B be a bi-ideal of S and $b \in B$. As S is regular, $b \in b\Gamma S\Gamma b \subseteq B\Gamma S\Gamma B$. Therefore $B \subseteq B\Gamma S\Gamma B$. Hence $B = B\Gamma S\Gamma B$.
(2) $\Rightarrow$ (3) As every quasi-ideal is a bi-ideal, implication (2) $\Rightarrow$ (3) holds.
(3) $\Rightarrow$ (1) Let R be a right ideal and L be a left ideal of S. Then $R \cap L$ is a quasi-ideal of S. Hence by assumption $R\cap L = (R \cap L)\,\Gamma S\Gamma\,(R \cap L) \subseteq (R\Gamma S)\,\Gamma L \subseteq R\Gamma L$. Therefore $R \cap L = R\Gamma L$. Thus S is a regular Γ-semiring by Theorem 1. □

Theorem 3. *In S the following statements are equivalent.*
(1) S is regular.
(2) For every bi-ideal B and an ideal I of S, $B \cap I = B\Gamma I\Gamma B$.
(3) For every quasi-ideal Q and an ideal I of S, $Q \cap I = Q\Gamma I\Gamma Q$.

Proof. (1) $\Rightarrow$ (2) Let B be a bi-ideal and I be an ideal of S. Now $B\Gamma I\Gamma B \subseteq B\Gamma S\Gamma B \subseteq B$ and $B\Gamma I\Gamma B \subseteq I$. Therefore $B\Gamma I\Gamma B \subseteq B\cap I$. For the reverse inclusion, let $a \in B \cap I$. As S is regular, $a \in a\Gamma S\Gamma a$. Then $a\Gamma S\Gamma a \subseteq (a\Gamma S\Gamma a)\Gamma S\Gamma a\Gamma S\Gamma a) \subseteq (B\Gamma S\Gamma B)\,\Gamma(S\Gamma I\Gamma S)\Gamma B \subseteq B\Gamma I\Gamma B$. Therefore $a \in Q\Gamma I\Gamma Q$. Hence we have $B \cap I \subseteq B\Gamma I\Gamma B$. Thus we get $B\Gamma I\Gamma B = B \cap I$.
(2) $\Rightarrow$ (3) Implication follows as every quasi-ideal of S is a bi-ideal.
(3) $\Rightarrow$ (1) Let R be a right ideal and L be a left ideal of S. Then by assumption we have, $R = R\cap S = R\Gamma S\Gamma R \subseteq R\Gamma R$ and $L\cap S = L\Gamma S\Gamma L \subseteq L\Gamma L$. Also $R\cap L = R\Gamma L$ is a quasi-ideal of S. Hence by Theorem 1, S is a regular Γ-semiring. □

Proof of the following theorem is straightforward.

Theorem 4. *In S the following statements are equivalent.*
(1) S is regular.
(2) For every bi-ideal B and a left ideal L of S, $B \cap L \subseteq B\Gamma L$.
(3) For every quasi-ideal Q and a left ideal L of S, $Q \cap L \subseteq Q\Gamma\mathrm{L}$.
(4) For every bi-ideal B and a right ideal R of S, $B \cap R \subseteq R\Gamma B$.
(5) For every right ideal R and a quasi-ideal Q of S, $R \cap Q \subseteq R\Gamma Q$.
(6) For every left ideal L, every right ideal R and every bi-ideal B of S, $L \cap R \cap B \subseteq R\Gamma B\Gamma L$.
(7) For every left ideal, every right ideal R and every quasi-ideal Q of S, $L\cap R\cap Q \subseteq R\Gamma Q\Gamma L$.

Theorem 5. *In S the following conditions are equivalent.*
(1) S is regular.
(2) $I \cap Q = Q\Gamma I\Gamma Q$, for an ideal I and a quasi-ideal Q of S.
(3) $I \cap Q = Q\Gamma I\Gamma Q$, for an interior ideal I and a quasi-ideal Q of S.

Proof. (1) $\Rightarrow$ (2) Let Q be a quasi-ideal and I be an ideal of S. Now $Q\Gamma I\Gamma Q \subseteq Q\Gamma S\Gamma Q \subseteq Q\Gamma S$ by Proposition 1. Similarly we get $Q\Gamma I\Gamma Q \subseteq S\Gamma Q$. Therefore $Q\Gamma I\Gamma Q \subseteq (S\Gamma Q)\cap(Q\Gamma S) \subseteq Q$, since Q is a quasi-ideal. Also $Q\Gamma I\Gamma Q \subseteq I$ as I is an ideal. Therefore $Q\Gamma I\Gamma Q \subseteq Q\cap I$. For the reverse inclusion, let $a \in Q\cap I$. As S is regular, $a \in a\Gamma S\Gamma a$. We have $a \in (a\Gamma S\Gamma a)\,\Gamma S\Gamma\,(a\Gamma S\Gamma a) \subseteq (Q\Gamma S\Gamma Q)\,\Gamma(S\Gamma I\Gamma S)\Gamma Q \subseteq Q\Gamma I\Gamma Q$. Hence $Q \cap I \subseteq Q\Gamma I\Gamma Q$. Therefore $Q\Gamma I\Gamma Q = Q \cap I$.
(2) $\Rightarrow$ (1) Let Q be a quasi-ideal of S. By (2), $Q\Gamma S\Gamma Q = Q\cap S$. Hence $Q\Gamma S\Gamma Q = Q$. Therefore S is regular by Theorem 2.
(1) $\Rightarrow$ (3) Let Q be a quasi-ideal and I be an interior ideal of S. Now $Q\Gamma I\Gamma Q \subseteq Q\Gamma S\Gamma Q \subseteq Q\Gamma S$ by Proposition 1. Similarly we get $Q\Gamma I\Gamma Q \subseteq S\Gamma Q$. Therefore $Q\Gamma I\Gamma Q \subseteq (S\Gamma Q)\cap(Q\Gamma S) \subseteq Q$. Also $Q\Gamma I\Gamma Q \subseteq I$ as I is an interior ideal. Therefore $Q\Gamma I\Gamma Q \subseteq Q\cap I$. For the reverse inclusion, let $a \in Q\cap I$. As S is regular, $a \in a\Gamma S\Gamma a$. Therefore $a \in (a\Gamma S\Gamma a)\,\Gamma S\Gamma\,(a\Gamma S\Gamma a) \subseteq (Q\Gamma S\Gamma Q)\,\Gamma(S\Gamma I\Gamma S)\Gamma Q \subseteq Q\Gamma I\Gamma Q$. Therefore $Q \cap I \subseteq Q\Gamma I\Gamma Q$. Hence $Q\Gamma I\Gamma Q = Q \cap I$.
(3) $\Rightarrow$ (1) Let Q be a quasi-ideal of S. By (3), $Q\Gamma S\Gamma Q = Q\cap S$. Hence $Q\Gamma S\Gamma Q = Q$. Hence by Theorem 2, S is regular. □

