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Uniqueness of certain power of a meromorphic function sharing a set with its differential monomial

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Abstract. In this paper we are mainly devoted to find out the specific form of a meromorphic function when it shares a set of small functions with its differential monomial counterpart. Our results will improve and extend some of the recent results due to Zhang-Yang [J. L. Zhang and L. Z. Yang, A power of a meromorphic function sharing a small function with its derivative, Ann. Acad. Sci. Fenn. Math. 34(2009), 249–260] and Xu-Yi-Yang [H. Y. Xu, C. F. Yi and H. Wang, On a conjecture of R. Bruck concerning meromorphic function sharing small functions, Revista de Matematica Teoria y Aplicaciones, 23(1)(2016), 291-308]. We provide some examples to show that certain conditions used in the paper can not be removed.

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1 Introduction, Definitions and Results

Let f be a non-constant meromorphic function in the whole complex plane $\mathbb{C}$. We shall use the following standard notations of the value distribution theory:

$$T(r,f), m(r,f), N(r,f), \overline{N}(r,f), \ldots$$

([11, 19, 23]). We denote by $S(r,f)$ any quantity satisfying

$$S(r,f) = o(T(r,f)),$$

as $r \to +\infty$, possibly outside of a set of finite measure. A meromorphic function $a \equiv a(z)$ is called a small function with respect to f if $T(r,a) = S(r,f)$. Let $\mathcal{S}(f)$ be the set of meromorphic functions in the complex plane $\mathbb{C}$ which are small functions with respect to f.

Let f be a non-constant meromorphic function and $a \in \mathcal{S}(f) \cup \{\infty\}$ and $\mathcal{S} \subset \mathcal{S}(f) \cup \{\infty\}$. Define

$$E(\mathcal{S}, f) = \bigcup_{a \in \mathcal{S}} \{z : f(z) - a = 0, \text{Counting Multiplicity}\},$$

$$\overline{E}(\mathcal{S}, f) = \bigcup_{a \in \mathcal{S}} \{z : f(z) - a = 0, \text{Ignoring Multiplicity}\},$$

If $E(\mathcal{S}, f) = E(\mathcal{S}, g)$, we say that f and g share the set $\mathcal{S}$ CM; if $\overline{E}(\mathcal{S}, f) = \overline{E}(\mathcal{S}, g)$, we say that f and g share the set $\mathcal{S}$ IM. Especially, when $\mathcal{S} = \{a\}$, we say that f and g share the value a CM if $E(\mathcal{S}, f) = E(\mathcal{S}, g)$; and we say that f and g share the value a IM if $\overline{E}(\mathcal{S}, f) = \overline{E}(\mathcal{S}, g)$ [11].

Nowadays the problems relative to a meromorphic function f and its derivative $f^{(k)}$ sharing some value or small functions have been studied rigorously by many researchers. Readers are requested to make a glance at [9, 15, 24, 27].

In 1996, *Brück* [7] proposed the following famous conjecture.

Conjecture 1.1. *Let f be a non-constant entire function. Suppose that $\rho_1(f)$ is not a positive integer or infinite. If f and f' share one finite value a CM, then*

$$\frac{f' - a}{f - a} = c,$$

for some non-zero constant c, where $\rho_1(f)$ is the first iterated order of f which is defined by

$$\rho_1(f) = \limsup_{r \to \infty} \frac{\log \log T(r, f)}{\log r}.$$

In 1996, *Brück* [7] proved that the conjecture is true when $a = 0$ or $N(r, 1/f') = S(r, f)$ and later many researchers like *Gundersen* and *Yang* [10] proved that the conjecture is true when f is of finite order [10]. A few years later, *Chen* and *Shon* [8] proved that the conjecture is true for entire function of first order $\rho_1(f) < \frac{1}{2}$. However, the conjecture fails in general for meromorphic functions, shown by *Gundersen* and *Yang* [10], while it remains true in the case that $N(r, 1/f') = S(r, f)$, shown by *Al-Kahaladi* [1].

In 2008, *Yang* and *Zhang* [20] obtained the following results.

Theorem 1.1 (see [20]). *Let f be a non-constant entire function, $n \geq 7$ be an integer. Denote $\mathcal{F} = f^n$. If $\mathcal{F}$ and $\mathcal{F}'$ share 1 CM, then $\mathcal{F} \equiv \mathcal{F}'$, and f assumes the form*

$$f(z) = ce^{\frac{z}{n}},$$

where c is a non-zero constant.

Theorem 1.2 (see [20]). *Let f be a non-constant meromorphic function and $n \geq 12$ be an integer. Denote $\mathcal{F} = f^n$. If $\mathcal{F}$ and $\mathcal{F}'$ share 1 CM, then $\mathcal{F} \equiv \mathcal{F}'$, and f assumes the form*

$$f(z) = ce^{\frac{z}{n}},$$

where c is a non-zero constant.

In 2009, *Zhang* and *Yang* [25] improved Theorem 1.1 and Theorem 1.2 to a large extent and obtained the following results.

Theorem 1.3 (see [25]). *Let f be a non-constant entire function, n, k be positive integers and $a(z)$ be a small function of f such that $a(z) \not\equiv 0, \infty$. If $f^n - a$ and $(f^n)^{(k)} - a$ share the value 0 CM and $n \geq k+2$, then $f^n \equiv (f^n)^{(k)}$ and f assumes the form*

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a non-zero constant and $\lambda^k = 1$.

Theorem 1.4 (see [25]). *Let f be a non-constant meromorphic function, n, k be positive integers and $a(z)$ be a small function of f such that $a(z) \not\equiv 0, \infty$. If $f^n - a$ and $(f^n)^{(k)} - a$ share the value 0 CM and $n > k+1+\sqrt{k+1}$, then $f^n \equiv (f^n)^{(k)}$ and f assumes the form*

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a non-zero constant and $\lambda^k = 1$.

Theorem 1.5 (see [25]). *Let f be a non-constant entire function, n, k be positive integers and $a(z)$ be a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$. If $f^n - a$ and $(f^n)^{(k)} - a$ share the value 0 IM and $n > 2k+3$, then $f^n \equiv (f^n)^{(k)}$ and f assumes the form*

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a non-zero constant with $\lambda^k = 1$.

Theorem 1.6 (see [25]). *Let f be a non-constant meromorphic function, n, k be positive integers and $a(z)$ be a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$. If $f^n - a$ and $(f^n)^{(k)} - a$ share the value 0 IM and*

$$n > 2k+3+\sqrt{(k+3)(2k+3)},$$

then $f^n \equiv (f^n)^{(k)}$ and f assumes the form

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a non-zero constant with $\lambda^k = 1$.

Though the standard definitions and notations of the value distribution theory are available in [3, 22], we explain the following definitions and notations which are used in the paper.

Definition 1.1 (see [3, 22]). *When f and g share 1 IM, we denote by $\overline{N}_L(r,1;f)$ the counting function of the 1-points of g. Similarly, we have $\overline{N}_L(r,1;g)$. Let z_0 be a zero of $f-1$ of multiplicity p and a zero of $g-1$ of multiplicity q, we also denote by $N_{11}(r,1;f)$ the counting function of those 1-points of f where $p=q=1$; $\overline{N}_E^{(2}(r,1;f)$ denotes the counting function of those $1-$points of f where $p=q\geq 2$, each point in these counting functions is counted only once. In the same way, one can define $N_{11}(r,1;g)$, $\overline{N}_E^{(2}(r,1;g)$.*

Definition 1.2 (see [5]). *For $a \in \mathbb{C} \cup \{\infty\}$ and p a positive integer, let f be a non-constant meromorphic function, we denote by $N(r, a; f \mid= 1)$ the counting function of simple a-points of f, denote by $N(r, a; f \mid\leq p)$ $(N(r, a; f \mid\geq p))$ the counting functions of those a-points of f whose multiplicities are not greater (less) than p where each a-point is counted according to its multiplicities. $\overline{N}(r, a; f \mid\leq p)$ $(\overline{N}(r, a; f \mid\geq p))$ are defined similarly, where in counting the a-points of f we ignore the multiplicities.*

Definition 1.3 (see [5]). *For $a \in \mathbb{C} \cup \{\infty\}$ and a positive integer p we denote by*

$$N_p(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2) + \ldots + \overline{N}(r, a; f \mid\geq p).$$

Clearly $N_1(r, a; f) = \overline{N}(r, a; f)$.

