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Some Properties of Meromorphic Solutions of Logarithmic Order to Higher Order Linear Difference Equations

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Abstract. This paper is devoted to the study of the growth of solutions of the linear difference equation

$$A_n(z) f(z+n) + A_{n-1}(z) f(z+n-1)$$
$$+ \cdots + A_1(z) f(z+1) + A_0(z) f(z) = 0,$$

where $A_n(z), \cdots, A_0(z)$ are entire or meromorphic functions of finite logarithmic order. We extend some precedent results due to Liu and Mao, Zheng and Tu, Chen and Shon and others.

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1 Introduction and main results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna theory of meromorphic functions [9, 16]. We use the notations $\mu(f)$, $\rho(f)$ to denote the lower order and the order of a meromorphic function f. Since Halburd–Korhonen [7] and Chiang–Feng [5], independently, have given a difference version of the logarithmic derivative lemma, and Halburd–Korhonen [8] subsequently showed how all key results of the Nevanlinna theory have corresponding difference variants as well, some interest appeared to investigate solutions of difference equations in the complex domain by making use of this variant of the value distribution theory, see [1, 3, 12–15].

Definition 1 (see [9])**.** Let f be an entire function of order ρ $(0 < \rho < \infty)$, the type of f is defined as

$$\tau(f) = \limsup_{r\to+\infty} \frac{\log M(r,f)}{r^{\rho}}.$$

Similarly the lower type of an entire function f of lower order μ $(0 < \mu < \infty)$ is defined by

$$\underline{\tau}(f) = \liminf_{r\to+\infty} \frac{\log M(r,f)}{r^{\mu}}.$$

We recall the following definitions. The linear measure of a set $E \subset (0,+\infty)$ is defined as $m(E) = \int_0^{+\infty} \chi_E(t)\,dt$ and the logarithmic measure of a set $F \subset (1,+\infty)$ is defined by $lm(F) = \int_1^{+\infty} \frac{\chi_F(t)}{t}dt$, where $\chi_H(t)$ is the characteristic function of a set H. The upper density of a set $E \subset (0,+\infty)$ is defined by

$$\overline{dens}E = \limsup_{r\longrightarrow+\infty} \frac{m(E\cap[0,r])}{r}.$$

The upper logarithmic density of a set $F \subset (1,+\infty)$ is defined by

$$\overline{\log dens}(F) = \limsup_{r\longrightarrow+\infty} \frac{lm(F\cap[1,r])}{\log r}.$$

Proposition 1. *For all $H \subset [1,+\infty)$ the following statements hold :*
i) If $lm(H) = \infty$, then $m(H) = \infty$;
ii) If $\overline{dens}H > 0$, then $m(H) = \infty$;
iii) If $\overline{\log dens}H > 0$, then $lm(H) = \infty$.

Proof. i) Since we have $\frac{\chi_H(t)}{t} \leq \chi_H(t)$ for all $t \in H \subset [1,+\infty)$, then

$$m(H) \geq lm(H).$$

So, if $lm(H) = \infty$, then $m(H) = \infty$. We can easily prove the results ii) and iii) by applying the definition of the limit and the properties $m(H\cap[0,r]) \leq m(H)$ and $lm(H\cap[1,r]) \leq lm(H)$. □

Definition 2 (see [9,16])**.** For $a \in \overline{\mathbb{C}} = \mathbb{C}\cup\{\infty\}$, the deficiency of a with respect to a meromorphic function f is defined as

$$\delta(a,f) = \liminf_{r\to+\infty} \frac{m\left(r,\frac{1}{f-a}\right)}{T(r,f)} = 1 - \limsup_{r\to+\infty} \frac{N\left(r,\frac{1}{f-a}\right)}{T(r,f)}.$$

In recent paper [5], Chiang and Feng investigated meromorphic solutions of the linear difference equation

$$A_n(z)f(z+n) + A_{n-1}(z)f(z+n-1)$$
$$+\cdots + A_1(z)f(z+1) + A_0(z)f(z) = 0, \tag{1}$$

where $A_n(z),\cdots,A_0(z)$ are entire functions such that $A_n(z)A_0(z) \not\equiv 0$, and proved the following result.

Theorem 1 (see [5])**.** *Let $A_0(z), A_1(z),\cdots,A_n(z)$ be entire functions such that there exists an integer l, $0 \leq l \leq n$ such that*

$$\rho(A_l) > \max_{0\leq j\leq n, j\neq l}\{\rho(A_j)\}.$$

If $f(z)$ is a meromorphic solution of (1), then $\rho(f) \geq \rho(A_l) + 1$.

Note that in Theorem 1, equation (1) has only one dominating coefficient A_l. For the case when there is more than one coefficients which have the maximal order, Laine and Yang [12] obtained the following result.

Theorem 2 (see [12]). *Let $A_0(z), A_1(z), \cdots, A_n(z)$ be entire functions of finite order such that among those having the maximal order $\rho = \max_{0\leq j\leq n}\{\rho(A_j)\}$, one has exactly its type strictly greater than the others. Then for any meromorphic solution of (1), we have $\rho(f) \geq \rho + 1$.*

Recently, Liu and Mao [13], Zheng and Tu [15] investigated the growth of solutions of equation (1) and proved the following results.

