Uniqueness of certain power of a meromorphic function sharing a set with its differential monomial

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Abstract. In this paper we are mainly devoted to find out the specific form of a meromorphic function when it shares a set of small functions with its differential monomial counterpart. Our results will improve and extend some of the recent results due to Zhang-Yang [J. L. Zhang and L. Z. Yang, A power of a meromorphic function sharing a small function with its derivative, Ann. Acad. Sci. Fenn. Math. 34(2009), 249–260] and Xu-Yi-Yang [H. Y. Xu, C. F. Yi and H. Wang, On a conjecture of R. Bruck concerning meromorphic function sharing small functions, Revista de Matematica Teoria y Aplicaciones, 23(1)(2016), 291-308]. We provide some examples to show that certain conditions used in the paper cannot be removed.

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1 Introduction, Definitions and Results

Let \( f \) be a non-constant meromorphic function in the whole complex plane \( \mathbb{C} \). We shall use the following standard notations of the value distribution theory:

\[
T(r, f), m(r, f), N(r, f), \overline{N}(r, f), \ldots
\]

([11,19,23]). We denote by \( S(r, f) \) any quantity satisfying

\[
S(r, f) = o(T(r, f)),
\]

as \( r \to +\infty \), possibly outside of a set of finite measure. A meromorphic function \( a \equiv a(z) \) is called a small function with respect to \( f \) if \( T(r, a) = S(r, f) \). Let \( S(f) \) be the set of meromorphic functions in the complex plane \( \mathbb{C} \) which are small functions with respect to \( f \).

Let \( f \) be a non-constant meromorphic function and \( a \in S(f) \cup \{\infty\} \) and \( S \subset S(f) \cup \{\infty\} \). Define

\[
E(S, f) = \bigcup_{a \in S} \{z : f(z) - a = 0, \text{Counting Multiplicity}\},
\]

\[
\overline{E}(S, f) = \bigcup_{a \in S} \{z : f(z) - a = 0, \text{Ignoring Multiplicity}\},
\]

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If \( E(S, f) = E(S, g) \), we say that \( f \) and \( g \) share the set \( S \) CM; if \( \overline{E}(S, f) = \overline{E}(S, g) \), we say that \( f \) and \( g \) share the set \( S \) IM. Especially, when \( S = \{a\} \), we say that \( f \) and \( g \) share the value \( a \) CM if \( E(S, f) = E(S, g) \); and we say that \( f \) and \( g \) share the value \( a \) IM if \( \overline{E}(S, f) = \overline{E}(S, g) \) [11].

Nowadays the problems relative to a meromorphic function \( f \) and its derivative \( f^{(k)} \) sharing some value or small functions have been studied rigorously by many researchers. Readers are requested to make a glance at [9, 15, 24, 27].

In 1996, Brück [7] proposed the following famous conjecture.

**Conjecture 1.1.** Let \( f \) be a non-constant entire function. Suppose that \( \rho_1(f) \) is not a positive integer or infinite. If \( f \) and \( f' \) share one finite value \( a \) CM, then

\[
\frac{f' - a}{f - a} = c,
\]

for some non-zero constant \( c \), where \( \rho_1(f) \) is the first iterated order of \( f \) which is defined by

\[
\rho_1(f) = \limsup_{r \to \infty} \frac{\log \log T(r, f)}{\log r}.
\]

In 1996, Brück [7] proved that the conjecture is true when \( a = 0 \) or \( N(r, 1/f') = S(r, f) \) and later many researchers like Gundersen and Yang [10] proved that the conjecture is true when \( f \) is of finite order [10]. A few years later, Chen and Shon [8] proved that the conjecture is true for entire function of first order \( \rho_1(f) < \frac{1}{2} \). However, the conjecture fails in general for meromorphic functions, shown by Gundersen and Yang [10], while it remains true in the case that \( N(r, 1/f') = S(r, f) \), shown by Al-Kahaladi [1].

In 2008, Yang and Zhang [20] obtained the following results.

**Theorem 1.1 (see [20]).** Let \( f \) be a non-constant entire function, \( n \geq 7 \) be an integer. Denote \( F = f^n \). If \( F \) and \( F' \) share \( 1 \) CM, then \( F \equiv F' \), and \( f \) assumes the form

\[
f(z) = ce^{\frac{z}{n}},
\]

where \( c \) is a non-zero constant.

**Theorem 1.2 (see [20]).** Let \( f \) be a non-constant meromorphic function and \( n \geq 12 \) be an integer. Denote \( F = f^n \). If \( F \) and \( F' \) share \( 1 \) CM, then \( F \equiv F' \), and \( f \) assumes the form

\[
f(z) = ce^{\frac{z}{n}},
\]

where \( c \) is a non-zero constant.

In 2009, Zhang and Yang [25] improved Theorem 1.1 and Theorem 1.2 to a large extent and obtained the following results.
**Theorem 1.3** (see [25]). Let \( f \) be a non-constant entire function, \( n, k \) be positive integers and \( a(z) \) be a small function of \( f \) such that \( a(z) \neq 0, \infty \). If \( f^n - a \) and \( (f^n)^{(k)} - a \) share the value 0 CM and \( n \geq k + 2 \), then \( f^n \equiv (f^n)^{(k)} \) and \( f \) assumes the form

\[
f(z) = ce^{\lambda z},
\]

where \( c \) is a non-zero constant and \( \lambda^k = 1 \).