Next we recall the following definition of weighted sharing of values which generally measures how closed a shared value is to being sharing IM or CM, as follows.

Definition 1.4 (see [13, 14]). *Let p be a non-negative integer or infinity. For $c \in \mathbb{C} \cup \{\infty\}$, we denote by $E_f(a, p)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq p$ and $p + 1$ times if $m > p$. If $E_f(a, p) = E_g(a, p)$, we say that f, g share the value a with weight p.*

We write f, g share (a, p) to mean that f, g share the value a with weight p. Clearly if f, g share (a, p), then f, g share (a, q) for all integer q $(0 \leq q < p)$. Also, we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ and (a, ∞) respectively.

Let $\mathcal{S}$ be a subset of $\mathcal{S}(f) \cup \{\infty\}$, we can get the definition of $E_f(\mathcal{S}, p)$ as

$$E_f(\mathcal{S}, p) = \bigcup_{a \in \mathcal{S}} E_f(a, p).$$

Very recently in [18], for further investigations, *Xu*, *Yi* asked the following questions:

Question 1.1 (see [18]). *Can the nature of sharing 1 or $a(z)$ CM be further relaxed in Theorem 1.1 and Theorem 1.3?*

Question 1.2 (see [18]). *What will happen when 1 or $a(z)$ are replaced by the set*

$$\mathcal{S}_m = \left\{ a(z), a(z)\zeta, a(z)\zeta^2, ..., a(z)\zeta^{m-1} \right\}$$

of small functions in Theorems 1.1 – 1.4, where $\zeta = \cos \dfrac{2\pi}{m} + i \sin \dfrac{2\pi}{m}$ and m is a positive integer?

To answer their question *Xu*, *Yi* and *Wang* [18] obtained the following two results which in turn improve Theorem 1.3 and Theorem 1.4.

Theorem 1.7 (see [18]). *Let f be a non-constant entire function, n, k, p, m be positive integers and $a(z)$ be a small function of f such that $a(z) \not\equiv 0, \infty$. If $E_{f^n}(\mathcal{S}_m, p) = E_{(f^n)^{(k)}}(\mathcal{S}_m, p)$ and*

$$n > \max\left\{k+1, k+\frac{\eta}{pm}\right\},$$

where $\eta = k+p+2$, then $f^n \equiv t(f^n)^{(k)}$ with $t^m = 1$ and f assumes the form

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a non-zero constant and $\lambda^{km} = 1$.

Theorem 1.8 (see [18]). *Let f be a non-constant meromorphic function, n, k, p, m be positive integers and $a(z)$ be a small function of f such that $a(z) \not\equiv 0, \infty$. If $E_{f^n}(\mathcal{S}_m, p) = E_{(f^n)^{(k)}}(\mathcal{S}_m, p)$ and*

$$n > \max\left\{k+1, \frac{p(m+1)k+2\eta}{2pm} + \frac{\sqrt{4\eta(\eta+pk)+(m-1)^2p^2k^2}}{2pm}\right\},$$

where $\eta = k+p+2$, then $f^n \equiv t(f^n)^{(k)}$ with $t^m = 1$ and f assumes the form

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a non-zero constant and $\lambda^{km} = 1$.

We observe from the above discussions that the research have gradually been shifted towards finding the relation between the power of a meromorphic function and its certain derivative. Since derivative's natural extension is a differential monomial it will be quite natural to expect the extension and improvement of Theorems 1.1 – 1.8 up to a relation between a power of a meromorphic function and a general differential monomial sharing set of small functions.

Next we present the following well known definition.

Definition 1.5 (see [5]). *Let $n_{0j}, n_{1j}, \ldots, n_{kj}$ be nonnegative integers and $g = f^n$.*

The expression $M_j[g] = (g)^{n_{0j}}(g^{(1)})^{n_{1j}} \ldots (g^{(k)})^{n_{kj}}$ is called a differential monomial generated by g of degree $d_{M_j} = d(M_j) = \sum_{i=0}^{k} n_{ij}$ and weight $\Gamma_{M_j} = \sum_{i=0}^{k}(i+1)n_{ij}$.

The sum $\mathcal{P}[g] = \sum_{j=1}^{t} b_j M_j[g]$ is called a differential polynomial generated by g of degree $\overline{d}(\mathcal{P}) = max\{d(M_j) : 1 \le j \le t\}$ and weight $\Gamma_{\mathcal{P}} = max\{\Gamma_{M_j} : 1 \le j \le t\}$, where $T(r, b_j) = S(r, g)$ for $j = 1, 2, \ldots, t$.

The numbers $\underline{d}(\mathcal{P}) = min\{d(M_j) : 1 \le j \le t\}$ and k (the highest order of the derivative of g in $\mathcal{P}[g]$) are called respectively the lower degree and order of $\mathcal{P}[g]$.

$\mathcal{P}[g]$ is said to be homogeneous if $\overline{d}(\mathcal{P})$=$\underline{d}(\mathcal{P})$. $\mathcal{P}[g]$ is called a linear differential polynomial generated by g if $\overline{d}(\mathcal{P}) = 1$. Otherwise $\mathcal{P}[g]$ is called a non-linear differential polynomial.

We denote by $Q = max\ \{\Gamma_{M_j} - d(M_j) : 1 \leq j \leq t\} = max\ \{n_{1j} + 2n_{2j} + \ldots + kn_{kj} : 1 \leq j \leq t\}$.

Also for the sake of convenience for a differential monomial $M[g]$ we denote by $d_{_M} = d(M)$ and $Q_{_M} = \Gamma_{_M} - d_{_M}$.

Next we pose the following questions which have great significance towards the further extension and improvement of all the above mentioned theorems.

Question 1.3. *Is it possible to extend $(f^n)^{(k)}$ to a differential monomial $M[f^n]$ to get the same conclusion as in Theorem 1.7 and Theorem 1.8?*

Question 1.4. *Like Theorem 1.7 and Theorem 1.8, is it possible to find out the specific form of the function f?*

Question 1.5. *Can the lower bound of n be further reduced in Theorem 1.7 and Theorem 1.8?*

Our main intention of writing this paper is to find out the possible affirmative answer of all the above questions such that Theorems 1.1 – 1.8 can be accommodated under a single theorem which extends and improves all of them. Henceforth we need the following notations throughout the paper for the sake of convenience.

Let

$$\alpha = 2Q_{_M} + 3, \quad \beta = mQ_{_M} + (k+1)d_{_M} + 2 \quad \text{and} \quad \gamma_m^p = mQ_{_M} + 1 + \frac{1}{p},$$

where p, m and k are three positive integers.

The following two theorems are the main results of this paper answering all the above mentioned questions affirmatively.