Theorem 6. *In S the following statements are equivalent.*
(1) S is regular.
(2) $Q \cap L \subseteq Q\Gamma L$, for a quasi-ideal Q and a left ideal L of S.
(3) $Q \cap R \subseteq R\Gamma Q$, for a quasi-ideal Q and a right ideal R of S.

Theorem 7. *S is regular if and only if $R \cap Q \cap L \subseteq R\Gamma Q\Gamma L$, for a right ideal R, quasi-ideal Q and a left ideal L of S.*

Proof. Suppose that S is a regular Γ-semiring. Let R be a right ideal, Q be a quasi-ideal and L be a left ideal of S. Let $a \in R \cap Q \cap L$. As S is regular, $a \in a\Gamma S\Gamma a$. Therefore $a \in (a\Gamma S\Gamma a)\,\Gamma S\Gamma a \subseteq (R\Gamma S)\,\Gamma Q\Gamma\,(S\Gamma L) \subseteq R\Gamma Q\Gamma L$. Hence $R \cap Q \cap L \subseteq R\Gamma Q\Gamma L$. Conversely, let R be a right ideal and L be a left ideal of S. By assumption $R \cap S \cap L \subseteq R\Gamma S\Gamma L$. Therefore $R \cap L \subseteq R\Gamma L$. Thus we have $R \cap L = R\Gamma L$. Hence S is regular by Theorem 1. □

4 Intra-regular Γ-semiring

Now we give the definition of an intra-regular Γ-semiring.

Definition 10. A Γ-semiring S is said to be an intra-regular Γ-semiring if for any $x \in S$, $x \in S\Gamma x\Gamma x\Gamma S$.

Theorem 8. *S is intra-regular if and only if each right ideal R and left ideal L of S satisfy $R \cap L \subseteq L\Gamma R$.*

Proof. Suppose that S is an intra-regular Γ-semiring and R and L be a right ideal and a left ideal of S respectively. Let $a \in R \cap L$. As S is intra-regular, $a \in S\Gamma a\Gamma a\Gamma S$. Now $S\Gamma a\Gamma a\Gamma S = (S\Gamma a)\,\Gamma(a\Gamma \mathrm{S}) \subseteq (S\Gamma L)\,\Gamma(R\Gamma S) \subseteq L\Gamma R$. Therefore $R \cap L \subseteq L\Gamma R$. Conversely, for $a \in S$, $(a)_l = N_0 a + S\Gamma a$, $(a)_r = N_0 a + a\Gamma S$. By assumption $(a)_r \cap (a)_l \subseteq (a)_l\Gamma(a)_r$. Then $(a)_r \cap (a)_l \subseteq (a)_l\Gamma(a)_r = (N_0 a + S\Gamma a)\,\Gamma\,(N_0 a + a\Gamma S)$. Also by assumption we have $(a)_r \subseteq S\Gamma a\ + S\Gamma a\Gamma S$ and $(a)_l \subseteq a\Gamma S\ + S\Gamma a\Gamma S$. Hence we have $(a)_r \subseteq S\Gamma a\ + S\Gamma a\Gamma S \subseteq S\Gamma a\Gamma a\Gamma S$. Therefore we get $a \in S\Gamma a\Gamma a\Gamma S$. Thus any $a \in S$ is an intra-regular element of S. Therefore S is an intra-regular Γ-semiring. □

Theorem 9. *In S the following statements are equivalent.*
(1) S is intra-regular.
(2) For bi-ideals B_1 and B_2 of S, $B_1 \cap B_2 \subseteq S\Gamma B_1\Gamma B_2\Gamma S$.
(3) For every bi-ideal B and a quasi-ideal Q of S, $B \cap Q \subseteq (S\Gamma Q\Gamma B\Gamma S) \cap (S\Gamma B\Gamma Q\Gamma S)$.
(4) For every quasi-ideals Q_1 and Q_2 of S, $Q_1 \cap Q_2 \subseteq S\Gamma Q_1\Gamma Q_2\Gamma S$.