Theorem 3 (see [15]). *Let $A_0(z), \cdots, A_n(z)$ be entire functions such that there exists an integer l $(0 \leq l \leq n)$ satisfying*

$$\max\{\rho(A_j) : j = 0, 1, \cdots, n, j \neq l\} \leq \mu(A_l) < \infty$$

and

$$\max\{\tau(A_j) : \rho(A_j) = \mu(A_l) : j = 0, 1, \cdots, n, j \neq l\} < \underline{\tau}(A_l).$$

Then every meromorphic solution f of equation (1) satisfies $\mu(f) \geq \mu(A_l) + 1$.

Theorem 4 (see [13]). *Let H be a set of complex numbers satisfying $\overline{dens}\{|z| : z \in H\} > 0$, and let $A_0(z), \cdots, A_n(z)$ be entire functions satisfying $\max\{\rho(A_j) : j = 0, 1, \cdots, n\} \leq \rho$. If there exists an integer l $(0 \leq l \leq n)$ such that for some constants $0 \leq \beta < \alpha$ and $\varepsilon > 0$ sufficiently small, we have*

$$|A_l(z)| \geq \exp\left\{\alpha r^{\rho-\varepsilon}\right\}$$

and

$$|A_j(z)| \leq \exp\left\{\beta r^{\rho-\varepsilon}\right\} \quad (j \neq l)$$

as $|z| = r \to +\infty$ for $z \in H$, then every meromorphic solution $f \not\equiv 0$ of equation (1) satisfies $\rho(f) \geq \rho(A_l) + 1$.

When the coefficients $A_0(z), A_1(z), \cdots, A_n(z)$ are meromorphic, Chen and Shon extended the result of Theorem 1 and obtained.

Theorem 5 (see [3]). *Let $A_0(z), \cdots, A_n(z)$ be meromorphic functions such that there exists an integer l $(0 \leq l \leq n)$ such that $\rho(A_l) > \max\{\rho(A_j) : j = 0, 1, \cdots, n, j \neq l\}$, $\delta(\infty, A_l) > 0$. Then every meromorphic solution $f \not\equiv 0$ of equation (1) satisfies $\rho(f) \geq \rho(A_l) + 1$.*

Obviously, we have $\rho(A_l) > 0$ and $\rho > 0$ in Theorems 1, 2 and 5. Thus, a natural question arises : How to express the growth of solutions of (1) when all coefficients $A_0(z), A_1(z), \cdots, A_n(z)$ are entire or meromorphic functions and of order zero in $\mathbb{C}$?

The main purpose of this paper is to make use of the concept of finite logarithmic order due to Chern [4] to extend previous results for meromorphic solutions to equation (1) of zero order in $\mathbb{C}$.

Definition 3 (see [4])**.** The logarithmic order of a meromorphic function f is defined as

$$\rho_{\log}(f) = \limsup_{r\to+\infty} \frac{\log T(r,f)}{\log\log r}.$$

If f is an entire function, then

$$\rho_{\log}(f) = \limsup_{r\to+\infty} \frac{\log\log M(r,f)}{\log\log r}.$$

Remark 1. Obviously, the logarithmic order of any non-constant rational function f is one, and thus, any transcendental meromorphic function in the plane has logarithmic order no less than one. However, a function of logarithmic order one is not necessarily a rational function. Constant functions have zero logarithmic order, while there are no meromorphic functions of logarithmic order between zero and one. Moreover, any meromorphic function with finite logarithmic order in the plane is of order zero.

Definition 4. The logarithmic lower order of a meromorphic function f is defined as

$$\mu_{\log}(f) = \liminf_{r\to+\infty} \frac{\log T(r,f)}{\log\log r}.$$

If f is an entire function, then

$$\mu_{\log}(f) = \liminf_{r\to+\infty} \frac{\log\log M(r,f)}{\log\log r}.$$

Definition 5 (see [2])**.** The logarithmic type of an entire function f with $1 \leq \rho_{\log}(f) < +\infty$ is defined by

$$\tau_{\log}(f) = \limsup_{r\to+\infty} \frac{\log M(r,f)}{(\log r)^{\rho_{\log}(f)}}.$$

Similarly the logarithmic lower type of an entire function f with $1 \leq \mu_{\log}(f) < +\infty$ is defined by

$$\underline{\tau}_{\log}(f) = \liminf_{r\to+\infty} \frac{\log M(r,f)}{(\log r)^{\mu_{\log}(f)}}.$$

Remark 2. It is evident that the logarithmic type of any non-constant polynomial P equals its degree $\deg(P)$; that any non-constant rational function is of finite logarithmic type, and that any transcendental meromorphic function whose logarithmic order equals one in the plane must be of infinite logarithmic type.

Recently, the concept of logarithmic order has been used to investigate the growth and the oscillation of solutions of linear differential equations in the complex plane [2] and complex linear difference and q-difference equations in the complex plane and in the unit disc [1,10,11,14]. In the following, we continue to consider growth estimates of meromorphic solutions to higher order linear difference equations, and we obtain the following results.