Theorem 1.9. *Let f be a non-constant meromorphic function, n, k, p, m be positive integers and $a(z)$ be a small function of f such that $a(z) \not\equiv 0, \infty$. If $E_{f^{nd_{_M}}}(\mathcal{S}_m, p) = E_{M[f^n]}(\mathcal{S}_m, p)$ and if*

1. *$p \geq 2$ and $n > \dfrac{\gamma_m^p + \gamma_1^p + \sqrt{(\gamma_m^p - \gamma_1^p)^2 + 4C}}{2md_{_M}}$, or if*
2. *$p = 0$ and $n > \dfrac{\alpha + \beta + \sqrt{(\alpha - \beta)^2 + 4D}}{2md_{_M}}$,*

where $C = \dfrac{(p+1)(p(k+1)d_{_M} + 1)}{p^2}$ and $D = (Q_{_M} + 3)(2(k+1)d_{_M} + 1)$,

then $f^{nd_{_M}} \equiv tM[f^n]$ with $t^m = 1$ and f assumes the form

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a non-zero constant with $\lambda^{mQ_{_M}} = 1$.

Theorem 1.10. *Let f be a non-constant entire function, n, k, p, m be positive integers and $a(z)$ be a small function of f such that $a(z) \not\equiv 0, \infty$. If $E_{f^{nd_M}}(\mathcal{S}_m, p) = E_{M[f^n]}(\mathcal{S}_m, p)$ and if*

1. *$p \geq 2$ and $n > \dfrac{pmQ_M + p + 1}{pmd_M}$, or if*
2. *$p = 0$ and $n > \dfrac{mQ_M + (k+1)d_M + 2}{md_M}$,*

then $f^{nd_M} \equiv tM[f^n]$ with $t^m = 1$ and f assumes the form

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a non-zero constant with $\lambda^{mQ_M} = 1$.

2 Some Corollaries

In Theorem 1.9 and Theorem 1.10, if we take $M[f^n] = (f^n)^{(k)}$, where $n > k$, then it is clear that $d_M = 1$, $Q_M = k$. The following are some corollaries of the main results of this paper. What worth noticing here is that the lower bound of n is reduced as compare to Theorem 1.7 and Theorem 1.8.

Corollary 1. *Let f be a non-constant meromorphic function and n, m, p, k be positive integers and $a(z)$ be a small meromorphic function of f such that $a(z) \neq 0, \infty$. If $E_{f^n}(\mathcal{S}_m, p) = E_{(f^n)^{(k)}}(\mathcal{S}_m, p)$ and if*

1. *$p \geq 2$ and $n > \dfrac{2p + p(m+1)k + 2}{2pm} + \dfrac{\sqrt{4(p+1)(pk+p+1) + (m-1)^2p^2k^2}}{2pm}$, or if*
2. *$p = 0$ and $n > \dfrac{(m+3)k + 6}{2m} + \dfrac{\sqrt{4(k+3)(2k+3) + (m-1)^2k^2}}{2m}$,*

then $f^n \equiv t(f^n)^{(k)}$, where $t^m = 1$ and f assumes the form

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a non-zero constant and $\lambda^{mk} = 1$.

Corollary 2. *Let f be a non-constant entire function and n, m, p, k be positive integers and $a(z)$ be a small meromorphic function of f such that $a(z) \neq 0, \infty$. If $E_{f^n}(\mathcal{S}_m, p) = E_{(f^n)^{(k)}}(\mathcal{S}_m, p)$ and if*

1. *$p \geq 2$ and $n > k + \dfrac{p+1}{pm}$, or if*
2. *$p = 0$ and $n > k + \dfrac{k+3}{m}$,*

then $f^n \equiv t(f^n)^{(k)}$, *where* $t^m = 1$ *and* f *assumes the form*

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c *is a non-zero constant and* $\lambda^{mk} = 1$.

Corollary 3. *Let* f *be a non-constant entire function and* n, p, k *be positive integers and* $a \equiv a(z)$ *is a small meromorphic function of* f *and* $E_{f^n}(\mathcal{S}_1, p) = E_{(f^n)^{(k)}}(\mathcal{S}_1, p)$, *then if*

1. $p \geq 2$ *and* $n > k + \dfrac{p+1}{p}$, *or if*
2. $p = 0$ *and* $n > 2k + 3$,

then $f^n \equiv (f^n)^{(k)}$ *and* f *assumes the form*

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c *is a non-zero constant and* $\lambda^k = 1$.

3 Examples

The following examples show that conditions 1. and 2. in Corollary 1 and Corollary 2 are essential in order to get the conclusions.

Example 3.1. *For* $n \geq 2$, *let the principal branch of* f *be given by* $f(z) = \left(e^{\theta z} + 2a\right)^{\frac{1}{n}}$, *where* $a \neq 0$ *is a constant and* θ *is a root of the equation* $z^n + 1 = 0$. *Let* $\mathcal{S}_m = \{a\}$ *and* $M[f^n] = (f^n)^{(n)}$. *Clearly* $f^n = e^{\theta z} + 2a$ *and* $M[f^n] = -e^{\theta z}$ *and* $d_{_M} = 1$. *Therefore we see that* $E_{f^{nd_M}}(\mathcal{S}_m, \infty) = E_{M[f^n]}(\mathcal{S}_m, \infty)$ *and*

$$n \leq \min\left\{k + \frac{p+1}{pm}, k + \frac{k+3}{m}\right\} = \max\left\{n+1, 2n+3\right\} = n+1.$$

Here it is clear that

$$f^n \not\equiv tM[f^n]$$

with $t^m = 1$. *Also we see that* f *does not assume the form*

$$f(z) = ce^{\frac{\lambda}{n}z}$$

with $\lambda^{mQ_M} = 1$.

The following example shows that the conditions 1. and 2. used in Corollary 1 and Corollary 2 are not necessary but sufficient.

Example 3.2. *Let* $\mathcal{S}_m = \{-1, 1, -i, i\}$ *and* f *be given by* $f(z) = e^{\frac{\lambda}{3}z}$, *where* λ *is a root of the equation* $z^3 + 1 = 0$. *Let* $M[f^3] = (f^3)^{(3)}$. *It is clear that* $f^3(z) = e^{\lambda z}$ *and* $M[f^3] = -e^{\lambda z}$. *Also* $E_{{f^n}^d{}_M}(\mathcal{S}_m, \infty) = E_{M[f^n]}(\mathcal{S}_m, \infty)$ *and*

$$n \leq \min\left\{k + \frac{p+1}{pm},\ k + \frac{k+3}{m}\right\} = \min\left\{\frac{13}{4},\ \frac{9}{2}\right\} = \frac{13}{4}.$$

But we see that $f^3 \equiv tM[f^3]$ *with* $t^m = (-1)^4 = 1$. *Also here* f *assumes the form*

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where $c = 1$ *and* $\lambda^{mQ_M} = \lambda^{12} = 1$.

The following examples show that the set $\mathcal{S}_m$ in Theorems *1.9 – 1.10* can not be replaced by an arbitrary set.

Example 3.3. *Let* $\mathcal{S}_m = \left\{\frac{a\omega}{2}, \frac{a\omega}{3}, \frac{2a\omega}{3}\right\}$, *where* a *is an arbitrary non-zero complex number. Let* $f^n = \mathcal{B}e^{\theta z} + a\omega$, *where* $n \leq 16$ *is a positive integer and* θ *and* ω *are roots of the equations* $z^{n-5} + 1 = 0$ *and* $z^3 - 1 = 0$ *respectively and* $\mathcal{B} \in \mathbb{C} - \{0\}$. *Let* $M[f^n] = (f^n)^{(n-5)}$, *then we see that* $M[f^n] = -\mathcal{B}e^{\theta z}$. *It is clear that* $E_{{f^n}^d{}_M}(\mathcal{S}_m, \infty) = E_{M[f^n]}(\mathcal{S}_m, \infty)$ *and*

$$n > \max\left\{k + \frac{p+1}{pm},\ k + \frac{k+3}{m}\right\}.$$

But we see that $f^n \not\equiv tM[f^n]$ *with* $t^m = 1$*and hence* f *does not assume the form*

$$f(z) = ce^{\frac{\lambda}{n}z}$$

with $\lambda^{mQ_M} = 1$.