**Theorem 1.4** (see [25]). Let \( f \) be a non-constant meromorphic function, \( n, k \) be positive integers and \( a(z) \) be a small function of \( f \) such that \( a(z) \neq 0, \infty \). If \( f^n - a \) and \( (f^n)^{(k)} - a \) share the value 0 CM and \( n > k + 1 + \sqrt{k + 1} \), then \( f^n \equiv (f^n)^{(k)} \) and \( f \) assumes the form

\[
f(z) = ce^{\lambda z},
\]

where \( c \) is a non-zero constant and \( \lambda^k = 1 \).

**Theorem 1.5** (see [25]). Let \( f \) be a non-constant entire function, \( n, k \) be positive integers and \( a(z) \) be a small meromorphic function of \( f \) such that \( a(z) \neq 0, \infty \). If \( f^n - a \) and \( (f^n)^{(k)} - a \) share the value 0 IM and \( n > 2k + 3 \), then \( f^n \equiv (f^n)^{(k)} \) and \( f \) assumes the form

\[
f(z) = ce^{\lambda z},
\]

where \( c \) is a non-zero constant with \( \lambda^k = 1 \).

**Theorem 1.6** (see [25]). Let \( f \) be a non-constant meromorphic function, \( n, k \) be positive integers and \( a(z) \) be a small meromorphic function of \( f \) such that \( a(z) \neq 0, \infty \). If \( f^n - a \) and \( (f^n)^{(k)} - a \) share the value 0 IM and

\[
n > 2k + 3 + \sqrt{(k + 3)(2k + 3)},
\]

then \( f^n \equiv (f^n)^{(k)} \) and \( f \) assumes the form

\[
f(z) = ce^{\lambda z},
\]

where \( c \) is a non-zero constant with \( \lambda^k = 1 \).

Though the standard definitions and notations of the value distribution theory are available in [3, 22], we explain the following definitions and notations which are used in the paper.

**Definition 1.1** (see [3, 22]). When \( f \) and \( g \) share 1 IM, we denote by \( N_L(r, 1; f) \) the counting function of the 1-points of \( g \). Similarly, we have \( N_L(r, 1; g) \). Let \( z_0 \) be a zero of \( f - 1 \) of multiplicity \( p \) and a zero of \( g - 1 \) of multiplicity \( q \), we also denote by \( N_{11}(r, 1; f) \) the counting function of those 1-points of \( f \) where \( p = q = 1 \); \( N_E^2(r, 1; f) \) denotes the counting function of those 1-points of \( f \) where \( p = q \geq 2 \), each point in these counting functions is counted only once. In the same way, one can define \( N_{11}(r, 1; g), N_E^2(r, 1; g) \).
Definition 1.2 (see [5]). For $a \in \mathbb{C} \cup \{\infty\}$ and $p$ a positive integer, let $f$ be a non-constant meromorphic function, we denote by $N(r, a; f | = 1)$ the counting function of simple $a$-points of $f$, denote by $N(r, a; f | \leq p)$ ($N(r, a; f | \geq p)$) the counting functions of those $a$-points of $f$ whose multiplicities are not greater (less) than $p$ where each $a$-point is counted according to its multiplicities. $\overline{N}(r, a; f | \geq p)$ are defined similarly, where in counting the $a$-points of $f$ we ignore the multiplicities.

Definition 1.3 (see [5]). For $a \in \mathbb{C} \cup \{\infty\}$ and a positive integer $p$ we denote by

$$N_p(r, a; f) = N(r, a; f | \leq 1) + N(r, a; f | \geq 2) + \ldots + N(r, a; f | \geq p).$$

Clearly $N_1(r, a; f) = N(r, a; f)$.

Next we recall the following definition of weighted sharing of values which generally measures how closed a shared value is to being sharing IM or CM, as follows.

Definition 1.4 (see [13, 14]). Let $p$ be a non-negative integer or infinity. For $c \in \mathbb{C} \cup \{\infty\}$, we denote by $E_f(a, p)$ the set of all $a$-points of $f$ where an $a$-point of multiplicity $m$ is counted $m$ times if $m \leq p$ and $p + 1$ times if $m > p$. If $E_f(a, p) = E_g(a, p)$, we say that $f, g$ share the value $a$ with weight $p$.

We write $f, g$ share $(a, p)$ to mean that $f, g$ share the value $a$ with weight $p$. Clearly if $f, g$ share $(a, p)$, then $f, g$ share $(a, q)$ for all integer $q$ ($0 \leq q < p$). Also, we note that $f, g$ share a value $a$ IM or CM if and only if $f, g$ share $(a, 0)$ and $(a, \infty)$ respectively.

Let $S$ be a subset of $S(f) \cup \{\infty\}$, we can get the definition of $E_f(S, p)$ as

$$E_f(S, p) = \bigcup_{a \in S} E_f(a, p).$$

Very recently in [18], for further investigations, Xu, Yi asked the following questions:

Question 1.1 (see [18]). Can the nature of sharing 1 or $a(z)$ CM be further relaxed in Theorem 1.1 and Theorem 1.3?

Question 1.2 (see [18]). What will happen when 1 or $a(z)$ are replaced by the set

$$S_m = \{a(z), a(z)\zeta, a(z)\zeta^2, \ldots, a(z)\zeta^{m-1}\}$$

of small functions in Theorems 1.1 – 1.4, where $\zeta = \cos \frac{2\pi}{m} + i \sin \frac{2\pi}{m}$ and $m$ is a positive integer?