Theorem 6. *Let* $A_0(z), \cdots, A_n(z)$ *be entire functions such that there exists an integer* l $(0 \le l \le n)$ *satisfying*

$$\max\{\rho_{\log}(A_j) : j = 0, 1, \cdots, n, j \neq l\} \le \mu_{\log}(A_l) < \infty \tag{2}$$

and

$$\max\{\tau_{\log}(A_j) : \rho_{\log}(A_j) = \mu_{\log}(A_l) : j = 0, 1, \cdots, n, j \neq l\} < \underline{\tau}_{\log}(A_l). \tag{3}$$

Then every meromorphic solution $f \not\equiv 0$ *of equation* (1) *satisfies* $\mu_{\log}(f) \ge \mu_{\log}(A_l) + 1$.

Theorem 7. *Let* H *be a set of complex numbers satisfying* $\overline{\log dens}\{|z| : z \in H\} > 0$, *and let* $A_0(z), \cdots, A_n(z)$ *be entire functions satisfying* $\max\{\rho_{\log}(A_j) : j = 0, 1, \cdots, n\} \le \rho$ *with* $\rho > 1$. *If there exists an integer* l $(0 \le l \le n)$ *such that for some constants* $0 \le \beta < \alpha$ *and* ε $(0 < \varepsilon < \rho)$ *sufficiently small, we have*

$$|A_l(z)| \ge \exp\left\{\alpha [\log r]^{\rho-\varepsilon}\right\} \tag{4}$$

and

$$|A_j(z)| \le \exp\left\{\beta [\log r]^{\rho-\varepsilon}\right\} \quad (j \neq l) \tag{5}$$

as $|z| = r \to +\infty$ *for* $z \in H$, *then every meromorphic solution* $f \not\equiv 0$ *of equation* (1) *satisfies* $\rho_{\log}(f) \ge \rho_{\log}(A_l) + 1$.

Remark 3. By the assumptions of Theorem 7, we obtain $\rho_{\log}(A_l) = \rho$. Indeed, we have $\rho_{\log}(A_l) \le \rho$. Suppose that $\rho_{\log}(A_l) = \eta < \rho$. Then, by Definition 3 and (4), we have for any given ε $\left(0 < \varepsilon < \frac{\rho-\eta}{2}\right)$

$$\exp\left\{\alpha [\log r]^{\rho-\varepsilon}\right\} \le |A_l(z)| \le \exp\left\{[\log r]^{\eta+\varepsilon}\right\}$$

as $|z| = r \to +\infty$ for $z \in H$. By ε $\left(0 < \varepsilon < \frac{\rho-\eta}{2}\right)$ this is a contradiction as $r \to +\infty$. Hence $\rho_{\log}(A_l) = \rho$.

Theorem 8. *Let* $A_0(z), \cdots, A_n(z)$ *be entire functions of finite logarithmic order such that there exists an integer* l $(0 \le l \le n)$ *satisfying*

$$\limsup_{r\to+\infty} \sum_{\substack{j=0\\ j\neq l}}^{n} \frac{m(r, A_j)}{m(r, A_l)} < 1. \tag{6}$$

Then every meromorphic solution $f \not\equiv 0$ *of equation* (1) *satisfies* $\rho_{\log}(f) \ge \rho_{\log}(A_l) + 1$.

The following theorems investigate the logarithmic order of meromorphic solutions of (1) in the case when the coefficients are meromorphic functions.

Theorem 9. *Let* $A_0(z), \cdots, A_n(z)$ *be meromorphic functions such that there exists an integer* l $(0 \le l \le n)$ *satisfying* $\rho_{\log}(A_l) > \max\{\rho_{\log}(A_j) : j = 0, 1, \cdots, n, j \neq l\}$, $\delta(\infty, A_l) > 0$. *Then every meromorphic solution* $f \not\equiv 0$ *of equation* (1) *satisfies* $\rho_{\log}(f) \ge \rho_{\log}(A_l) + 1$.

Theorem 10. *Let $A_0(z), \cdots, A_n(z)$ be meromorphic functions of finite logarithmic order such that there exists an integer l $(0 \leq l \leq n)$ satisfying* $\limsup_{r\to+\infty} \sum_{\substack{j=0 \\ j\neq l}}^{n} \frac{m(r,A_j)}{m(r,A_l)} < 1$, $\delta(\infty, A_l) > 0$. *Then every meromorphic solution $f \not\equiv 0$ of equation* (1) *satisfies* $\rho_{\log}(f) \geq \rho_{\log}(A_l) + 1$.

2 Some lemmas

Lemma 1 (see [1]). *Let η_1, η_2 be two arbitrary complex numbers such that $\eta_1 \neq \eta_2$ and let $f(z)$ be a finite logarithmic order meromorphic function. Let ρ be the logarithmic order of $f(z)$. Then for each $\varepsilon > 0$, we have*

$$m\left(r, \frac{f(z+\eta_1)}{f(z+\eta_2)}\right) = O\left((\log r)^{\rho-1+\varepsilon}\right).$$

Lemma 2 (see [5]). *Let f be a meromorphic function, η a non-zero complex number, and let $\gamma > 1$, and $\varepsilon > 0$ be given real constants. Then there exists a subset $E_1 \subset (1, \infty)$ of finite logarithmic measure, and a constant A depending only on γ and η, such that for all $|z| = r \notin [0,1] \cup E_1$, we have*

$$\left|\log\left|\frac{f(z+\eta)}{f(z)}\right|\right| \leq A\left(\frac{T(\gamma r, f)}{r} + \frac{n(\gamma r)}{r}\log^{\gamma} r \log^{+} n(\gamma r)\right), \tag{7}$$

where $n(t) = n(t, \infty, f) + n(t, \infty, 1/f)$

Lemma 3 (see [6]). *Let f be a transcendental meromorphic function, let j be non-negative integer, let a be a value in the extended complex plane, and let $\alpha > 1$ be a real constant. Then there exists a constant $R > 0$ such that for all $r > R$, we have*

$$n\left(r, a, f^{(j)}\right) \leq \frac{2j+6}{\log \alpha} T(\alpha r, f). \tag{8}$$