Proof. (1) ⇒ (2) Suppose that S is intra-regular. Let B_1 and B_2 be bi-ideals of S. Let $a \in B_1 \cap B_2$. As S is intra-regular, $a \in S\Gamma a\Gamma a\Gamma S$. $a \in S\Gamma a\Gamma a\Gamma S \subseteq S\Gamma B_1\Gamma B_2\Gamma S$. Therefore $B_1 \cap B_2 \subseteq S\Gamma B_1\Gamma B_2\Gamma S$.
(2) ⇒ (3), (3) ⇒ (4) Implications follow as every quasi-ideal is a bi-ideal.

(4) $\Rightarrow$ (1) Let L be a left ideal and R be a right ideal of S. Then R and L both are quasi-ideals of S. By (4), $R \cap L \subseteq S\Gamma L\Gamma R\Gamma S = (S\Gamma L)\,\Gamma\,(R\Gamma S) \subseteq L\Gamma R$. Therefore we get $R \cap L \subseteq L\Gamma R$. Thus by Theorem 8, S is an intra-regular Γ-semiring.
Thus we have proved (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$(4) $\Rightarrow$ (1). $\square$

Theorem 10. *In S the following statements are equivalent.*
(1) S is intra-regular.
(2) For a left ideal L and a bi-ideal B of S, $L \cap B \subseteq L\Gamma B\Gamma S$.
(3) For a left ideal L and a quasi-ideal Q of S, $L \cap Q \subseteq L\Gamma Q\Gamma S$.
(4) For a right ideal R and a bi-ideal B of S, $R \cap B \subseteq S\Gamma B\Gamma R$.
(5) For a right ideal R and a quasi-ideal Q of S, $R \cap Q \subseteq S\Gamma Q\Gamma R$.

Proof. (1) $\Rightarrow$ (2) Suppose that S is intra-regular. Let L be a left ideal and B be a bi-ideal of S. Let $a \in B \cap L$. As S is intra-regular, $a \in S\Gamma a\Gamma a\Gamma S$. $a \in S\Gamma a\Gamma a\Gamma S \subseteq S\Gamma L\Gamma B\Gamma S \subseteq L\Gamma B\Gamma S$. Hence $B \cap L \subseteq L\Gamma B\Gamma S$.
(2) $\Rightarrow$ (3), (4) $\Rightarrow$ (5) As every quasi-ideal is a bi-ideal, implications follow.
(3) $\Rightarrow$ (1) Let L be a left ideal and R be a right ideal of S. Then R is a quasi-ideal of S. By (3), $R \cap L \subseteq L\Gamma R\Gamma S \subseteq L\Gamma R$. Therefore we get $R \cap L \subseteq L\Gamma R$. Thus by Theorem 8, S is an intra-regular Γ-semiring.
(1) $\Rightarrow$ (4) Suppose that S is intra-regular. Let R be a right ideal and B be a bi-ideal of S. Let $a \in B \cap R$. As S is intra-regular, $a \in S\Gamma a\Gamma a\Gamma S$. Hence $a \in S\Gamma a\Gamma a\Gamma S \subseteq S\Gamma B\Gamma R\Gamma S \subseteq S\Gamma B\Gamma R$. This shows that $B \cap R \subseteq S\Gamma B\Gamma R$.
(5) $\Rightarrow$ (1) Let L be a left ideal and R be a right ideal of S. By (5), $R\cap L \subseteq S\Gamma L\Gamma R \subseteq L\Gamma R$, since L is a quasi-ideal of S. Therefore we get $R \cap L \subseteq L\Gamma R$. This shows that S is an intra-regular Γ-semiring by Theorem 8. $\square$

Theorem 11. *In S the following statements are equivalent.*
(1) S is intra-regular.
(2) $K \cap B \cap R \subseteq K\Gamma B\Gamma R$, for a bi-ideal B, a right ideal R and an interior ideal K of S.
(3) $I \cap B \cap R \subseteq I\Gamma B\Gamma R$, for a bi-ideal B, a right ideal R and an ideal I of S.
(4) $K \cap Q \cap R \subseteq K\Gamma Q\Gamma R$, for a quasi-ideal Q, a right ideal R and an interior ideal K of S.
(5) $I \cap Q \cap R \subseteq I\Gamma Q\Gamma R$, for a quasi-ideal Q, a right ideal R and an ideal I of S.

Proof. (1) $\Rightarrow$ (2) Suppose that S is intra-regular. Let R be a right ideal, K be an interior ideal and B be a bi-ideal of S. Let $a \in K \cap B \cap R$. As S is intra-regular, $a \in S\Gamma a\Gamma a\Gamma S$. Therefore $a \in S\Gamma a\Gamma a\Gamma S \subseteq (S\Gamma K\Gamma S)\,\Gamma B\Gamma\,(R\Gamma S\Gamma S) \subseteq K\Gamma B\Gamma R$. Thus we have $K \cap B \cap R \subseteq K\Gamma B\Gamma R$.
(2) $\Rightarrow$ (3), (4) $\Rightarrow$ (5) As every ideal is an interior ideal, implications follow.
(2) $\Rightarrow$ (4), (3) $\Rightarrow$ (5) Clearly implications follow, since quasi-ideal is a bi-ideal.
(5)$\Rightarrow$ (1) Let L be a left ideal and R be a right ideal of S. As L is a quasi-ideal of S, by (5) we have $S \cap L \cap R \subseteq S\Gamma L\Gamma R \subseteq L\Gamma R$. Therefore we have $R \cap L \subseteq L\Gamma R$. Hence by Theorem 8, S is an intra-regular Γ-semiring. $\square$

Theorem 12. *In S the following statements are equivalent.*
(1) S is intra-regular.
(2) $I \cap B \cap L \subseteq L\Gamma B\Gamma I$, for a bi-ideal B, a left ideal L and an interior ideal I of S.
(3) $I \cap B \cap L \subseteq L\Gamma B\Gamma I$, for a bi-ideal B, a left ideal L and an ideal I of S.
(4) $I \cap Q \cap L \subseteq L\Gamma Q\Gamma I$, for a quasi-ideal Q, a left ideal L and an interior ideal I of S.
(5) $I \cap Q \cap L \subseteq L\Gamma Q\Gamma I$, for a quasi-ideal Q, a left ideal L and an ideal I of S.