Example 3.4. *Let* $\mathcal{S}_m = \left\{\frac{1}{r}\mathcal{A}, \frac{r-1}{r}\mathcal{A} : 2 \leq r \leq \frac{m+3}{2}\right\}$, *where* $\mathcal{A}$ *is an arbitrary non-zero complex number and* $m, r \in \mathbb{N}$ *where* m *is odd and* $m > n+2$. *Let* $f^n = \mathcal{A}e^{\theta z} + \mathcal{A}$, *where* $n \geq 2$ *is a positive integer and* θ *is a root of the equation* $z^{n-1} + 1 = 0$ *and* $\mathcal{A} \in \mathbb{C} - \{0\}$. *Let* $M[f^n] = (f^n)^{(n-1)}$, *then we see that* $M[f^n] = -\mathcal{A}e^{\theta z}$. *It is clear that* $E_{{f^n}^d{}_M}(\mathcal{S}_m, \infty) = E_{M[f^n]}(\mathcal{S}_m, \infty)$ *and*

$$n > \max\left\{k + \frac{p+1}{pm}, k + \frac{k+3}{m}\right\}.$$

But we see that $f^n \not\equiv tM[f^n]$ *with* $t^m = 1$. *Also we see that* f *does not assume the form*

$$f(z) = ce^{\frac{\lambda}{n}z}$$

with $\lambda^{mQ_M} = 1$.

The following example shows that if the conditions of Theorem 1.9 and Theorem 1.10 are satisfied, then the conclusions hold.

Example 3.5. *Let* $\mathcal{S}_m = \{-1, 1, -i, i\}$ *and* f *be given by* $f(z) = e^{\frac{\lambda}{5}z}$, *where* λ *is a root of the equation* $z^3 + 1 = 0$. *Let* $M[f^n] = (f^n)^{(k)}$. *It is clear that* $f^n(z) = e^{\lambda z}$ *and* $M[f^n] = -e^{\lambda z}$ *with* $n = 5$, $k = 3$, $m = 4$ *and* $d_M = 1$. *Also we see that* $E_{f^{nd_M}}(\mathcal{S}_m, \infty) = E_{M[f^n]}(\mathcal{S}_m, \infty)$ *and*

$$n > \max\left\{k + \frac{p+1}{pm}, k + \frac{k+3}{m}\right\} = \max\left\{\frac{13}{4}, \frac{9}{2}\right\} = \frac{9}{2}.$$

Here we see that $f^{nd_M} \equiv tM[f^n]$ *with* $t^m = (-1)^4 = 1$. *Also here* f *assumes the form*

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where $c = 1$ *and* $\lambda^{mQ_M} = \lambda^{12} = 1$.

Example 3.6. *For a non-zero complex number* a, *let* $S = \left\{a, a\zeta, a\zeta^2, a\zeta^3, a\zeta^4\right\}$, *where* ζ *is the non-real* 5^{th} *root of unity and* f *is given by* $f(z) = e^{\zeta^{\frac{1}{nk}} z}$. *It is clear that* $f^n(z) = e^{\zeta^{\frac{1}{k}} z}$ *and* $M[f^n] = \zeta\, e^{\zeta^{\frac{1}{k}} z}$, *where* $M[f^n] = (f^n)^{(k)}$ *with* $n = 10$, $k = 7$, $m = 5$ *and* $d_M = 1$. *Also we see that* $E_{f^{nd_M}}(\mathcal{S}_m, \infty) = E_{M[f^n]}(\mathcal{S}_m, \infty)$ *and*

$$n > \max\left\{k + \frac{p+1}{pm}, k + \frac{k+3}{m}\right\} = \max\left\{\frac{36}{5}, 9\right\} = 9.$$

Here we see that $f^{nd_M} \equiv tM[f^n]$ *with* $t^m = (\frac{1}{\zeta})^5 = 1$. *Also here* f *assumes the form*

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where $c = 1$ *and* $\lambda^{mQ_M} = \lambda^{12} = 1$.

4 Lemmas

In this section we present some Lemmas which will be needed in the sequel. Let $\mathcal{F}$, $\mathcal{G}$ be two non-constant meromorphic functions. Henceforth we shall denote by $\mathcal{H}$ the following function

$$\mathcal{H} = \left(\frac{\mathcal{F}''}{\mathcal{F}'} - \frac{2\mathcal{F}'}{\mathcal{F}-1}\right) - \left(\frac{\mathcal{G}''}{\mathcal{G}'} - \frac{2\mathcal{G}'}{\mathcal{G}-1}\right). \tag{4.1}$$

$$\mathcal{V} = \left(\frac{\mathcal{F}'}{\mathcal{F}-1} - \frac{\mathcal{F}'}{\mathcal{F}}\right) - \left(\frac{\mathcal{G}'}{\mathcal{G}-1} - \frac{\mathcal{G}'}{\mathcal{G}}\right) \tag{4.2}$$

and

$$\mathcal{U} = \frac{\mathcal{F}'}{\mathcal{F}-1} - \frac{\mathcal{G}'}{\mathcal{G}-1}. \tag{4.3}$$

Lemma 1 (see [18])**.** *Let f be a non-constant meromorphic function and k, p are positive integers. Then*

$$N_p(r,0;f^{(k)}) \leq T(r,f^{(k)}) - T(r,f) + N_{p+k}(r,0;f) + S(r,f).$$

$$N_p(r,0;f^{(k)}) \leq k\overline{N}(r,\infty;f) + N_{p+k}(r,0;f) + S(r,f).$$

Lemma 2. *Let f be a non-constant meromorphic function and $M[f^n]$ be a differential monomial of degree d_M and weight Γ_M. Then*

$$N(r,0;M[f^n]) \leq T(r,M) - nd_M T(r,f) + nd_M N(r,0;f) + S(r,f).$$

Proof. This can be proved in the line of the proof of ([6, Lemma 2.3]). □

Lemma 3. *Let f be a non-constant meromorphic function and $M[f^n]$ be a differential monomial of degree d_M and weight Γ_M. Then*

$$N(r,0;M[f^n]) \leq nd_M N(r,0;f) + Q_{_M}\overline{N}(r,\infty;f) + S(r,f).$$

Proof. This can be proved in the line of the proof of ([6, Lemma 2.4]). □

Lemma 4. *For the differential monomial $M[f^n]$,*

$$N_p(r,0;M[f^n]) \leq d_M N_{p+k}(r,0;f^n) + Q_{_M}\overline{N}(r,\infty;f) + S(r,f).$$

Proof. This can be proved in the line of the proof of ([6, Lemma 2.9]). □

Lemma 5 (see [21])**.** *Let f be a non-constant meromorphic function and $P(f) = a_n f^n + a_{n-1}f^{n-1} + ... + a_0$, where $a_0, a_1, ..., a_n$ are constants with $a_n \neq 0$. Then*

$$T(r,P(f)) = nT(r,f) + S(r,f).$$

Lemma 6 (see [21])**.** *Let $\mathcal{H}$ be given by (4.1), $\mathcal{F}$ and $\mathcal{G}$ be two non-constant meromorphic functions. If $\mathcal{H} \not\equiv 0$, then*

$$N_{11}(r,1;\mathcal{F}) \leq N(r,\mathcal{H}) + S(r,\mathcal{F}) + S(r,\mathcal{G}).$$

Lemma 7. *Let f be a non-constant meromorphic function and $a \equiv a(z)$ be a small meromorphic functions of f such that $a(z) \not\equiv 0, \infty$ and let $\mathcal{F}_1 = \dfrac{f^{nd_M}}{a}$ and $\mathcal{G}_1 = \dfrac{M[f^n]}{a}$. Let $\mathcal{V}$ be given by (4.2) and $\mathcal{F} = \mathcal{F}_1^m$ and $\mathcal{G} = \mathcal{G}_1^m$. If n, m and k are positive integers such that $n > k$ and $\mathcal{V} \equiv 0$, then $f^{nd_M} \equiv tM[f^n]$, where $t^m = 1$ and f assumes the form*

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a non-zero constant and $\lambda^{mQ_M} = 1$.