Lemma 4. *Let f be a meromorphic function with $1 \leq \mu_{\log}(f) < +\infty$. Then there exists a set $E_2 \subset (1, +\infty)$ with infinite logarithmic measure such that for all $r \in E_2 \subset (1, +\infty)$, we have*

$$T(r, f) < (\log r)^{\mu_{\log}(f)+\varepsilon}. \tag{9}$$

Proof. By definition of logarithmic lower order, there exists a sequence $\{r_n\}_{n=1}^{\infty}$ tending to ∞ satisfying $\left(1 + \frac{1}{n}\right) r_n < r_{n+1}$ and

$$\lim_{r_n\to+\infty} \frac{\log T(r_n, f)}{\log\log r_n} = \mu_{\log}(f).$$

Then for any given $\varepsilon > 0$, there exists an integer n_1 such that for all $n \geq n_1$,

$$T(r_n, f) < (\log r_n)^{\mu_{\log}(f)+\frac{\varepsilon}{2}}.$$

Set $E_2 = \bigcup\limits_{n=n_1}^{\infty} \left[\frac{n}{n+1} r_n, r_n\right]$. Then for $r \in E_2 \subset (1, +\infty)$, we obtain

$$T(r, f) \leq T(r_n, f) < (\log r_n)^{\mu_{\log}(f)+\frac{\varepsilon}{2}} \leq \left(\log \frac{n+1}{n} r\right)^{\mu_{\log}(f)+\frac{\varepsilon}{2}} < (\log r)^{\mu_{\log}(f)+\varepsilon},$$

and $lm(E_2) = \sum\limits_{n=n_1}^{\infty} \int\limits_{\frac{n}{n+1} r_n}^{r_n} \frac{dt}{t} = \sum\limits_{n=n_1}^{\infty} \log\left(1 + \frac{1}{n}\right) = \infty$. Thus, Lemma 4 is proved. □

Lemma 5. *Let f be a meromorphic function, η a non-zero complex number, and $\varepsilon > 0$ be given real constants. Then there exists a subset $E_3 \subset (1, \infty)$ of finite logarithmic measure, such that if f has finite logarithmic order ρ, then for all $|z| = r \notin [0, 1] \cup E_3$, we have*

$$\exp\left\{-\frac{(\log r)^{\rho+\varepsilon}}{r}\right\} \leq \left|\frac{f(z+\eta)}{f(z)}\right| \leq \exp\left\{\frac{(\log r)^{\rho+\varepsilon}}{r}\right\}. \tag{10}$$

Proof. By using (7) and (8), we obtain

$$\left|\log\left|\frac{f(z+\eta)}{f(z)}\right|\right| \leq A\left(\frac{T(\gamma r, f)}{r}\right.$$

$$\left.+\frac{12}{\log \alpha} \frac{T(\alpha\gamma r, f)}{r} \log^{\gamma} r \log^{+}\left(\frac{12}{\log \alpha} T(\alpha\gamma r, f)\right)\right)$$

$$\leq B\left(\frac{T(\beta r, f)}{r} + \frac{\log^{\beta} r}{r} T(\beta r, f) \log T(\beta r, f)\right), \tag{11}$$

for all $|z| = r \notin [0, 1] \cup E_3$ with $lm(E_3) < +\infty$, where $B > 0$ is some constant and $\beta = \alpha\gamma > 1$. Since $f(z)$ has finite logarithmic order $\rho_{\log}(f) = \rho < +\infty$, so given ε, $0 < \varepsilon < 2$, we have for sufficiently large r

$$T(r, f) < (\log r)^{\rho+\frac{\varepsilon}{2}}. \tag{12}$$

Then by using (11) and (12), we obtain

$$\left|\log\left|\frac{f(z+\eta)}{f(z)}\right|\right| \leq B\left(\frac{T(\beta r, f)}{r} + \frac{\log^{\beta} r}{r} T(\beta r, f) \log T(\beta r, f)\right)$$

$$\leq B\left(\frac{(\log \beta r)^{\rho+\frac{\varepsilon}{2}}}{r} + \frac{\log^{\beta} r}{r} (\log \beta r)^{\rho+\frac{\varepsilon}{2}} \log(\log \beta r)^{\rho+\frac{\varepsilon}{2}}\right) \leq \frac{(\log r)^{\rho+\varepsilon}}{r}. \tag{13}$$

From (13), we easily obtain (10). □

Lemma 6. *Let η_1, η_2 be two arbitrary complex numbers such that $\eta_1 \neq \eta_2$ and let $f(z)$ be a meromorphic function of finite logarithmic order ρ. Let $\varepsilon > 0$ be given, then there exists a subset $E_4 \subset (1, \infty)$ with finite logarithmic measure such that for all $|z| = r \notin [0,1] \cup E_4$, we have*

$$\exp\left\{-\frac{(\log r)^{\rho+\varepsilon}}{r}\right\} \leq \left|\frac{f(z+\eta_1)}{f(z+\eta_2)}\right| \leq \exp\left\{\frac{(\log r)^{\rho+\varepsilon}}{r}\right\}. \tag{14}$$