To answer their question Xu, Yi and Wang [18] obtained the following two results which in turn improve Theorem 1.3 and Theorem 1.4.
**Theorem 1.7** (see [18]). Let \( f \) be a non-constant entire function, \( n, k, p, m \) be positive integers and \( a(z) \) be a small function of \( f \) such that \( a(z) \neq 0, \infty \). If \( E_{f^n}(S_m, p) = E_{(f^n)(k)}(S_m, p) \) and

\[
 n > \max \left\{ k + 1, k + \frac{\eta}{pm} \right\},
\]

where \( \eta = k + p + 2 \), then \( f^n \equiv t(f^n)^{(k)} \) with \( t^m = 1 \) and \( f \) assumes the form

\[
f(z) = ce^\lambda z,
\]

where \( c \) is a non-zero constant and \( \lambda^{km} = 1 \).

**Theorem 1.8** (see [18]). Let \( f \) be a non-constant meromorphic function, \( n, k, p, m \) be positive integers and \( a(z) \) be a small function of \( f \) such that \( a(z) \neq 0, \infty \). If \( E_{f^n}(S_m, p) = E_{(f^n)(k)}(S_m, p) \) and

\[
 n > \max \left\{ k + 1, \frac{p(m+1)k+2\eta}{2pm} + \frac{\sqrt{4\eta(pk + (m-1)^2p^2k^2)}}{2pm} \right\},
\]

where \( \eta = k + p + 2 \), then \( f^n \equiv t(f^n)^{(k)} \) with \( t^m = 1 \) and \( f \) assumes the form

\[
f(z) = ce^\lambda z,
\]

where \( c \) is a non-zero constant and \( \lambda^{km} = 1 \).

We observe from the above discussions that the research have gradually been shifted towards finding the relation between the power of a meromorphic function and its certain derivative. Since derivative’s natural extension is a differential monomial it will be quite natural to expect the extension and improvement of Theorems 1.1 – 1.8 up to a relation between a power of a meromorphic function and a general differential monomial sharing set of small functions.

Next we present the following well known definition.

**Definition 1.5** (see [5]). Let \( n_{ij}, n_{ij}, \ldots, n_{kj} \) be nonnegative integers and \( g = f^n \).

The expression \( M_j[g] = (g)^{n_{ij}}(g^{(1)})^{n_{ij}} \ldots (g^{(k)})^{n_{kj}} \) is called a differential monomial generated by \( g \) of degree \( d_{M_j} = d(M_j) = \sum_{i=0}^{k} n_{ij} \) and weight \( \Gamma_{M_j} = \sum_{i=0}^{k} (i+1)n_{ij} \).

The sum \( P[g] = \sum_{j=1}^{t} b_j M_j[g] \) is called a differential polynomial generated by \( g \) of degree \( d(P) = \max\{d(M_j) : 1 \leq j \leq t\} \) and weight \( \Gamma_P = \max\{\Gamma_{M_j} : 1 \leq j \leq t\} \), where \( T(r, b_j) = S(r, g) \) for \( j = 1, 2, \ldots, t \).

The numbers \( d(P) = \min\{d(M_j) : 1 \leq j \leq t\} \) and \( k \) (the highest order of the derivative of \( g \) in \( P[g] \)) are called respectively the lower degree and order of \( P[g] \).
\( \mathcal{P}[g] \) is said to be homogeneous if \( \mathcal{P}(\mathcal{P}) = d(\mathcal{P}) \). \( \mathcal{P}[g] \) is called a linear differential polynomial generated by \( g \) if \( \mathcal{P}(\mathcal{P}) = 1 \). Otherwise \( \mathcal{P}[g] \) is called a non-linear differential polynomial.

We denote by \( Q = \max \{ \Gamma_{M_j} - d(M_j) : 1 \leq j \leq t \} = \max \{ n_1 + 2n_2 + \ldots + kn_{k_j} : 1 \leq j \leq t \} \).

Also for the sake of convenience for a differential monomial \( \mathcal{M}[g] \) we denote by \( d_M = d(M) \) and \( Q_M = \Gamma_M - d_M \).

Next we pose the following questions which have great significance towards the further extension and improvement of all the above mentioned theorems.

**Question 1.3.** Is it possible to extend \((f^n)^{(k)}\) to a differential monomial \( \mathcal{M}[f^n] \) to get the same conclusion as in Theorem 1.7 and Theorem 1.8?

**Question 1.4.** Like Theorem 1.7 and Theorem 1.8, is it possible to find out the specific form of the function \( f \)?

**Question 1.5.** Can the lower bound of \( n \) be further reduced in Theorem 1.7 and Theorem 1.8?

Our main intention of writing this paper is to find out the possible affirmative answer of all the above questions such that Theorems 1.1 – 1.8 can be accommodated under a single theorem which extends and improves all of them. Henceforth we need the following notations throughout the paper for the sake of convenience.

Let

\begin{align*}
\alpha &= 2Q_M + 3, \quad \beta = mQ_M + (k+1)d_M + 2 \quad \text{and} \quad \gamma^p_m = mQ_M + 1 + \frac{1}{p},
\end{align*}

where \( p, m \) and \( k \) are three positive integers.

The following two theorems are the main results of this paper answering all the above mentioned questions affirmatively.