Proof. (1) $\Rightarrow$ (2) Suppose that S is intra-regular. Let L be a left ideal, I be an interior ideal and B be a bi-ideal of S. Let $a \in I \cap B \cap L$. As S is intra-regular, $a \in S\Gamma a\Gamma a\Gamma S$. $a \in S\Gamma a\Gamma a\Gamma S \subseteq (S\Gamma S\Gamma L)\,\Gamma B\Gamma\,(S\Gamma I\Gamma S) \subseteq L\Gamma B\Gamma I$. Thus we have $I \cap B \cap L \subseteq L\Gamma B\Gamma I$.
(2) $\Rightarrow$ (3), (4) $\Rightarrow$ (5) Clearly implications follow, since an ideal is an interior ideal.
(2) $\Rightarrow$ (4), (4) $\Rightarrow$ (5) As every quasi-ideal is a bi-ideal, implications follow.
(5) $\Rightarrow$ (1) Let L be a left ideal and R be a right ideal of S. As right ideal R is a quasi-ideal, and S itself is an ideal of S, $S \cap R \cap L \subseteq L\Gamma R\Gamma S$ by (5). Therefore $L\Gamma R\Gamma S \subseteq L\Gamma R$. Thus we get $R \cap L \subseteq L\Gamma R$. Therefore S is an intra-regular Γ-semiring by Theorem 8. □

5 Regular and Intra-regular Γ-semiring

Theorem 13. *For S the following statements are equivalent.*
(1) S is regular and intra-regular.
(2) Each right ideal R and left ideal L of S satisfy $R \cap L = R\Gamma L \subseteq L\Gamma R$.
(3) Each bi-ideal B of S satisfies $B = B^2 = B\Gamma B$.
(4) Each quasi-ideal Q of S satisfies $Q = Q^2 = Q\Gamma Q$.

Proof. (1) $\Leftrightarrow$ (2) Proof follows from Theorems 1 and 8.
(1) $\Rightarrow$ (3) Suppose that S is regular and intra-regular. Let B be a bi-ideal of S. Then $B^2 = B\Gamma B \subseteq B$. For the reverse inclusion, let $a \in B$. As S is regular and intra-regular, we have $a \in a\Gamma S\Gamma a$ and $a \in S\Gamma a\Gamma a\Gamma S$. Hence $a \in a\Gamma S\Gamma a \subseteq a\Gamma S\Gamma\,(a\Gamma S\Gamma a) \subseteq a\Gamma S\Gamma\,(S\Gamma a\Gamma a\Gamma S)\,\Gamma S\Gamma a \subseteq.\ (B\Gamma S\Gamma B)\Gamma(B\Gamma S\Gamma B) \subseteq B\Gamma B$. Therefore $B \subseteq B\Gamma B$. Thus we get $B = B\Gamma B = B^2$.
(3) $\Rightarrow$ (4) As every quasi-ideal is a bi-ideal, implication follows.
(4) $\Rightarrow$ (1) Let L be a left ideal and R be a right ideal of S. Then $R \cap L$ is a quasi-ideal of S. By (4), $R \cap L = (R \cap L)^2 = (R \cap L)\,\Gamma(R \cap L) \subseteq L\Gamma R$. This shows that S is an intra-regular Γ-semiring by Theorem 8. Similarly $R \cap L = (R \cap L)^2 = (R \cap L)\,\Gamma(R \cap L) \subseteq R\Gamma L$. Hence we get $R \cap L = R\Gamma L$. Therefore S is a regular Γ-semiring by Theorem 1. □

Theorem 14. *In S the following statements are equivalent.*
(1) S is regular and intra-regular.
(2) For bi-ideals B_1 and B_2 of S, $B_1 \cap B_2 \subseteq (B_1\Gamma B_2) \cap (B_2\Gamma B_1)$.
(3) For every bi-ideal B and a quasi-ideal Q of S, $B \cap Q \subseteq (Q\Gamma B) \cap (B\Gamma Q)$.
(4) For quasi-ideals Q_1 and Q_2 of S, $Q_1 \cap Q_2 \subseteq (Q_1\Gamma Q_2) \cap (Q_2\Gamma Q_1)$.

(5) For every quasi-ideal Q and a generalized bi-ideal G of S, $G \cap Q \subseteq (G\Gamma Q) \cap (Q\Gamma G)$.
(6) For every left ideal L and a bi-ideal B of S, $B \cap L \subseteq (B\Gamma L) \cap (L\Gamma B)$.
(7) For every left ideal L and a quasi-ideal Q of S, $Q \cap L \subseteq (Q\Gamma L) \cap (L\Gamma Q)$.
(8) For every right ideal R and a bi-ideal B of S, $B \cap R \subseteq (B\Gamma R) \cap (R\Gamma B)$.
(9) For every quasi-ideal Q and a right ideal R of S, $R \cap Q \subseteq (R\Gamma Q) \cap (Q\Gamma R)$.
(10) For every left ideal L and a right ideal R of S, $R \cap L \subseteq (R\Gamma L) \cap (L\Gamma R)$.