Proof. Let $\mathcal{V} \equiv 0$. Then we get

$$1 - \frac{1}{\mathcal{F}_1^m} \equiv \mathcal{A} - \frac{\mathcal{A}}{\mathcal{G}_1^m}, \tag{4.4}$$

where $\mathcal{A}$ is a non-zero constant. We now consider the following cases.

Case 1. Let $N(r,\infty;f) = S(r,f)$. If $\mathcal{A} \neq 1$, then from (4.4) we have

$$\overline{N}\left(r, \frac{1}{1-\mathcal{A}}; \mathcal{F}_1^m\right) = \overline{N}(r,\infty;\mathcal{G}_1^m) = S(r,f).$$

By the Second Fundamental Theorem and definitions of $\mathcal{F}_1, \mathcal{G}_1$, we have

$$\begin{aligned} & T(r,\mathcal{F}_1^m) \\ \leq\ & \overline{N}(r,\infty;\mathcal{F}_1^m) + \overline{N}(r,0;\mathcal{F}_1^m) + \overline{N}\left(r, \frac{1}{1-\mathcal{A}}; \mathcal{F}_1^m\right) + S(r,f). \end{aligned}$$

i.e.,

$$mnd_{_M} T(r,f) \leq \overline{N}(r,0;f) + S(r,f),$$

which is not possible.

Case 2. Let $N(r,\infty;f) \neq S(r,f)$. Then there exists a z_0 which is not a zero or pole of $a(z)$ such that $\dfrac{1}{f(z_0)} = 0$, so $\dfrac{1}{\mathcal{F}_1(z_0)} = \dfrac{1}{\mathcal{G}_1(z_0)} = 0$. Therefore, from (4.4) we get $\mathcal{A} = 1$.

Thus, by (4.4) and $\mathcal{A} = 1$, then $\mathcal{F}_1^m = \mathcal{G}_1^m$, i.e.,

$$f^{nd_M} \equiv tM[f^n], \tag{4.5}$$

where $t^m = 1$. Now if z_0 be a zero of f with multiplicity q, then z_0 is a zero of f^{nd_M} with multiplicity $nqd_{_M}$ and a zero of $M[f^n]$ with multiplicity $nqd_{_M} - Q_{_M}$. Therefore,

$$nqd_{_M} = nqd_{_M} - Q_{_M},$$

which is not possible. Thus it is obvious that 0 is a Picard exceptional value of f. Similarly we can get that ∞ is also a Picard exceptional value of f. Then from (4.5) we have

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a non-zero constant and $\lambda^{nQ_M} = 1$. □

Lemma 8. *Let $\mathcal{V}$ be given by* (4.2) *and $\mathcal{F}, \mathcal{G}, \mathcal{F}_1$ and $\mathcal{G}_1$ be given by Lemma 7 and n, m be positive integers. If $\mathcal{V} \not\equiv 0$, then*

$$(mnd_{_M} - 1)\overline{N}(r,\infty;f) \leq N(r,\infty;\mathcal{V}) + S(r,f).$$

Proof. From (4.2) and the definitions of $\mathcal{F}, \mathcal{G}$, we see that if z_0 is a pole of f with the multiplicity q such that $a(z_0) \neq 0$ and $a(z_0) \neq \infty$, then z_0 is a zero of $\frac{\mathcal{F}'}{\mathcal{F}-1} - \frac{\mathcal{F}'}{\mathcal{F}}$ with the multiplicity $mnqd_{_M} - 1$ and a zero of $\frac{\mathcal{G}'}{\mathcal{G}-1} - \frac{\mathcal{G}'}{\mathcal{G}}$ with the multiplicity $m(nqd_{_M} + Q_{_M}) - 1$. Therefore z_0 is zero of $\mathcal{V}$ with multiplicity

$$p \geq \min\left\{mnd_{_M} - 1, m(nd_{_M} + Q_{_M}) - 1\right\} = mnd_{_M} - 1.$$

Also note that $m(r, \mathcal{V}) = S(r, f)$. Therefore

$$\begin{aligned} & (mnd_{_M} - 1)\overline{N}(r, \infty; f) \\ \leq\ & N(r, 0; \mathcal{V}) + S(r, f) \\ \leq\ & T(r, \mathcal{V}) + S(r, f) \\ \leq\ & N(r, \infty; \mathcal{V}) + S(r, f). \end{aligned}$$

□

Lemma 9. *Let $\mathcal{U}$ be given by (4.3) and $\mathcal{F}, \mathcal{G}, \mathcal{F}_1$ and $\mathcal{G}_1$ be given by Lemma 7. If n, m are psotive integers such that $n > k$ and $\mathcal{U} \equiv 0$, then*

$$f^{nd_M} \equiv tM[f^n],$$

where $t^m = 1$ and f assumes the form

$$f(z) = ce^{\frac{\lambda}{n}}z,$$

where c is a non-zero constant and $\lambda^{mQ_M} = 1$.

Proof. Since $\mathcal{U} = 0$, we get

$$\mathcal{F} \equiv \mathcal{B}\mathcal{G} + 1 - \mathcal{B}, \tag{4.6}$$

where $\mathcal{B}$ is a non-zero constant. By the definitions of $\mathcal{F}, \mathcal{G}, \mathcal{F}_1$ and $\mathcal{G}_1$, we get $N(r, \infty; f) = S(r, f)$. We discuss the following cases.

Case 1. Let $\mathcal{B} = 1$. Then we see that $\mathcal{F} \equiv \mathcal{G}$, i.e., $\mathcal{F}_1^m \equiv \mathcal{G}_1^m$. Then we have

$$f^{nd_M} \equiv tM[f^n],$$

where $t^m = 1$. Then f assumes the form

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a non-zero constant with $\lambda^{mQ_M} = 1$.

Case 2. Let $\mathcal{B} \neq 1$. If $N(r, 0; f) \neq S(r, f)$, then there exists a point z_0 for which $f(z_0) = 0$ but $a(z_0) \neq 0$. Since $n > k$, then it is clear that $F(z_0) = 0 = G(z_0)$. Now from (4.6), we get $B = 1$, which is clearly absurd.

Again if $N(r,0;f)=S(r,f)$, then from (4.6) and using Lemma 3, we get

$$\begin{aligned}&\overline{N}(r,1-\mathcal{B};\mathcal{F})=\overline{N}(r,0;\mathcal{G})\\ \leq\ & nd_{_M}N(r,0;f)+Q_{_M}\overline{N}(r,\infty;f)\\ \leq\ & S(r,f).\end{aligned}$$

Now using Second Fundamental Theorem and $N(r,0;f)=N(r,\infty;f)=S(r,f)$, we have

$$\begin{aligned}&mnd_{_M}T(r,f)\leq T(r,\mathcal{F})+S(r,f)\\ \leq\ & \overline{N}(r,\infty;\mathcal{F})+\overline{N}(r,0;\mathcal{F})+\overline{N}(r,1-\mathcal{B};\mathcal{F})+S(r,f)\\ \leq\ & \overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,0;\mathcal{G})+S(r,f)\\ \leq\ & S(r,f),\end{aligned}$$

which is not possible. □

Lemma 10. *Let* $\mathcal{U}$ *be given by* (4.3) *and* $\mathcal{F},\mathcal{G},\mathcal{F}_1$ *and* $\mathcal{G}_1$ *be given by Lemma 7. If* n,m *and* k *are positive integers such that* $n>k$ *and* $\mathcal{U}\not\equiv 0$, *then*

$$[(nd_{_M}-Q_{_M})m-1]\overline{N}(r,0;f)\leq N(r,\infty;\mathcal{U})+S(r,f).$$

Proof. Let z_0 is a zero of f with multiplicity $q(\geq 1)$ such that $a(z_0)\neq 0,\infty$. Then z_0 is a zero of $\dfrac{\mathcal{F}'}{\mathcal{F}-1}$ with the multiplicity $nmqd_{_M}-1$ and z_0 is also a zero of $\dfrac{\mathcal{G}'}{\mathcal{G}-1}$ of multiplicity $(nqd_{_M}-Q_{_M})m-1$. Therefore z_0 is a zero of $\mathcal{U}$ of multiplicity at least $(nqd_{_M}-Q_{_M})m-1$. Since $m(r,\mathcal{U})=S(r,f)$, we have

$$\begin{aligned}&[(nd_{_M}-Q_{_M})m-1]\overline{N}(r,0;f)\leq N(r,0;\mathcal{U})+S(r,f)\\ \leq\ & T(r,\mathcal{U})+S(r,f)\\ \leq\ & N(r,\infty;\mathcal{U})+S(r,f).\end{aligned}$$