Proof. We can write

$$\left|\frac{f(z+\eta_1)}{f(z+\eta_2)}\right| = \left|\frac{f(z+\eta_2+\eta_1-\eta_2)}{f(z+\eta_2)}\right| \quad (\eta_1 \neq \eta_2).$$

Then by using Lemma 5, we obtain for any given $\varepsilon > 0$ and all $|z+\eta_2| = R \notin [0,1] \cup E_3$, such that $lm(E_3) < \infty$

$$\exp\left\{-\frac{(\log r)^{\rho+\varepsilon}}{r}\right\} \leq \exp\left\{-\frac{(\log(|z|+|\eta_2|))^{\rho+\frac{\varepsilon}{2}}}{|z+\eta_2|}\right\}$$
$$= \exp\left\{-\frac{(\log R)^{\rho+\frac{\varepsilon}{2}}}{R}\right\} \leq \left|\frac{f(z+\eta_1)}{f(z+\eta_2)}\right|$$
$$= \left|\frac{f(z+\eta_2+\eta_1-\eta_2)}{f(z+\eta_2)}\right| \leq \exp\left\{\frac{(\log R)^{\rho+\frac{\varepsilon}{2}}}{R}\right\}$$
$$\leq \exp\left\{\frac{(\log(|z|+|\eta_2|))^{\rho+\frac{\varepsilon}{2}}}{|z+\eta_2|}\right\} \leq \exp\left\{\frac{(\log r)^{\rho+\varepsilon}}{r}\right\},$$

where $|z| = r \notin [0,1] \cup E_4$ and E_4 is a set of finite logarithmic measure. □

By using Lemmas 2–4, we can generalize Lemma 6 into finite logarithmic lower order case as following.

Lemma 7. *Let η_1, η_2 be two arbitrary complex numbers such that $\eta_1 \neq \eta_2$ and let $f(z)$ be a meromorphic function of finite logarithmic lower order μ. Let $\varepsilon > 0$ be given, then there exists a subset $E_5 \subset (1, \infty)$ with infinite logarithmic measure such that for all $|z| = r \in E_5$, we have*

$$\exp\left\{-\frac{(\log r)^{\mu+\varepsilon}}{r}\right\} \leq \left|\frac{f(z+\eta_1)}{f(z+\eta_2)}\right| \leq \exp\left\{\frac{(\log r)^{\mu+\varepsilon}}{r}\right\}.$$

Lemma 8 (see [1])**.** *Let f be a meromorphic function with $\rho_{\log}(f) \geq 1$. Then there exists a set $E_6 \subset (1, +\infty)$ with infinite logarithmic measure such that*

$$\lim_{\substack{r\to+\infty \\ r\in E_6}} \frac{\log T(r,f)}{\log\log r} = \rho.$$

Lemma 9 (see [1]). *Let f_1, f_2 be meromorphic functions satisfying $\rho_{\log}(f_1) > \rho_{\log}(f_2)$. Then there exists a set $E_7 \subset (1, +\infty)$ having infinite logarithmic measure such that for all $r \in E_7$, we have*

$$\lim_{r \to +\infty} \frac{T(r, f_2)}{T(r, f_1)} = 0.$$

Lemma 10. *Let f be an entire function with $1 \leq \mu_{\log}(f) < +\infty$. Then there exists a set $E_8 \subset (1, +\infty)$ with infinite logarithmic measure such that*

$$\underline{\tau}_{\log}(f) = \lim_{\substack{r \to +\infty \\ r \in E_8}} \frac{\log M(r, f)}{(\log r)^{\mu_{\log}(f)}}.$$

Proof. By the definition of the logarithmic lower type, there exists a sequence $\{r_n\}_{n=1}^{\infty}$ tending to ∞ satisfying $\left(1 + \frac{1}{n}\right) r_n < r_{n+1}$, and

$$\underline{\tau}_{\log}(f) = \lim_{r_n \to +\infty} \frac{\log M(r_n, f)}{(\log r_n)^{\mu_{\log}(f)}}.$$

Then for any given $\varepsilon > 0$, there exists an n_1 such that for $n \geq n_1$ and any $r \in \left[\frac{n}{n+1} r_n, r_n\right]$, we have

$$\frac{\log M(\frac{n}{n+1} r_n, f)}{(\log r_n)^{\mu_{\log}(f)}} \leq \frac{\log M(r, f)}{(\log r)^{\mu_{\log}(f)}} \leq \frac{\log M(r_n, f)}{(\log \frac{n}{n+1} r_n)^{\mu_{\log}(f)}}.$$

It follows that

$$\left(\frac{\log \frac{n}{n+1} r_n}{\log r_n}\right)^{\mu_{\log}(f)} \frac{\log M(\frac{n}{n+1} r_n, f)}{(\log \frac{n}{n+1} r_n)^{\mu_{\log}(f)}} \leq \frac{\log M(r, f)}{(\log r)^{\mu_{\log}(f)}}$$

$$\leq \frac{\log M(r_n, f)}{(\log r_n)^{\mu_{\log}(f)}} \left(\frac{\log r_n}{\log \frac{n}{n+1} r_n}\right)^{\mu_{\log}(f)}.$$