**Theorem 1.9.** Let \( f \) be a non-constant meromorphic function, \( n, k, p, m \) be positive integers and \( a(z) \) be a small function of \( f \) such that \( a(z) \not\equiv 0, \infty \). If \( E_{f^{nd_M}} (\mathcal{S}_m, p) = E_{\mathcal{M}[f^n]} (\mathcal{S}_m, p) \) and if

1. \( p \geq 2 \) and \( n > \frac{\gamma^p_m + \gamma^p_1 + \sqrt{(\gamma^p_m - \gamma^p_1)^2 + 4C}}{2md_M} \), or if
2. \( p = 0 \) and \( n > \frac{\alpha + \beta + \sqrt{(\alpha - \beta)^2 + 4D}}{2md_M} \),

where \( C = \frac{(p + 1)(p(k + 1)d_M + 1)}{p^2} \) and \( D = (Q_M + 3)(2(k + 1)d_M + 1) \),

then \( f^{nd_M} \equiv tM[f^n] \) with \( t^m = 1 \) and \( f \) assumes the form

\[ f(z) = ce^{\lambda m Q_M}, \]

where \( c \) is a non-zero constant with \( \lambda^{mQ_M} = 1 \).
Theorem 1.10. Let $f$ be a non-constant entire function, $n, k, p, m$ be positive integers and $a(z)$ be a small function of $f$ such that $a(z) \neq 0, \infty$. If $E_{f^n}(S_m, p) = E_{M[f^n]}(S_m, p)$ and if

1. $p \geq 2$ and $n > \frac{pmQ_M + p + 1}{pmd_M}$, or if
2. $p = 0$ and $n > \frac{mQ_M + (k+1)d_M + 2}{md_M}$,

then $f^{nd_M} \equiv tM[f^n]$ with $t^m = 1$ and $f$ assumes the form

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where $c$ is a non-zero constant with $\lambda^{mQ_M} = 1$.

2 Some Corollaries

In Theorem 1.9 and Theorem 1.10, if we take $M[f^n] = (f^n)^{(k)}$, where $n > k$, then it is clear that $d_M = 1$, $Q_M = k$. The following are some corollaries of the main results of this paper. What worth noticing here is that the lower bound of $n$ is reduced as compare to Theorem 1.7 and Theorem 1.8.

Corollary 1. Let $f$ be a non-constant meromorphic function and $n, m, p, k$ be positive integers and $a(z)$ be a small meromorphic function of $f$ such that $a(z) \neq 0, \infty$. If $E_{f^n}(S_m, p) = E_{(f^n)^{(k)}}(S_m, p)$ and if

1. $p \geq 2$ and $n > \frac{2p + p(m+1)k + 2}{2pm} + \frac{\sqrt{4(p+1)(pk + p + 1) + (m-1)^2p^2k^2}}{2pm}$, or if
2. $p = 0$ and $n > \frac{(m+3)k + 6}{2m} + \frac{\sqrt{4(k+3)(2k+3) + (m-1)^2k^2}}{2m}$,

then $f^n \equiv t(f^n)^{(k)}$, where $t^m = 1$ and $f$ assumes the form

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where $c$ is a non-zero constant and $\lambda^{mk} = 1$.

Corollary 2. Let $f$ be a non-constant entire function and $n,m,p,k$ be positive integers and $a(z)$ be a small meromorphic function of $f$ such that $a(z) \neq 0, \infty$. If $E_{f^n}(S_m, p) = E_{(f^n)^{(k)}}(S_m, p)$ and if

1. $p \geq 2$ and $n > k + \frac{p + 1}{pm}$, or if
2. $p = 0$ and $n > k + \frac{k + 3}{m}$,
then \( f^n \equiv t(f^n)^{(k)} \), where \( t^m = 1 \) and \( f \) assumes the form
\[
f(z) = ce^{\lambda_n z},
\]
where \( c \) is a non-zero constant and \( \lambda^{mk} = 1 \).

**Corollary 3.** Let \( f \) be a non-constant entire function and \( n, p, k \) be positive integers and \( a \equiv a(z) \) is a small meromorphic function of \( f \) and \( E_{f^n}(S_1, p) = E_{(f^n)^{(k)}}(S_1, p) \), then if

1. \( p \geq 2 \) and \( n > k + \frac{p+1}{p} \), or if
2. \( p = 0 \) and \( n > 2k + 3 \),

then \( f^n \equiv (f^n)^{(k)} \) and \( f \) assumes the form
\[
f(z) = ce^{\lambda_n z},
\]
where \( c \) is a non-zero constant and \( \lambda^{k} = 1 \).

### 3 Examples

The following examples show that conditions 1. and 2. in Corollary 1 and Corollary 2 are essential in order to get the conclusions.

**Example 3.1.** For \( n \geq 2 \), let the principal branch of \( f \) be given by \( f(z) = (e^{\theta z} + 2a)^{\frac{1}{n}} \), where \( a \neq 0 \) is a constant and \( \theta \) is a root of the equation \( z^n + 1 = 0 \). Let \( S_m = \{a\} \) and \( M[f^n] = (f^n)^{(m)} \). Clearly \( f^n = e^{\theta z} + 2a \) and \( M[f^n] = -e^{\theta z} \) and \( d_M = 1 \). Therefore we see that \( E_{f^n}d_M(S_m, \infty) = E_{M[f^n]}(S_m, \infty) \) and
\[
n \leq \min\left\{ k + \frac{p+1}{pm}, k + \frac{k+3}{m} \right\} = \max\left\{ n+1, 2n+3 \right\} = n+1.
\]
Here it is clear that
\[
f^n \not\equiv tM[f^n]
\]
with \( t^m = 1 \). Also we see that \( f \) does not assume the form
\[
f(z) = ce^{\lambda_n z}
\]
with \( \lambda^{nQ_m} = 1 \).