□

Lemma 11. *Let* $\mathcal{F}$, $\mathcal{G}$, $\mathcal{F}_1$, $\mathcal{G}_1$ *be as in Lemma 7 and* $\mathcal{V}$ *as in (4.2). Now if* $n>k$ *and* $E_p(1,\mathcal{F})=E_p(1;\mathcal{G})$ *and* $\mathcal{V}\not\equiv 0$, *then the following hold:*

1. *When* $p\geq 2$, *then*

$$\left\{mnd_{_M}-1-Q_{_M}-\frac{1}{p}\right\}\overline{N}(r,\infty;f)\leq\left\{(k+1)d_{_M}+\frac{1}{p}\right\}\overline{N}(r,0;f)+S(r,f).\tag{4.7}$$

2. *When* $p=0$, *then*

$$\left\{mnd_{_M}-1-2(Q_{_M}+1)\right\}\overline{N}(r,\infty;f)\leq\left\{2(k+1)d_{_M}+1\right\}\overline{N}(r,0;f)+S(r,f).\tag{4.8}$$

Proof. Let $p \geq 2$ and $\mathcal{V} = \dfrac{\mathcal{F}'}{\mathcal{F}(\mathcal{F}-1)} - \dfrac{\mathcal{G}'}{\mathcal{G}(\mathcal{G}-1)}$. Now since $E_p(1;\mathcal{F}) = E_p(1;\mathcal{G})$, so we have

$$N(r,\infty;\mathcal{V}) \leq \overline{N}(r,0;\mathcal{G}) + \overline{N}_{(p+1}(r,1;\mathcal{F}) + S(r,f),$$

where

$$\begin{aligned} \overline{N}_{(p+1}(r,1;\mathcal{F}) &\leq \frac{1}{p} N\left(r, \frac{\mathcal{F}}{\mathcal{F}'}\right) \\ &\leq \frac{1}{p} N\left(r, \frac{\mathcal{F}'}{\mathcal{F}}\right) + S(r,f) \\ &\leq \frac{1}{p}\overline{N}(r,\infty;\mathcal{F}) + \frac{1}{p}\overline{N}(r,0;\mathcal{F}) + S(r,f) \\ &\leq \frac{1}{p}\overline{N}(r,\infty;f) + \frac{1}{p}\overline{N}(r,0;f) + S(r,f). \end{aligned}$$

Now by applying Lemma 8 and Lemma 4 we get

$$\begin{aligned} (mnd_{M}-1)\overline{N}(r,\infty;f) &\leq \frac{1}{p}\overline{N}(r,0;f) + \frac{1}{p}\overline{N}(r,\infty;f) + \overline{N}(r,0;\mathcal{G}) + S(r,f) \\ &\leq \frac{1}{p}\overline{N}(r,0;f) + \frac{1}{p}\overline{N}(r,\infty;f) + d_{M} N_{k+1} N(r,0;f^n) + Q_{M}\overline{N}(r,\infty;f) + S(r,f), \end{aligned}$$

i.e.,

$$\left\{mnd_{M} - 1 - Q_{M} - \frac{1}{p}\right\}\overline{N}(r,\infty;f) \leq \left\{(k+1)d_{M} + \frac{1}{p}\right\}\overline{N}(r,0;f) + S(r,f).$$

Let $p = 0$, then

$$N(r,\infty;\mathcal{V}) \leq \overline{N}(r,0;\mathcal{G}) + \overline{N}_L(r,1;\mathcal{F}) + \overline{N}_L(r,1;\mathcal{G}) + S(r,f),$$

where

$$\begin{aligned} \overline{N}_L(r,1;\mathcal{F}) &\leq N\left(r,\frac{\mathcal{F}}{\mathcal{F}'}\right) \leq N\left(r,\frac{\mathcal{F}'}{\mathcal{F}}\right) + S(r,f) \\ &\leq \overline{N}(r,\infty;\mathcal{F}) + \overline{N}(r,0;\mathcal{F}) + S(r,f) \\ &\leq \overline{N}(r,\infty;f) + \overline{N}(r,0;f) + S(r,f). \end{aligned}$$

Similarly, applying Lemma 4 and proceeding as above, we get

$$\begin{aligned} \overline{N}_L(r,1;\mathcal{G}) &\leq \overline{N}(r,\infty;\mathcal{G}) + \overline{N}(r,0;\mathcal{G}) + S(r,f) \\ &\leq (Q_{M}+1)\overline{N}(r,\infty;f) + (k+1)d_{M}\overline{N}(r,0;f) + S(r,f). \end{aligned}$$

Now by Lemma 8 and Lemma 4, we get

$$(mnd_{M}-1)\overline{N}(r,\infty;f) \leq \{2(k+1)d_{M}+1\}\overline{N}(r,0;f) + 2(Q_{M}+1)\overline{N}(r,\infty;f) + S(r,f).$$

i.e.,

$$\left\{mnd_{M}-1-2(Q_{M}+1)\right\}\overline{N}(r,\infty;f)\leq\left\{2(k+1)d_{M}+1\right\}\overline{N}(r,0;f)+S(r,f).$$

□

Lemma 12. *Let $\mathcal{F}$, $\mathcal{G}$, $\mathcal{F}_1$, $\mathcal{G}_1$ be as in Lemma 7 and $\mathcal{U}$ as in (4.3). Now if $n>k$ and $E_p(1,\mathcal{F})=E_p(1;\mathcal{G})$ and $\mathcal{U}\not\equiv 0$, then the following holds:*

1. *When $p\geq 2$, then*
$$\left\{(nd_{M}-Q_{M})m-1-\frac{1}{p}\right\}\overline{N}(r,0;f)\leq\left\{1+\frac{1}{p}\right\}\overline{N}(r,\infty;f)+S(r,f). \tag{4.9}$$
2. *When $p=0$, then*
$$\left\{(nd_{M}-Q_{M})m-(k+1)d_{M}-2\right\}\overline{N}(r,0;f)\leq\left\{Q_{M}+3\right\}\overline{N}(r,\infty;f)$$
$$+S(r,f). \tag{4.10}$$

Proof. Let $p\geq 2$, then we have

$$\begin{aligned} & N(r,\infty;\mathcal{U})\leq\overline{N}(r,\infty;\mathcal{F})+\overline{N}_{(p+1}(r,1;\mathcal{F})+S(r,f)\\ \leq\; & \overline{N}(r,\infty;f)+\left\{\frac{1}{p}\overline{N}(r,0;f)+\frac{1}{p}\overline{N}(r,\infty;f)\right\}+S(r,f)\\ \leq\; & \frac{1}{p}\overline{N}(r,0;f)+\left(1+\frac{1}{p}\right)\overline{N}(r,\infty;f)+S(r,f).\end{aligned}$$