Set

$$E_8 = \bigcup_{n=n_1}^{\infty} \left[\frac{n}{n+1} r_n, r_n\right].$$

Then, we have

$$\lim_{\substack{r \to +\infty \\ r \in E_8}} \frac{\log M(r, f)}{(\log r)^{\mu_{\log}(f)}} = \lim_{r_n \to +\infty} \frac{\log M(r_n, f)}{(\log r_n)^{\mu_{\log}(f)}} = \underline{\tau}_{\log}(f)$$

and $lm(E_8) = \int\limits_{E_8} \frac{dr}{r} = \sum\limits_{n=n_1}^{\infty} \int\limits_{\frac{n}{n+1} r_n}^{r_n} \frac{dt}{t} = \sum\limits_{n=n_1}^{\infty} \log\left(1 + \frac{1}{n}\right) = \infty.$ □

3 Proofs of Theorems

3.1 Proof of Theorem 6

Let $f \not\equiv 0$ be a meromorphic solution of (1). We suppose $\mu_{\log}(f) < \mu_{\log}(A_l)+1 < \infty$. We divide through equation (1) by $f(z+l)$ to get

$$|A_l(z)| \leq |A_n(z)| \left|\frac{f(z+n)}{f(z+l)}\right| + \cdots + |A_{l-1}(z)| \left|\frac{f(z+l-1)}{f(z+l)}\right|$$

$$+ |A_{l+1}(z)| \left|\frac{f(z+l+1)}{f(z+l)}\right| + \cdots + |A_1(z)| \left|\frac{f(z+1)}{f(z+l)}\right| + |A_0(z)| \left|\frac{f(z)}{f(z+l)}\right|. \quad (15)$$

In relation to (2) and (3), we set

$$\rho = \max\{\rho_{\log}(A_j) : j = 0, 1, \cdots, n, j \neq l\},$$

and

$$\tau = \max\{\tau_{\log}(A_j) : \rho_{\log}(A_j) = \mu_{\log}(A_l) : j = 0, 1, \cdots, n, j \neq l\}.$$

Then for sufficiently large r, we have

$$|A_j(z)| \leq \exp\left\{(\log r)^{\rho+\varepsilon}\right\} \quad (j \neq l) \quad (16)$$

if $\rho_{\log}(A_j) < \mu_{\log}(A_l)$, and

$$|A_j(z)| \leq \exp\left\{(\tau+\varepsilon)(\log r)^{\mu_{\log}(A_l)}\right\} \quad (j \neq l) \quad (17)$$

if $\rho_{\log}(A_j) = \mu_{\log}(A_l)$. By Lemma 7, for any given $\varepsilon > 0$, there exists a set $E_5 \subset (1, \infty)$ with infinite logarithmic measure such that for all $|z| = r \in E_5$, we have

$$\left|\frac{f(z+j)}{f(z+l)}\right| \leq \exp\left\{\frac{(\log r)^{\mu_{\log}(f)+\varepsilon}}{r}\right\} \quad (j = 0, 1, \cdots, n, j \neq l). \quad (18)$$

Then we can choose ε $(0 < \varepsilon < 1)$ sufficiently small to satisfy

$$\tau + 2\varepsilon < \underline{\tau}_{\log}(A_l), \;\; \max\{\rho, \mu_{\log}(f) - 1\} + 2\varepsilon < \mu_{\log}(A_l). \quad (19)$$

Substituting (16), (17) and (18) into (15), we get for $|z| = r \in E_5$,

$$M(r, A_l) \leq \exp\left\{\frac{(\log r)^{\mu_{\log}(f)+\varepsilon}}{r}\right\} O\left(\exp\left\{(\tau+\varepsilon)(\log r)^{\mu_{\log}(A_l)}\right\}\right.$$

$$\left. + \exp\left\{(\log r)^{\rho+\varepsilon}\right\}\right). \quad (20)$$

By (19) and (20) and Lemma 10, we get

$$\underline{\tau}_{\log}(A_l) = \liminf_{\substack{r \to +\infty \\ r \in E_5}} \frac{\log M(r, A_l)}{(\log r)^{\mu_{\log}(A_l)}} \leq \tau + \varepsilon < \underline{\tau}_{\log}(A_l) - \varepsilon$$

which is a contradiction. Hence $\mu_{\log}(f) \geq \mu_{\log}(A_l) + 1$. □

3.2 Proof of Theorem 7

By Remark 3, we know that $\rho_{\log}(A_l) = \rho$. Let $f \not\equiv 0$ be a meromorphic solution of (1). Next we suppose $\rho_{\log}(f) < \rho_{\log}(A_l)+1 = \rho+1 < +\infty$. From the conditions of Theorem 7, there is a set H of complex numbers satisfying $\overline{\log dens}\{|z| : z \in H\} > 0$ such that for $z \in H$, we have (4) and (5) as $|z| = r \to +\infty$. Set $H_1 = \{r = |z| : z \in H\}$, since $\overline{\log dens}\{|z| : z \in H\} > 0$, then by Proposition 1, H_1 is a set with $\int_{H_1} \frac{dr}{r} = \infty$. By Lemma 6, for any given ε $\left(0 < \varepsilon < \frac{\rho-\rho_{\log}(f)+1}{2}\right)$, there exists a set $E_4 \subset (1,\infty)$ with finite logarithmic measure such that for all $|z| = r \notin [0,1] \cup E_4$, we have

$$\left|\frac{f(z+j)}{f(z+l)}\right| \le \exp\left\{\frac{(\log r)^{\rho_{\log}(f)+\varepsilon}}{r}\right\} \quad (j = 0, 1, \cdots, n, j \ne l). \tag{21}$$