The following example shows that the conditions 1. and 2. used in Corollary 1 and Corollary 2 are not necessary but sufficient.
Example 3.2. Let $S_m = \{-1, 1, -i, i\}$ and $f$ be given by $f(z) = e^{\frac{\lambda}{3}z}$, where $\lambda$ is a root of the equation $z^3 + 1 = 0$. Let $M[f^3] = (f^3)^{(3)}$. It is clear that $f^3(z) = e^{\lambda z}$ and $M[f^3] = -e^{\lambda z}$. Also $E_{f^{3d}} (S_m, \infty) = E_M[f^n] (S_m, \infty)$ and

$$n \leq \min \left\{ k + \frac{p + 1}{pm}, k + \frac{k + 3}{m} \right\} = \min \left\{ \frac{13}{4}, \frac{9}{2} \right\} = \frac{13}{4}.$$

But we see that $f^3 \equiv tM[f^3]$ with $t^m = (-1)^4 = 1$. Also here $f$ assumes the form

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where $c = 1$ and $\lambda^m Q_M = \lambda^{12} = 1$.

The following examples show that the set $S_m$ in Theorems 1.9 – 1.10 can not be replaced by an arbitrary set.

Example 3.3. Let $S_m = \left\{ \frac{a\omega}{2}, \frac{a\omega}{3}, \frac{2a\omega}{3} \right\}$, where $a$ is an arbitrary non-zero complex number. Let $f^n = Be^{\theta z} + a\omega$, where $n \leq 16$ is a positive integer and $\theta$ and $\omega$ are roots of the equations $z^n - 1 = 0$ and $z^3 - 1 = 0$ respectively and $B \in \mathbb{C} - \{0\}$. Let $M[f^n] = (f^n)^{(n-5)}$, then we see that $M[f^n] = -Be^{\theta z}$. It is clear that

$$E_{f^{nd}} (S_m, \infty) = E_M[f^n] (S_m, \infty)$$

and

$$n > \max \left\{ k + \frac{p + 1}{pm}, k + \frac{k + 3}{m} \right\}.$$

But we see that $f^n \not\equiv tM[f^n]$ with $t^m = 1$ and hence $f$ does not assume the form

$$f(z) = ce^{\frac{\lambda}{n}z}$$

with $\lambda^m Q_M = 1$.

Example 3.4. Let $S_m = \left\{ \frac{1}{r}A, \frac{r - 1}{r}A : 2 \leq r \leq \frac{m + 3}{2} \right\}$, where $A$ is an arbitrary non-zero complex number and $m, r \in \mathbb{N}$ where $m$ is odd and $m > n + 2$. Let $f^n = Ae^{\theta z} + A$, where $n \geq 2$ is a positive integer and $\theta$ is a root of the equation $z^{n-1} + 1 = 0$ and $A \in \mathbb{C} - \{0\}$. Let $M[f^n] = (f^n)^{(n-1)}$, then we see that $M[f^n] = -Ae^{\theta z}$. It is clear that $E_{f^{nd}} (S_m, \infty) = E_M[f^n] (S_m, \infty)$ and

$$n > \max \left\{ k + \frac{p + 1}{pm}, k + \frac{k + 3}{m} \right\}.$$

But we see that $f^n \not\equiv tM[f^n]$ with $t^m = 1$. Also we see that $f$ does not assume the form

$$f(z) = ce^{\frac{\lambda}{n}z}$$

with $\lambda^m Q_M = 1$. 

The following example shows that if the conditions of Theorem 1.9 and Theorem 1.10 are satisfied, then the conclusions hold.

**Example 3.5.** Let $S_m = \{-1, 1, -i, i\}$ and $f$ be given by $f(z) = e^{\frac{\lambda}{5} z}$, where $\lambda$ is a root of the equation $z^3 + 1 = 0$. Let $M[f^n] = (f^n)^{(k)}$. It is clear that $f^n(z) = e^{\lambda z}$ and $M[f^n] = -e^{\lambda z}$ with $n = 5$, $k = 3$, $m = 4$ and $d_M = 1$. Also we see that $E_{f^{nd_M}}(S_m, \infty) = E_{M[f^n]}(S_m, \infty)$ and

$$n > \max \left\{ k + \frac{p + 1}{pm}, k + \frac{k + 3}{m}\right\} = \max \left\{ \frac{13}{4}, \frac{9}{2}\right\} = \frac{9}{2}.$$ 

Here we see that $f^{nd_M} \equiv tM[f^n]$ with $t^m = (-1)^4 = 1$. Also here $f$ assumes the form

$$f(z) = ce^{\frac{\lambda}{5} z},$$

where $c = 1$ and $\lambda^{m Q_M} = \lambda^{12} = 1$.

**Example 3.6.** For a non-zero complex number $a$, let $S = \{a, a\zeta, a\zeta^2, a\zeta^3, a\zeta^4\}$, where $\zeta$ is the non-real $5^{th}$ root of unity and $f$ is given by $f(z) = e^{\frac{1}{\sqrt{5}} z}$. It is clear that $f^n(z) = e^{\frac{1}{\sqrt{5}} z}$ and $M[f^n] = \zeta e^{\frac{1}{\sqrt{5}} z}$, where $M[f^n] = (f^n)^{(k)}$ with $n = 10$, $k = 7$, $m = 5$ and $d_M = 1$. Also we see that $E_{f^{nd_M}}(S_m, \infty) = E_{M[f^n]}(S_m, \infty)$ and

$$n > \max \left\{ k + \frac{p + 1}{pm}, \frac{k + 3}{m}\right\} = \max \left\{ \frac{36}{5}, 9\right\} = 9.$$ 

Here we see that $f^{nd_M} \equiv tM[f^n]$ with $t^m = \left(\frac{1}{\sqrt{5}}\right)^5 = 1$. Also here $f$ assumes the form

$$f(z) = ce^{\frac{\lambda}{5} z},$$

where $c = 1$ and $\lambda^{m Q_M} = \lambda^{12} = 1$.