Proof. (1) $\Rightarrow$ (2) Suppose that S is regular and intra-regular. Let B_1 and B_2 be bi-ideals of S. Let $a \in B_1 \cap B_2$. As S is regular and intra-regular, $a \in a\Gamma S\Gamma a$ and $a \in S\Gamma a\Gamma a\Gamma S$. Hence $a \in a\Gamma S\Gamma a \subseteq (a\Gamma S\Gamma S\Gamma a)\,\Gamma\,(a\Gamma S\Gamma S\Gamma a) \subseteq (B_1\Gamma S\Gamma B_1)\Gamma(B_2\Gamma S\Gamma B_2) \subseteq B_1\Gamma B_2$.

Similarly we can show that $a \in B_2\Gamma B_1$. Therefore $a \in B_1 \cap B_2$ implies $a \in B_1\Gamma B_2$ and $a \in B_2\Gamma B_1$. This gives $B_1 \cap B_2 \subseteq (B_1\Gamma B_2) \cap (B_2\Gamma B_1)$.

(2) $\Rightarrow$ (3), (3) $\Rightarrow$ (4) Implications follow as every quasi-ideal is a bi-ideal.
(4) $\Rightarrow$ (1) Let L be a left ideal and R be a right ideal of S. Then R and L both are quasi-ideals of S. By (4) , $R \cap L \subseteq (R\Gamma L) \cap (L\Gamma R)$. $R \cap L \subseteq L\Gamma R$ implies S is an intra-regular Γ-semiring by Theorem 8. Also $R \cap L \subseteq R\Gamma L$. Therefore we get $R \cap L = R\Gamma L$. Hence by Theorem 1, S is a regular Γ-semiring.
(1) $\Rightarrow$ (5) Suppose that S is regular and intra-regular. Let G be a generalized bi-ideal and Q be quasi-ideal of S. Let $a \in G \cap Q$. As S is regular and intra-regular, $a \in a\Gamma S\Gamma a$ and $a \in S\Gamma a\Gamma a\Gamma S$. Therefore $a \in a\Gamma S\Gamma a \subseteq a\Gamma S\Gamma\,(a\Gamma S\Gamma a) \subseteq (a\Gamma S\Gamma S\Gamma a)\,\Gamma\,(a\Gamma S\Gamma S\Gamma a) \subseteq (G\Gamma S\Gamma G)\Gamma(Q\Gamma S\Gamma Q) \subseteq G\Gamma Q$. Hence $a \in G\Gamma Q$. Similarly we can show that $a \in Q\Gamma G$.Therefore $a \in G\cap Q$ implies $a \in G\Gamma Q$ and $a \in Q\Gamma G$, which gives $G \cap Q \subseteq (G\Gamma Q) \cap (Q\Gamma G)$.
(5) $\Rightarrow$ (1) Let L be a left ideal and R be a right ideal of S respectively. As R is a generalized bi-ideal and L is a quasi-ideal of S, proof follows from (4) $\Rightarrow$(1).
(1) $\Rightarrow$ (6) Suppose that S is regular and intra-regular. Let B be a bi-ideal and L be a left ideal of S. Let $a \in B \cap L$. As S is regular and intra-regular, $a \in a\Gamma S\Gamma a$ and $a \in S\Gamma a\Gamma a\Gamma S$. $a \in a\Gamma S\Gamma a \subseteq a\Gamma S\Gamma\,(a\Gamma S\Gamma a) \subseteq (a\Gamma S\Gamma S\Gamma a)\,\Gamma\,(a\Gamma S\Gamma S\Gamma a) \subseteq (B\Gamma S\Gamma B)\Gamma(S\Gamma S\Gamma S\Gamma L) \subseteq B\Gamma L$. Therefore we get $a \in B\Gamma L$. Similarly we can show that $a \in L\Gamma B$. Therefore $a \in B \cap L$ implies $a \in B\Gamma L$ and $a \in L\Gamma B$. Hence $B \cap L \subseteq (B\Gamma L) \cap (L\Gamma B)$.
(6) $\Rightarrow$ (7) As every quasi-ideal is a bi-ideal, implication follows.
(7) $\Rightarrow$ (1) Let L be a left ideal and R be a right ideal of S respectively. As R is a quasi-ideal of S, proof follows from (4) $\Rightarrow$ (1).
(1) $\Rightarrow$ (8) Suppose that S is regular and intra-regular. Let R be right ideal and B be a bi-ideals of S. Let $a \in B \cap R$. As S is regular and intra-regular, $a \in a\Gamma S\Gamma a$ and $a \in S\Gamma a\Gamma a\Gamma S$. Therefore $a \in a\Gamma S\Gamma a \subseteq a\Gamma S\Gamma(a\Gamma S\Gamma a) \subseteq (a\Gamma S\Gamma S\Gamma a)\,\Gamma\,(a\Gamma S\Gamma S\Gamma a) \subseteq (B\Gamma S\Gamma B)\Gamma(R\Gamma S\Gamma S\Gamma S) \subseteq B\Gamma R$. Therefore we get $a \in B\Gamma R$. Similarly we can show that $a \in R\Gamma B$. Therefore $a \in B \cap R$ implies $a \in B\Gamma R$ and $a \in R\Gamma B$, which gives $B \cap R \subseteq (B\Gamma R) \cap (R\Gamma B)$.
(8) $\Rightarrow$ (9), (9) $\Rightarrow$ (10) Implications follow as every left ideal is a quasi-ideal.