Substituting (4), (5) and (21) into (15), we get for $|z| = r \in H_1 \backslash ([0,1] \cup E_4)$,

$$\exp\left\{\alpha [\log r]^{\rho-\varepsilon}\right\} \le n \exp\left\{\beta [\log r]^{\rho-\varepsilon}\right\} \exp\left\{\frac{(\log r)^{\rho_{\log}(f)+\varepsilon}}{r}\right\},$$

it follows that

$$\exp\left\{(\alpha-\beta) [\log r]^{\rho-\varepsilon}\right\} \le n \exp\left\{\frac{(\log r)^{\rho_{\log}(f)+\varepsilon}}{r}\right\}. \tag{22}$$

By ε $\left(0 < \varepsilon < \frac{\rho-\rho_{\log}(f)+1}{2}\right)$ and $\alpha - \beta > 0$, we obtain a contradiction from (22). Hence, we get $\rho_{\log}(f) \ge \rho + 1 = \rho_{\log}(A_l) + 1$. □

3.3 Proof of Theorem 8

Let $f \not\equiv 0$ be a meromorphic solution of (1). If $\rho_{\log}(f) = \infty$, then the result is trivial. Next we suppose $\rho_{\log}(f) = \rho < \infty$. We divide through equation (1) by $f(z+l)$ to get

$$A_l(z) = -\left(A_n(z)\frac{f(z+n)}{f(z+l)} + \cdots + A_{l-1}(z)\frac{f(z+l-1)}{f(z+l)}\right.$$

$$\left.+A_{l+1}(z)\frac{f(z+l+1)}{f(z+l)} + \cdots + A_1(z)\frac{f(z+1)}{f(z+l)} + A_0(z)\frac{f(z)}{f(z+l)}\right). \tag{23}$$

It follows

$$m(r, A_l) \le \sum_{\substack{j=0\\ j\ne l}}^{n} m(r, A_j) + \sum_{\substack{j=0\\ j\ne l}}^{n} m\left(r, \frac{f(z+j)}{f(z+l)}\right) + O(1). \tag{24}$$

Suppose that

$$\limsup_{r\to+\infty} \sum_{\substack{j=0\\ j\ne l}}^{n} \frac{m(r, A_j)}{m(r, A_l)} = \mu < \lambda < 1. \tag{25}$$

Then for sufficiently large r, we have

$$\sum_{\substack{j=0\\ j\neq l}}^{n} m\left(r,A_j\right) < \lambda m\left(r,A_l\right). \tag{26}$$

By Lemma 1, we have for sufficiently large r and any given $\varepsilon>0$

$$m\left(r,\frac{f\left(z+j\right)}{f\left(z+l\right)}\right) = O\left(\left(\log r\right)^{\rho_{\log}(f)-1+\varepsilon}\right),\ j=0,\cdots,n,\ j\neq l. \tag{27}$$

Thus, by substituting (26) and (27) into (24), we obtain for sufficiently large r and any given $\varepsilon>0$

$$m\left(r,A_l\right) \leq \sum_{\substack{j=0\\ j\neq l}}^{n} m\left(r,A_j\right) + \sum_{\substack{j=0\\ j\neq l}}^{n} m\left(r,\frac{f\left(z+j\right)}{f\left(z+l\right)}\right) + O\left(1\right)$$

$$\leq \lambda m\left(r,A_l\right) + O\left(\left(\log r\right)^{\rho_{\log}(f)-1+\varepsilon}\right). \tag{28}$$

From (28), it follows that

$$\left(1-\lambda\right) m\left(r,A_l\right) \leq O\left(\left(\log r\right)^{\rho_{\log}(f)-1+\varepsilon}\right). \tag{29}$$

By (29), we obtain $\rho_{\log}\left(f\right) \geq \rho_{\log}\left(A_l\right)+1$. Thus, Theorem 8 is proved. □

3.4 Proof of Theorem 9

Clearly, (1) has no nonzero rational solution. If $\rho_{\log}\left(f\right)=\infty$, then the result is trivial. Now suppose that f is a transcendental meromorphic solution of (1) with $\rho_{\log}\left(f\right)<\infty$. Set

$$\delta\left(\infty,A_l\right) = \liminf_{r\to+\infty}\frac{m\left(r,A_l\right)}{T\left(r,A_l\right)} = \delta > 0. \tag{30}$$

Thus from (30), we have for sufficiently large r

$$m\left(r,A_l\right) > \frac{1}{2}\delta T\left(r,A_l\right). \tag{31}$$

Thus, by substituting (27) and (31) into (24), we obtain for sufficiently large r and any given $\varepsilon>0$

$$\frac{\delta}{2}T\left(r,A_l\right) < m\left(r,A_l\right) \leq \sum_{\substack{j=0\\ j\neq l}}^{n} m\left(r,A_j\right) + \sum_{\substack{j=0\\ j\neq l}}^{n} m\left(r,\frac{f\left(z+j\right)}{f\left(z+l\right)}\right) + O\left(1\right)$$

$$\leq \sum_{\substack{j=0\\ j\neq l}}^{n} T\left(r,A_j\right) + O\left(\left(\log r\right)^{\rho_{\log}(f)-1+\varepsilon}\right). \tag{32}$$