### 4 Lemmas

In this section we present some Lemmas which will be needed in the sequel. Let $\mathcal{F}, \mathcal{G}$ be two non-constant meromorphic functions. Henceforth we shall denote by $\mathcal{H}$ the following function

$$\mathcal{H} = \left( \frac{\mathcal{F}''}{\mathcal{F}' - \frac{2\mathcal{F}'}{\mathcal{F}} - \frac{2\mathcal{G}'}{\mathcal{G}'} - \frac{2\mathcal{G}'}{\mathcal{G}'} \right).$$

(4.1)

$$\mathcal{V} = \left( \frac{\mathcal{F}'}{\mathcal{F}' - \frac{2\mathcal{F}'}{\mathcal{F}} - \frac{2\mathcal{G}'}{\mathcal{G}'} - \frac{2\mathcal{G}'}{\mathcal{G}'} \right)$$

(4.2)

and

$$\mathcal{U} = \frac{\mathcal{F}'}{\mathcal{F}'} - \frac{\mathcal{G}'}{\mathcal{G}'}.$$

(4.3)
Lemma 1 (see [18]). Let \( f \) be a non-constant meromorphic function and \( k, p \) are positive integers. Then
\[
N_p(r, 0; f^{(k)}) \leq T(r, f^{(k)}) - T(r, f) + N_{p+k}(r, 0; f) + S(r, f).
\]
\[
N_p(r, 0; f^{(k)}) \leq kN(r, \infty; f) + N_{p+k}(r, 0; f) + S(r, f).
\]

Lemma 2. Let \( f \) be a non-constant meromorphic function and \( M[f^n] \) be a differential monomial of degree \( d_M \) and weight \( \Gamma_M \). Then
\[
N(r, 0; M[f^n]) \leq T(r, M) - nd_M T(r, f) + nd_M N(r, 0; f) + S(r, f).
\]

Proof. This can be proved in the line of the proof of ([6, Lemma 2.3]). \( \square \)

Lemma 3. Let \( f \) be a non-constant meromorphic function and \( M[f^n] \) be a differential monomial of degree \( d_M \) and weight \( \Gamma_M \). Then
\[
N(r, 0; M[f^n]) \leq nd_M N(r, 0; f) + Q_M N(r, \infty; f) + S(r, f).
\]

Proof. This can be proved in the line of the proof of ([6, Lemma 2.4]). \( \square \)

Lemma 4. For the differential monomial \( M[f^n] \),
\[
N_p(r, 0; M[f^n]) \leq d_M N_{p+k}(r, 0; f^n) + Q_M N(r, \infty; f) + S(r, f).
\]

Proof. This can be proved in the line of the proof of ([6, Lemma 2.9]). \( \square \)

Lemma 5 (see [21]). Let \( f \) be a non-constant meromorphic function and \( P(f) = a_n f^n + a_{n-1} f^{n-1} + \ldots + a_0 \), where \( a_0, a_1, \ldots, a_n \) are constants with \( a_n \neq 0 \). Then
\[
T(r, P(f)) = nT(r, f) + S(r, f).
\]

Lemma 6 (see [21]). Let \( \mathcal{H} \) be given by (4.1), \( \mathcal{F} \) and \( \mathcal{G} \) be two non-constant meromorphic functions. If \( \mathcal{H} \neq 0 \), then
\[
N_{11}(r, 1; \mathcal{F}) \leq N(r, \mathcal{H}) + S(r, \mathcal{F}) + S(r, \mathcal{G}).
\]

Lemma 7. Let \( f \) be a non-constant meromorphic function and \( a \equiv a(z) \) be a small meromorphic functions of \( f \) such that \( a(z) \neq 0, \infty \) and let \( \mathcal{F}_1 = \frac{f^{nd_M}}{a} \) and \( \mathcal{G}_1 = \frac{M[f^n]}{a} \). Let \( \mathcal{V} \) be given by (4.2) and \( \mathcal{F} = \mathcal{F}_1^m \) and \( \mathcal{G} = \mathcal{G}_1^m \). If \( n, m \) and \( k \) are positive integers such that \( n > k \) and \( \mathcal{V} \equiv 0 \), then \( f^{nd_M} \equiv tM[f^n] \), where \( t^n = 1 \) and \( f \) assumes the form
\[
f(z) = ce^\lambda z,
\]
where \( c \) is a non-zero constant and \( \lambda^{mQ_M} = 1 \).
Proof. Let $V \equiv 0$. Then we get

$$1 - \frac{1}{\mathcal{F}_1^m} \equiv \mathcal{A} - \frac{\mathcal{A}}{\mathcal{G}_1^m},$$

(4.4)

where $\mathcal{A}$ is a non-zero constant. We now consider the following cases.