Now by applying Lemma 10 we get

$$\left\{(nd_{M}-Q_{M})m-1\right\}\overline{N}(r,0;f)\leq\frac{1}{p}\overline{N}(r,0;f)+\left(1+\frac{1}{p}\right)\overline{N}(r,\infty;f)+S(r,f),$$

i.e.,

$$\left\{(nd_{M}-Q_{M})m-1-\frac{1}{p}\right\}\overline{N}(r,0;f)\leq\left(1+\frac{1}{p}\right)\overline{N}(r,\infty;f)+S(r,f).$$

Let $p=0$, by applying Lemma 10 and Lemma 4 and proceeding in the same way as done in the proof of Lemma 11, we get

$$\begin{aligned} & N(r,\infty;\mathcal{U})\leq\overline{N}(r,\infty;\mathcal{F})+\overline{N}_{L}(r,1;\mathcal{F})+\overline{N}_{L}(r,1;\mathcal{G})+S(r,f)\\ \leq\; & \overline{N}(r,\infty;f)+\left\{\overline{N}(r,0;f)+\overline{N}(r,\infty;f)\right\}+\big\{(Q_{M}+1)\overline{N}(r,\infty;f)\\ & +(k+1)d_{M}\overline{N}(r,0;f)\big\}+S(r,f),\end{aligned}$$

i.e.,

$$\left\{(nd_{M}-Q_{M})m-(k+1)d_{M}-2\right\}\overline{N}(r,0;f)\leq\left\{Q_{M}+3\right\}\overline{N}(r,\infty;f)+S(r,f).$$

□

Lemma 13. *Let $\mathcal{F}$ and $\mathcal{G}$ be two non-constant meromorphic functions such that $E_p(1,\mathcal{F})=E_p(1,\mathcal{G})$ and $\mathcal{H}\not\equiv 0$ and $p=0$, then*

$$\begin{aligned}&T(r,\mathcal{F})+T(r,\mathcal{G})\\ \leq\ & 2N_2(r,0;\mathcal{F})+2N_2(r,0;\mathcal{G})+6\overline{N}(r,\infty;\mathcal{F})+3\overline{N}_L(r,1;\mathcal{F})+3\overline{N}_L(r,1;\mathcal{G})+S(r,\mathcal{F}).\end{aligned}$$

Proof. Noting that $S(r,\mathcal{F})=S(r,\mathcal{G})$, the lemma can be proved by using Lemma 2.1, Lemma 2.2 and Lemma 2.3 of [4]. ☐

Lemma 14. *Let $\mathcal{F}$ and $\mathcal{G}$ be two non-constant meromorphic functions such that $E_p(1,\mathcal{F})=E_p(1,\mathcal{G})$ and $\mathcal{H}\not\equiv 0$ and $p\geq 2$, then*

$$T(r,\mathcal{F})+T(r,\mathcal{G})\leq 2N_2(r,0;\mathcal{F})+2N_2(r,0;\mathcal{G})+6\overline{N}(r,\infty;\mathcal{F})+S(r,\mathcal{F}).$$

Proof. Since $\mathcal{F}$ and $\mathcal{G}$ share $(1,p)$ where $p\geq 2$, so it is clear that $\mathcal{F}$ and $\mathcal{G}$ share $(1,2)$. Then the lemma can be obtained from Lemma 13 of [2]. ☐

Lemma 15. *Let $\mathcal{H}$ be given by (4.1) and $\mathcal{F},\mathcal{G},\mathcal{F}_1$ and $\mathcal{G}_1$ be given by Lemma 7. If n,m and k are positive integers such that $n>k$ and*

$$\overline{N}(r,\infty;f)=N(r,0;f)=S(r,f)$$

and $\mathcal{H}\equiv 0$, then

$$f^{nd_M}\equiv tM[f^n],$$

where $t^m=1$ and f assumes the form

$$f(z)=ce^{\frac{\lambda}{n}z},$$

where c is a non-zero constant and $\lambda^{nd_M}=1$.

Proof. Since $\mathcal{H}\equiv 0$, by integration we obtain

$$\frac{1}{\mathcal{F}-1}\equiv\frac{\mathcal{C}}{\mathcal{G}-1}+\mathcal{D}, \tag{4.11}$$

where $\mathcal{C}(\neq 0)$ and $\mathcal{D}$ are constants. Now from (4.11) we have

$$\mathcal{G}\equiv\frac{(\mathcal{D}-\mathcal{C})\mathcal{F}+(\mathcal{C}-\mathcal{D}-1)}{\mathcal{D}\mathcal{F}-(\mathcal{D}+1)},$$

i.e.,

$$\mathcal{G}_1^m\equiv\frac{(\mathcal{D}-\mathcal{C})\mathcal{F}_1^m+(\mathcal{C}-\mathcal{D}-1)}{\mathcal{D}\mathcal{F}_1^m-(\mathcal{D}+1)}. \tag{4.12}$$

Now we discuss the following cases.

Case 1. Let $\mathcal{D}\neq 0,-1$. Therefore from (4.12) we have

$$\overline{N}\left(r,\frac{\mathcal{D}+1}{\mathcal{D}};\mathcal{F}_1^m\right)=\overline{N}(r,\infty;\mathcal{G}_1^m).$$

By applying the Second Fundamental Theorem with $S(r,\mathcal{F}) = S(r,f)$, we get

$$\begin{aligned}
& mnd_{_M}T(r,f) = T(r,\mathcal{F}_1^m) + S(r,f)\\
\leq\ & \overline{N}(r,\infty;\mathcal{F}_1^m) + \overline{N}(r,0;\mathcal{F}_1^m) + \overline{N}\left(r,\frac{\mathcal{D}+1}{\mathcal{D}};\mathcal{F}_1^m\right) + S(r,f)\\
\leq\ & \overline{N}(r,\infty;f) + \overline{N}(r,0;f) + \overline{N}(r,\infty;\mathcal{G}_1^m) + S(r,f)\\
\leq\ & S(r,f),
\end{aligned}$$

which is not possible.

Case 2. Suppose $\mathcal{D} = 0$. Then from (4.12), we have

$$\overline{N}\left(r,\frac{\mathcal{C}-1}{\mathcal{C}};\mathcal{F}_1^m\right) = \overline{N}(r,0;\mathcal{G}_1^m).$$

Subcase 2.1. Let $\mathcal{C} \neq 1$. Now by the Second Fundamental Theorem and using Lemma 3, we get

$$\begin{aligned}
& mnd_{_M}T(r,f) = T(r,\mathcal{F}_1^m) + S(r,f)\\
\leq\ & \overline{N}(r,\infty;\mathcal{F}_1^m) + \overline{N}(r,0;\mathcal{F}_1^m) + \overline{N}\left(r,\frac{\mathcal{C}-1}{\mathcal{C}};\mathcal{F}_1^m\right) + S(r,f)\\
\leq\ & \overline{N}(r,\infty;f) + \overline{N}(r,0;f) + \overline{N}(r,0;\mathcal{G}_1^m) + S(r,f)\\
\leq\ & (nd_{_M}+1)N(r,0;f) + (Q_{_M}+1)\overline{N}(r,\infty;f) + S(r,f)\\
\leq\ & S(r,f),
\end{aligned}$$

which is not possible.

Subcase 2.2. Let $\mathcal{C} = 1$. Then we have $\mathcal{F}_1^m \equiv \mathcal{G}_1^m$, i.e.,

$$f^{nd_M} \equiv tM[f^n].$$

Then f assumes the form

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a non-zero constant and λ^{mQ_M}.