Since $\max\{\rho_{\log}(A_j)\ \ (j=0,\cdots,n), j\neq l\} < \rho_{\log}(A_l)$, then by Lemma 9, there exists a set $E_7 \subset (1,+\infty)$ with infinite logarithmic measure such that

$$\max\left\{\frac{T(r,A_j)}{T(r,A_l)}\ \ (j=0,\cdots,n), j\neq l\right\} \to 0,\ r\to+\infty,\ r\in E_7. \tag{33}$$

Thus, by (32) and (33), we have for all $r\in E_7$, $r\to+\infty$

$$\left(\frac{\delta}{2}-o(1)\right)T(r,A_l) \le O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right). \tag{34}$$

So that, it follows from (34) and Lemma 8 that $\rho_{\log}(f) \ge \rho_{\log}(A_l)+1$. Thus, Theorem 9 is proved. □

3.5 Proof of Theorem 10

Let $f\not\equiv 0$ be a meromorphic solution of (1). If $\rho_{\log}(f)=\infty$, then the result is trivial. Next we suppose $\rho_{\log}(f)=\rho<\infty$. By substituting (26) and (27) into (24), we have for sufficiently large r and any given $\varepsilon>0$

$$(1-\lambda)\,m(r,A_l) \le O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right). \tag{35}$$

By Lemma 8, we have

$$\lim_{\substack{r\to+\infty\\ r\in E_6}} \frac{\log T(r,A_l)}{\log\log r} = \rho_{\log}(A_l), \tag{36}$$

where E_6 is a set of r of infinite logarithmic linear measure. Since $\delta(\infty,A_l) = \liminf\limits_{r\to+\infty}\frac{m(r,A_l)}{T(r,A_l)}>0$, then we obtain

$$\lim_{\substack{r\to+\infty\\ r\in E_6}} \frac{\log m(r,A_l)}{\log\log r} = \rho_{\log}(A_l). \tag{37}$$

Thus, by (35) and (37), we obtain $\rho_{\log}(f) \ge \rho_{\log}(A_l)+1$. Thus, Theorem 10 is proved. □

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References

[1] Belaïdi B. *Growth of meromorphic solutions of finite logarithmic order of linear difference equations.* Fasc. Math., 2015, No. 54, 5–20.

[2] Cao T. B., Liu K., Wang J. *On the growth of solutions of complex differential equations with entirecoefficients of finite logarithmic order.* Math. Reports 15(65), 2013, **3**, 249–269.

[3] Chen Z. X., Shon K. H. *On growth of meromorphic solutions for linear difference equations.* Abstr. Appl. Anal. 2013, Art. ID 619296, 1–6.

[4] Chern Peter T. Y. *On meromorphic functions with finite logarithmic order*. Trans. Amer. Math. Soc., 2006, **358**, No. 2, 473–489.

[5] Chiang Y. M., Feng S. J. *On the Nevanlinna characteristic of* $f(z+\eta)$ *and difference equations in the complex plane*. Ramanujan J., 2008, **16**, No. 1, 105–129.

[6] Gundersen G. G. *Estimates for the logarithmic derivative of a meromorphic function, plus similar estimates.* J. London Math. Soc. (2), 1988, **37**, No. 1, 88–104.

[7] Halburd R. G., Korhonen R. J. *Difference analogue of the lemma on the logarithmic derivative with applications to difference equations.* J. Math. Anal. Appl., 2006, **314**, No. 2, 477–487.

[8] Halburd R. G., Korhonen R. J. *Nevanlinna theory for the difference operator*. Ann. Acad. Sci. Fenn. Math., 2006, **31**, No. 2, 463–478.

[9] Hayman W. K. *Meromorphic functions*, Oxford Mathematical Monographs Clarendon Press, Oxford 1964.

[10] Heittokangas J., Wen Z. T. *Functions of finite logarithmic order in the unit disc*. Part I. J. Math. Anal. Appl., 2014, **415**, No. 1, 435–461.

[11] Heittokangas J., Wen Z. T. *Functions of finite logarithmic order in the unit disc*. Part II. Comput. Methods Funct. Theory, 2015, **15**, No. 1, 37–58.

[12] Laine I., Yang C. C. *Clunie theorems for difference and q-difference polynomials*. J. Lond. Math. Soc. (2), 2007, **76**, No. 3, 556–566.

[13] Liu H., Mao Z. *On the meromorphic solutions of some linear difference equations*. Adv. Difference Equ. 2013, 2013:133, 1–12.

[14] Wen Z. T. *Finite logarithmic order solutions of linear q-difference equations.* Bull. Korean Math. Soc., 2014, **51**, No. 1, 83–98.

[15] Zheng X. M., Tu J. *Growth of meromorphic solutions of linear difference equations*. J. Math. Anal. Appl., 2011, **384**, No. 2, 349–356.

[16] Yang C. C., Yi H. X. *Uniqueness theory of meromorphic functions*. Mathematics and its Applications, 557. Kluwer Academic Publishers Group, Dordrecht, 2003.

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