Case 1. Let $N(r, \infty; f) = S(r, f)$. If $\mathcal{A} \neq 1$, then from (4.4) we have

$$N\left(r, \frac{1}{1-\mathcal{A}}; \mathcal{F}_1^m\right) = N(r, \infty; \mathcal{G}_1^m) = S(r, f).$$

By the Second Fundamental Theorem and definitions of $\mathcal{F}_1, \mathcal{G}_1$, we have

$$T(r, \mathcal{F}_1^m) \leq N(r, \infty; \mathcal{F}_1^m) + N(r, 0; \mathcal{F}_1^m) + N\left(r, \frac{1}{1-\mathcal{A}}; \mathcal{F}_1^m\right) + S(r, f).$$

i.e.,

$$mnd_M T(r, f) \leq N(r, 0; f) + S(r, f),$$

which is not possible.

Case 2. Let $N(r, \infty; f) \neq S(r, f)$. Then there exists a $z_0$ which is not a zero or pole of $a(z)$ such that $\frac{1}{f(z_0)} = 0$, so $\frac{1}{\mathcal{F}_1(z_0)} = \frac{1}{\mathcal{G}_1(z_0)} = 0$. Therefore, from (4.4) we get $\mathcal{A} = 1$.

Thus, by (4.4) and $\mathcal{A} = 1$, then $\mathcal{F}_1^m = \mathcal{G}_1^m$, i.e.,

$$f^{nd_M} \equiv tM[f^m],$$

(4.5)

where $t^m = 1$. Now if $z_0$ be a zero of $f$ with multiplicity $q$, then $z_0$ is a zero of $f^{nd_M}$ with multiplicity $nqd_M$ and a zero of $M[f^m]$ with multiplicity $nqd_M - Q_M$. Therefore,

$$nqd_M = nqd_M - Q_M,$$

which is not possible. Thus it is obvious that $0$ is a Picard exceptional value of $f$. Similarly we can get that $\infty$ is also a Picard exceptional value of $f$. Then from (4.5) we have

$$f(z) = ce^{\lambda z},$$

where $c$ is a non-zero constant and $\lambda^n Q_M = 1$. \hfill \Box

Lemma 8. Let $V$ be given by (4.2) and $\mathcal{F}, \mathcal{G}, \mathcal{F}_1$ and $\mathcal{G}_1$ be given by Lemma 7 and $n, m$ be positive integers. If $V \neq 0$, then

$$(mnd_M - 1)N(r, \infty; f) \leq N(r, \infty; V) + S(r, f).$$
Proof. From (4.2) and the definitions of $F, G$, we see that if $z_0$ is a pole of $f$ with the multiplicity $q$ such that $a(z_0) \neq 0$ and $a(z_0) \neq \infty$, then $z_0$ is a zero of $\frac{F' - F'}{F - 1}$ with the multiplicity $mnqd_M - 1$ and a zero of $\frac{G' - G'}{G - 1}$ with the multiplicity $m(nqd_M + Q_M) - 1$. Therefore $z_0$ is zero of $V$ with multiplicity

$$p \geq \min \left\{ mnq - 1, m(nq + Q) - 1 \right\} = mnq - 1.$$ 

Also note that $m(r, V) = S(r, f)$. Therefore

$$(mnq - 1)N(r, \infty; f) \leq N(r, 0; V) + S(r, f) \leq T(r, V) + S(r, f).$$

\hfill $\Box$

Lemma 9. Let $U$ be given by (4.3) and $F, G, F_1$ and $G_1$ be given by Lemma 7. If $n, m$ are positive integers such that $n > k$ and $U \equiv 0$, then

$$f^{mnq} \equiv tM[f^n],$$

where $t^m = 1$ and $f$ assumes the form

$$f(z) = c e^{\lambda z},$$

where $c$ is a non-zero constant and $\lambda^m Q_M = 1$.

Proof. Since $U = 0$, we get

$$F \equiv BG + 1 - B,$$  \hfill (4.6)

where $B$ is a non-zero constant. By the definitions of $F, G, F_1$ and $G_1$, we get $N(r, \infty; f) = S(r, f)$. We discuss the following cases.

Case 1. Let $B = 1$. Then we see that $F \equiv G$, i.e., $F_1^{mnq} \equiv G_1^{mnq}$. Then we have

$$f^{mnq} \equiv tM[f^n],$$

where $t^m = 1$. Then $f$ assumes the form

$$f(z) = c e^{\lambda z},$$

where $c$ is a non-zero constant with $\lambda^m Q_M = 1$.

Case 2. Let $B \neq 1$. If $N(r, 0; f) \neq S(r, f)$, then there exists a point $z_0$ for which $f(z_0) = 0$ but $a(z_0) \neq 0$. Since $n > k$, then it is clear that $F(z_0) = 0 = G(z_0)$. Now from (4.6), we get $B = 1$, which is clearly absurd.
Again if \( N(r, 0; f) = S(r, f) \), then from (4.6) and using Lemma 3, we get
\[
\overline{N}(r, 1 - B; \mathcal{F}) = \overline{N}(r, 0; \mathcal{G}) \\
\leq nd_M N(r, 0; f) + Q_M \overline{N}(r, \infty; f) \\
\leq S(r, f).
\]

Now using Second Fundamental Theorem and \( N(r, 0; f) = N(r, \infty; f) = S(r, f) \), we have
\[
mnd_M T(r, f) \leq T(r, \mathcal{F}) + S(r, f) \\
\leq \overline{N}(r, \infty; \mathcal{F}) + \overline{N}(r, 0; \mathcal{F}) + \overline{N}(r, 1 - B; \mathcal{F}) + S(r, f) \\
\leq \overline{N}(r, 0; f) + \overline{N}(r, \infty; f) + \overline{N}(r, 0; \mathcal{G}) + S(r, f) \\
\leq S(r, f),
\]
which is not possible. \( \square \)