(10) $\Rightarrow$ (1) Let L be a left ideal and R be a right ideal of S. Proof follows from (4) $\Rightarrow$ (1). $\square$

Theorem 15. *In S the following statements are equivalent.*
(1) S is regular and intra-regular.
(2) For bi-ideals B_1 and B_2 of S, $B_1 \cap B_2 \subseteq (B_1\Gamma B_2\Gamma B_1) \cap (B_2\Gamma B_1\Gamma B_2)$.
(3) For a quasi-ideal Q and a bi-ideal B of S, $Q \cap B \subseteq (B\Gamma Q\Gamma B) \cap (Q\Gamma B\Gamma Q)$.
(4) For quasi-ideals Q_1 and Q_2 of S, $Q_1 \cap Q_2 \subseteq (Q_1\Gamma Q_2\Gamma Q_1) \cap (Q_2\Gamma Q_1\Gamma Q_2)$.
(5) For a bi-ideal B and a left ideal L of S, $B \cap L \subseteq B\Gamma L\Gamma B$.
(6) For a quasi-ideal Q and a left ideal L of S, $Q \cap L \subseteq Q\Gamma L\Gamma Q$.
(7) For a bi-ideal B and a right ideal R of S, $B \cap R \subseteq B\Gamma R\Gamma B$.
(8) For a quasi-ideal Q and a right ideal R of S, $Q \cap R \subseteq Q\Gamma R\Gamma Q$.
(9) For a quasi-ideal Q and a generalized bi-ideal G of S, $Q \cap G \subseteq (Q\Gamma G\Gamma Q) \cap (G\Gamma Q\Gamma G)$.

Proof. (1)$\Rightarrow$ (2) Suppose that S is regular and intra-regular. Let B_1 and B_2 be bi-ideals of S. Let $a \in B_1 \cap B_2$. As S is regular and intra-regular, $a \in a\Gamma S\Gamma a$ and $a \in S\Gamma a\Gamma a\Gamma S$. Hence $a \in a\Gamma S\Gamma a \subseteq a\Gamma S\Gamma(a\Gamma S\Gamma a) \subseteq (a\Gamma S\Gamma a)\,\Gamma\,(a\Gamma S\Gamma a)\,\Gamma\,(a\Gamma S\Gamma a) \subseteq (B_1\Gamma S\Gamma B_1)\,\Gamma\,(B_2\Gamma S\Gamma B_2)\,\Gamma\,(B_1\Gamma S\Gamma B_1) \subseteq B_1\Gamma B_2\Gamma B_1$. Therefore $B_1 \cap B_2 \subseteq B_1\Gamma B_2\Gamma B_1$. In the same manner we can show that $B_1 \cap B_2 \subseteq B_2\Gamma B_1\Gamma B_2$. Thus we get $B_1 \cap B_2 \subseteq (B_1\Gamma B_2\Gamma B_1) \cap (B_2\Gamma B_1\Gamma B_2)$.
(2)$\Rightarrow$ (3), (3)$\Rightarrow$ (4) Implications follow as every quasi-ideal is a bi-ideal.
(4) $\Rightarrow$ (1) Let L be a left ideal and R be a right ideal of S. Then $R \cap L$ is a quasi-ideal of S. By (4), $(R \cap L)\cap(R \cap L) \subseteq ((R \cap L)\,\Gamma\,(R \cap L)\,\Gamma\,(R \cap L)) \subseteq L\Gamma R\Gamma R \subseteq L\Gamma R$. Hence $R \cap L \subseteq L\Gamma R$. This shows that S is an intra-regular Γ-semiring by Theorem 8. Also $R \cap L \subseteq ((R \cap L)\,\Gamma\,(R \cap L)\,\Gamma\,(R \cap L))$ implies $R \cap L \subseteq R\Gamma L$. Therefore $R \cap L = R\Gamma L$. Thus S is a regular Γ-semiring by Theorem 1.
(1)$\Rightarrow$ (5) Suppose that S is regular and intra-regular. Let B be a bi-ideal and L be a left ideal of S. Let $a \in B \cap L$. As S is regular and intra-regular, $a \in a\Gamma S\Gamma a$ and $a \in S\Gamma a\Gamma a\Gamma S$. Therefore $a \in a\Gamma S\Gamma a \subseteq a\Gamma S\Gamma(a\Gamma S\Gamma a) \subseteq (a\Gamma S\Gamma a)\,\Gamma\,(S\Gamma a)\,\Gamma\,(a\Gamma S\Gamma a) \subseteq (B\Gamma S\Gamma B)\,\Gamma\,(S\Gamma L)\,\Gamma\,(B\Gamma S\Gamma B) \subseteq B\Gamma L\Gamma B$. Hence we have $B \cap L \subseteq B\Gamma L\Gamma B$.
(5) $\Rightarrow$ (6) As every quasi-ideal is a bi-ideal, implication follows.
(6) $\Rightarrow$ (7) Proof is similar to (4)$\Rightarrow$ (1).
(1) $\Rightarrow$(7) $\Rightarrow$ (8) $\Rightarrow$ (1) can be proved similarly to (1) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$(1). Proof of (1) $\Rightarrow$ (9) is similar to (1) $\Rightarrow$ (2) and proof of (9) $\Rightarrow$ (1) is parallel to (1) $\Rightarrow$ (4) $\Rightarrow$ (1).
Thus we have shown that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$(4) $\Rightarrow$ (1), (1) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$(1) and (1) $\Rightarrow$ (7) $\Rightarrow$ (8) $\Rightarrow$ (1) and (1) $\Rightarrow$ (9) $\Rightarrow$(1). $\square$

Theorem 16. *In S the following statements are equivalent.*
(1) S is regular and intra-regular.
(2) $B \cap R \cap L \subseteq B\Gamma R\Gamma L$, for a bi-ideal B, right ideal R and a left ideal L of S.
(3) $Q \cap R \cap L \subseteq Q\Gamma R\Gamma L$, for a quasi-ideal Q, right ideal R and left ideal R of S.