Case 3. Let $\mathcal{D} = -1$, then from (4.12), we get

$$\mathcal{G}_1^m \equiv \frac{(\mathcal{C}+1)\mathcal{F}_1^m - \mathcal{C}}{\mathcal{F}_1^m}.$$

Now proceeding exactly the same way as in Case 2, we get $\mathcal{F}_1^m\mathcal{G}_1^m \equiv 1$, i.e., $f^{nd_M}M[f^n] \equiv ta^2$, where $t^m = 1$. Again since $\overline{N}(r,\infty;f) = S(r,f) = N(r,0;f)$, so

$$\begin{aligned}
& 2T\left(r,\frac{f^{nd_M}}{a}\right) = T\left(r,\frac{ta^2}{f^{2nd_M}}\right) + O(1)\\
\leq\ & T\left(r,\frac{M[f^n]}{f^{nd_M}}\right) + O(1)
\end{aligned}$$

$$
\begin{aligned}
&\leq m\left(r, \frac{M[f^n]}{f^{nd_M}}\right) + N\left(r, \frac{M[f^n]}{f^{nd_M}}\right) + O(1) \\
&\leq N\left(r, \infty; M[f^n]\right) + N\left(r, 0; f^{nd_M}\right) + O(1) \\
&\leq (nd_M + Q_M)\overline{N}(r, \infty; f) + nd_M N(r, 0; f) + O(1) \\
&\leq S(r, f),
\end{aligned}
$$

which is not possible. □

5 Proofs of the Theorems

Proof of Theorem 1.9. Let $\mathcal{F}_1 = \dfrac{f^{nd_M}}{a}$ and $\mathcal{G}_1 = \dfrac{M[f^n]}{a}$ and $\mathcal{F} = \mathcal{F}_1^m$, $\mathcal{G} = \mathcal{G}_1^m$, where f is a non-constant meromorphic function. Now we discuss the following cases.

Case 1. If $\mathcal{UV} \equiv 0$, then by using Lemma 7 and Lemma 9, we get the conclusions of the Theorem 1.9.

Case 2. Let $\mathcal{UV} \not\equiv 0$, then from the assumption of Theorem 1.9, we see that $E_p(1, \mathcal{F}) = E_p(1, \mathcal{G})$.

Subcase 2.1. When $p \geq 2$, then by using Lemma 11 and Lemma 12, we get

$$
\begin{aligned}
&\left\{mnd_M - 1 - Q_M - \frac{1}{p}\right\}\left\{(nd_M - Q_M)m - 1 - \frac{1}{p}\right\}\overline{N}(r, \infty; f) \\
\leq\; &\left\{(k+1)d_M + \frac{1}{p}\right\}\left\{1 + \frac{1}{p}\right\}\overline{N}(r, \infty; f) + S(r, f)
\end{aligned} \tag{5.1}
$$

and

$$
\begin{aligned}
&\left\{mnd_M - 1 - Q_M - \frac{1}{p}\right\}\left\{(nd_M - Q_M)m - 1 - \frac{1}{p}\right\}\overline{N}(r, 0; f) \\
\leq\; &\left\{(k+1)d_M + \frac{1}{p}\right\}\left\{1 + \frac{1}{p}\right\}\overline{N}(r, 0; f) + S(r, f).
\end{aligned} \tag{5.2}
$$

Now from the equations (5.1) and (5.2), we get

$$
\left\{(mnd_M - \gamma_1^p)(mnd_M - \gamma_m^p) - C\right\}\overline{N}(r, \infty; f) \leq S(r, f) \tag{5.3}
$$

and

$$
\left\{(mnd_M - \gamma_1^p)(mnd_M - \gamma_m^p) - C\right\}\overline{N}(r, 0; f) \leq S(r, f), \tag{5.4}
$$

where $\gamma_m^p = mQ_M + 1 + \dfrac{1}{p}$ and $C = \left\{(k+1)d_M + \dfrac{1}{p}\right\}\left\{1 + \dfrac{1}{p}\right\}$.

Since

$$
\left\{mnd_M - \gamma_1^p\right\}\left\{mnd_M - \gamma_m^p\right\} - C
$$

$$
\begin{aligned}
&= m^2d_M^2n^2 - md_M\Big\{\gamma_1^p + \gamma_m^p\Big\}n + \Big\{\gamma_1^p\gamma_m^p - C\Big\}\\
&= m^2d_M^2\left\{n - \frac{\gamma_m^p + \gamma_1^p + \sqrt{(\gamma_m^p - \gamma_1^p)^2 + 4C}}{2md_M}\right\}\left\{n - \frac{\gamma_m^p + \gamma_1^p - \sqrt{(\gamma_m^p - \gamma_1^p)^2 + 4C}}{2md_M}\right\},
\end{aligned}
$$

in view of the assumptions of Theorem 1.9, we get a contradiction from (5.3) and (5.4).

Thus we obtained from above

$$\overline{N}(r,0;f) = S(r,f) = \overline{N}(r,\infty;f). \tag{5.5}$$

We now consider the following two cases:
Case 2.1.1. Let $\mathcal{H} \not\equiv 0$. Using Lemma 13 and Lemma 14 and (5.5), we get $T(r,f) = S(r,f)$, which is a contradiction.
Case 2.1.2. Let $\mathcal{H} \equiv 0$. Then from Lemma 15, we get the conclusion of Theorem 1.9. **Subcase 2.2.** When $p = 0$, using Lemma 11 and Lemma 12, we get

$$
\begin{aligned}
&\Big\{mnd_M - 1 - 2(Q_M + 1)\Big\}\Big\{(nd_M - Q_M)m - (k+1)d_M - 2\Big\}\overline{N}(r,\infty;f) \qquad (5.6)\\
&\leq \Big\{2(k+1)d_M + 1\Big\}\Big\{Q_M + 3\Big\}\overline{N}(r,\infty;f) + S(r,f)
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big\{mnd_M - 1 - 2(Q_M + 1)\Big\}\Big\{(nd_M - Q_M)m - (k+1)d_M - 2\Big\}\overline{N}(r,0;f) \qquad (5.7)\\
&\leq \Big\{2(k+1)d_M + 1\Big\}\Big\{Q_M + 3\Big\}\overline{N}(r,0;f) + S(r,f).
\end{aligned}
$$

Now using equations (5.6) and (5.7) and proceeding the same way as done in Subcase 2.1, the rest of the proof can be carried out. So we omit the detail. □

Proof of Theorem 1.10. Since f is an entire function, we have $N(r,\infty;f) = S(r,f)$. Now if $\mathcal{U} \equiv 0$, then using Lemma 9, we get the conclusion of Theorem 1.10.

If $\mathcal{U} \not\equiv 0$, then using Lemma 10 for $p \geq 2$ we get from (5.2) that

$$(mnd_M - \gamma_1^p)(mnd_M - \gamma_m^p)\overline{N}(r,0;f) \leq S(r,f).$$

Since $n > \dfrac{pmQ_M + p + 1}{pmd_M}$, we get a contradiction.

Again when $p = 0$, using Lemma 10 we get from (5.7)

$$\Big\{mnd_M - [2Q_M + 3]\Big\}\Big\{mnd_M - [mQ_M + (k+1)d_M + 2]\Big\}\overline{N}(r,0;f) \leq S(r,f),$$

which is a contradiction since $n > \dfrac{mQ_M + (k+1)d_M + 2}{md_M}$.

Therefore $\overline{N}(r,0;f) = S(r,f)$. Now the rest of the proof follows Case 1 and Case 2 of the proof of Theorem 1.9. □

6 Some Open Questions

Question 6.1. *Can we replace f^n by a general linear expression $P(f)$ in anyway in Theorem 1.9 and Theorem 1.10 to get the same specific form the function?*

Question 6.2. *Can we replace the differential monomial $M[f^n]$ by a differential polynomial $\mathcal{P}[f^n]$ in anyway in Theorem 1.9 and Theorem 1.10 to get the same specific form the function?*

Question 6.3. *Can the lower bound of n be further reduced in Theorem 1.9 and Theorem 1.10 to get the same conclusions?*

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