Proof. (1) $\Rightarrow$ (2) Suppose that S is regular and intra-regular. Let B be a bi-ideal, R be a right ideal and L be a left ideal of S. Let $a \in B \cap R \cap L$. As S is regular and intra-regular, $a \in a\Gamma S\Gamma a$ and $a \in S\Gamma a\Gamma a\Gamma S$. Hence $a \in a\Gamma S\Gamma a \subseteq a\Gamma S\Gamma a\Gamma S\Gamma a) \subseteq (a\Gamma S\Gamma S\Gamma a)\,\Gamma\,(a\Gamma S)\Gamma S\Gamma a) \subseteq (B\Gamma S\Gamma B)\,\Gamma\,(R\Gamma S)\,\Gamma S\Gamma L) \subseteq B\Gamma R\Gamma L$. Therefore $B \cap R \cap L \subseteq B\Gamma R\Gamma L$.
(2) $\Rightarrow$ (3) As every quasi-ideal is a bi-ideal, implication follows.
(3) $\Rightarrow$ (1) Let L be a left ideal and R be a right ideal of S. As R is a quasi-ideal and S itself a right ideal of S, by (3) we have $R \cap S \cap L \subseteq R\Gamma S\Gamma L \subseteq R\Gamma L$. Therefore $R \cap L \subseteq R\Gamma L$. Thus we get $R \cap L = R\Gamma L$. Hence S is a regular Γ-semiring by Theorem 1. Similarly L is a quasi-ideal and S itself a left ideal of S gives $L \cap R \cap S \subseteq L\Gamma R\Gamma S \subseteq L\Gamma R$ by (3). Thus $R \cap L \subseteq L\Gamma R$. This shows that S is an intra-regular Γ-semiring by Theorem 8. □

6 Duo Γ-semiring

Now we define the notion of a duo Γ-semiring as follows.

Definition 11. A Γ - semiring S is said to be a left (right) duo Γ - semiring if every left (right) ideal of S is a right (left) ideal.

A Γ -semiring S is said to be a duo Γ - semiring if every one-sided ideal of S is a two-sided ideal.
That is a Γ-semiring S is said to be a duo Γ-semiring if it is both left duo and right duo.

Theorem 17. *If S is regular, then S is left duo if and only if for any two left ideals A and B of S, $A \cap B = A\Gamma B$.*

Proof. Let S be a regular Γ-semiring . Assume that S is left duo. Let A and B be any two left ideals of S. As S is left duo, A is a right ideal of S. Then by Theorem 1, $A \cap B = A\Gamma B$. Conversely, suppose that the given condition holds. Let L be a left ideal of S. Then by assumption $L\Gamma S = L \cap S \subseteq L$. This shows that L is a right ideal of S. Therefore S is a left duo Γ-semiring. □

Proof of the following theorem is analogous to proof of Theorem 17.

Theorem 18. *If S is regular, then S is right duo if and only if for any two right ideals A and B of S, $A \cap B = A\Gamma B$*

Theorem 19. *If S is regular, then S is left duo if and only if every quasi-ideal of S is a right ideal of S.*

Proof. Let S be a regular Γ-semiring. Suppose that S is left duo. Let Q be any quasi-ideal of S. Then there exists a right ideal R and a left ideal L of S such that $Q = R \cap L$. Therefore $Q = R \cap L$ is a right ideal of S. Conversely, let L be a left ideal of S. Then L is a quasi-ideal of S. Hence by assumption L is a right ideal of S. Therefore S is a left duo Γ-semiring. □

Proofs of the following theorems are similar to proof of Theorem 19.

Theorem 20. *If S is regular, then S is right duo if and only if every quasi-ideal of S is a left ideal of S.*

Theorem 21. *If S is regular, then S is duo if and only if every quasi-ideal of S is an ideal of S.*

Theorem 22. *If S is regular, then S is duo if and only if every bi-ideal of S is a ideal of S.*

Theorem 23. *In S the following conditions are equivalent.*
(1) S is regular duo.
(2) $I \cap B = I\Gamma B\Gamma I$, for every ideal I and a bi-ideal B of S.
(3) $I \cap Q = I\Gamma Q\Gamma I$, for every ideal I and a quasi-ideal Q of S.

Proof. (1) ⇒ (2) Suppose that S is a regular duo Γ-semiring. Let I be an ideal and B be a bi-ideal of S. Then by Theorem 22, B is an ideal of S. Therefore $I\Gamma B\Gamma I \subseteq I$ and $I\Gamma B\Gamma I \subseteq B$, since I and B are ideals of S. Hence $I\Gamma B\Gamma I \subseteq I \cap B$. For the reverse inclusion, let $a \in I \cap B$. S is regular implies $a \in a\Gamma S\Gamma a$. $a \in a\Gamma S\Gamma a \subseteq a\Gamma S\Gamma a\Gamma S\Gamma a) \subseteq (I\Gamma S)\Gamma B\Gamma(S\Gamma I) \subseteq I\Gamma B\Gamma I$. Therefore $I \cap B \subseteq I\Gamma B\Gamma I$. Hence $I \cap B = I\Gamma B\Gamma I$.
(2) ⇒ (3) As every quasi-ideal of S is a bi-ideal of S, implication follows.
(3) ⇒ (1) Let L be a left ideal and R be a right ideal of S. Hence $S \cap L = S\Gamma L\Gamma S$ and $S \cap R = S\Gamma R\Gamma S$ by (3). Therefore $L = S\Gamma L\Gamma S$ and $R = S\Gamma R\Gamma S$. Now $L\Gamma S = S\Gamma L\Gamma S\Gamma S \subseteq S\Gamma L\Gamma S = L$ and $S\Gamma R = S\Gamma S\Gamma R\Gamma S \subseteq S\Gamma R\Gamma S = R$. Hence $L\Gamma S \subseteq L$ and $S\Gamma R \subseteq R$. This shows that L is a right ideal and R is a left ideal of S. Therefore S is a duo Γ-semiring by Definition 11. As S is a duo Γ-semiring, $R \cap L = R\Gamma L\Gamma R$ by (3). $R \cap L = R\Gamma L\Gamma R \subseteq R\Gamma L$. This shows that $R \cap L = R\Gamma L$. Hence by Theorem 1, S is regular. □

Theorem 24. *If S is a Γ- semiring then the following statements are equivalents.*
(1) S is regular duo.
(2) For every bi-ideals A and B of S, $A \cap B = A\Gamma B$.
(3) For every bi-ideal B and a quasi-ideal Q of S, $B \cap Q = B\Gamma Q$.
(4) For every bi-ideal B and a right ideal R of S, $B \cap R = B\Gamma R$.
(5) For every quasi-ideal Q and a bi-ideal B of S, $Q \cap B = Q\Gamma \mathrm{B}$.
(6) For every quasi-ideals Q_1 and Q_2 of S, $Q_1 \cap Q_2 = Q_1\Gamma Q_2$.
(7) For every quasi-ideal Q and a right ideal R of S, $Q \cap R = Q\Gamma R$.
(8) For every left ideal L and a bi-ideal B of S, $L \cap B = L\Gamma B$.
(9) For every left ideal L and a right ideal R of S, $L \cap R = L\Gamma R$.

Proof. We can prove the equivalence of statements such as (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (1), (1) ⇒ (5) ⇒ (6) ⇒ (7) ⇒ (1) and (1) ⇒ (8) ⇒ (9) ⇒ (1) . Proof of each implication is straightforward so omitted. □

References

[1] Chinram R. *A Note on Quasi-ideals in* Γ*-Semirings*. Int. Math. Forum, 2008, **26(3)**, 1253–1259.

[2] Chinram R. *On Quasi gamma-ideals in* Γ*-Semigroups*. Science Asia, 2006, **32**, 351–353.

[3] Dutta T. K., Sardar S. K. *Semi-prime Ideals and Irreducible Ideals of* Γ*-Semiring*. Novi Sad Jour. Math., 2000, **30(1)**, 97–108.

[4] Good R. A., Hughes D. R. *Associated Groups for a Semigroup*. Bull. Amer. Math. Soc., 1952, **58**, 624–625.

[5] Green J. A. *On the Structure of Semigroup*. Ann. of Math., 1951, **2(54)**, 163–172.

[6] Iseki K. *Quasi-ideals in Semirings without Zero*. Proc. Japan Acad., 1958, **34**, 79–84.

[7] Jagatap R. D., Pawar Y. S. *Quasi-ideals and Minimal Quasi-ideals in* Γ*-Semirings*. Novi Sad Jour. of Mathematics, 2009, **39(2)**, 79–87.

[8] Jagatap R. D., Pawar Y. S. *Quasi-ideals in Regular* Γ*-Semirings*. Bull. Kerala Math. Asso., 2010, **6(2)**, 51–61.

[9] Lajos S., Szasz F. *On the Bi-ideals in Associative Ring*. Proc. Japan Acad., 1970, **46**, 505–507.

[10] Lajos S., Szasz F. *On the Bi-ideals in Semigroups*. Proc. Japan Acad., 1969, **45**, 710–712.

[11] Lajos S., Szasz F. *On theBi-ideals in Semigroups II*. Proc. Japan Acad., 1971, **47**, 837–839.

[12] Neumann J. V. *On Regular Rings*. Proc. Nat. Acad. Sci. U.S.A., 1936, **22**, 707–713.

[13] Rao M. M. K. Γ*-Semirings 1*. Southeast Asian Bull. of Math., 1995, **19**, 49–54.

[14] Shabir M., Ali A., Batool S. *A Note on Quasi-ideals in Semirings*. Southeast Asian Bull. of Math., 2004, **27**, 923–928.

[15] Steinfeld O. *On Ideals Quotients and Prime Ideals*. Acta.Math. Acad.Sci.Hungar, 1953, **4**, 289–298.

[16] Steinfeld O. *Uher Die Quasi-ideals von Halbgruppen*. Publ.Math. Debrecen, 1956, **4**, 262–275.

[17] Zelznikov J. *Regular Semirings*. Semigroup Forum, 1981, **23**, 119–136.

R. D. Jagatap
Y. C. College of Science, Karad, India
E-mail: *ravindrajagatap@yahoo.co.in; jagatapravindra@gmail.com*

Received January 10, 2015

Y. S. Pawar
Department of Mathematics
Shivaji University, Kolhapur, India
E-mail: *yspawar1950@gmail.com*