**Lemma 10.** Let \( \mathcal{U} \) be given by (4.3) and \( \mathcal{F}, \mathcal{G}, \mathcal{F}_1 \) and \( \mathcal{G}_1 \) be given by Lemma 7. If \( n, m \) and \( k \) are positive integers such that \( n > k \) and \( \mathcal{U} \neq 0 \), then
\[
((nd_M - Q_M)m - 1)\overline{N}(r, 0; f) \leq N(r, \infty; \mathcal{U}) + S(r, f).
\]

**Proof.** Let \( z_0 \) is a zero of \( f \) with multiplicity \( q(\geq 1) \) such that \( a(z_0) \neq 0, \infty \). Then \( z_0 \) is a zero of \( \frac{\mathcal{F}'}{\mathcal{F} - 1} \) with the multiplicity \( nmqd_M - 1 \) and \( z_0 \) is also a zero of \( \frac{\mathcal{G}'}{\mathcal{G} - 1} \) of multiplicity \( (nqd_M - Q_M)m - 1 \). Therefore \( z_0 \) is a zero of \( \mathcal{U} \) of multiplicity at least \( (nqd_M - Q_M)m - 1 \). Since \( m(r, \mathcal{U}) = S(r, f) \), we have
\[
((nd_M - Q_M)m - 1)\overline{N}(r, 0; f) \leq N(r, 0; \mathcal{U}) + S(r, f) \\
\leq T(r, \mathcal{U}) + S(r, f) \\
\leq N(r, \infty; \mathcal{U}) + S(r, f).
\]
\( \square \)

**Lemma 11.** Let \( \mathcal{F}, \mathcal{G}, \mathcal{F}_1, \mathcal{G}_1 \) be as in Lemma 7 and \( \mathcal{V} \) as in (4.2). Now if \( n > k \) and \( E_p(1, \mathcal{F}) = E_p(1, \mathcal{G}) \) and \( \mathcal{V} \neq 0 \), then the following hold:

1. When \( p \geq 2 \), then
\[
\left\{mnd_M - 1 - Q_M - \frac{1}{p}\right\}\overline{N}(r, \infty; f) \leq \left\{(k + 1)d_M + \frac{1}{p}\right\}\overline{N}(r, 0; f) + S(r, f).
\] \hspace{1cm} (4.7)

2. When \( p = 0 \), then
\[
\left\{mnd_M - 1 - 2(Q_M + 1)\right\}\overline{N}(r, \infty; f) \leq \left\{2(k + 1)d_M + 1\right\}\overline{N}(r, 0; f) + S(r, f).
\] \hspace{1cm} (4.8)
Proof. Let \( p \geq 2 \) and \( V = \frac{F'}{F(F-1)} - \frac{G'}{G(G-1)} \). Now since \( E_p(1; F) = E_p(1; G) \), so we have

\[
N(r, \infty; V) \leq \overline{N}(r, 0; G) + \overline{N}_{(p+1)}(r, 1; F) + S(r, f),
\]

where

\[
\overline{N}_{(p+1)}(r, 1; F) \leq \frac{1}{p} N \left( r, \frac{F}{F'} \right)
\]

\[
\leq \frac{1}{p} N \left( r, \frac{F}{F'} \right) + S(r, f)
\]

\[
\leq \frac{1}{p} N(r, \infty; F) + \frac{1}{p} N(r, 0; F) + S(r, f)
\]

\[
\leq \frac{1}{p} N(r, \infty; f) + \frac{1}{p} N(r, 0; f) + S(r, f).
\]

Now by applying Lemma 8 and Lemma 4 we get

\[
(mnd_m - 1)\overline{N}(r, \infty; f) \leq \frac{1}{p} \overline{N}(r, 0; f) + \frac{1}{p} N(r, 0; F) + \overline{N}(r, 0; G) + S(r, f)
\]

\[
\leq \frac{1}{p} N(r, 0; f) + \frac{1}{p} N(r, \infty; f) + d_m N_{k+1} N(r, 0; f^n) + Q_M N(r, \infty; f) + S(r, f),
\]

i.e.,

\[
\left\{ mnd_m - 1 - Q_M - \frac{1}{p} \right\} \overline{N}(r, \infty; f) \leq \left\{ (k + 1)d_m + \frac{1}{p} \right\} \overline{N}(r, 0; f) + S(r, f).
\]

Let \( p = 0 \), then

\[
N(r, \infty; V) \leq \overline{N}(r, 0; G) + \overline{N}_L(r, 1; F) + \overline{N}_L(r, 1; G) + S(r, f),
\]

where

\[
\overline{N}_L(r, 1; F) \leq N \left( r, \frac{F}{F'} \right) \leq N \left( r, \frac{F'}{F} \right) + S(r, f)
\]

\[
\leq \overline{N}(r, \infty; F) + \overline{N}(r, 0; F) + S(r, f)
\]

\[
\leq \overline{N}(r, \infty; f) + \overline{N}(r, 0; f) + S(r, f).
\]

Similarly, applying Lemma 4 and proceeding as above, we get

\[
\overline{N}_L(r, 1; G) \leq \overline{N}(r, \infty; G) + \overline{N}(r, 0; G) + S(r, f)
\]

\[
\leq (Q_M + 1)\overline{N}(r, \infty; f) + (k + 1)d_m \overline{N}(r, 0; f) + S(r, f).
\]

Now by Lemma 8 and Lemma 4, we get

\[
(mnd_m - 1)\overline{N}(r, \infty; f) \leq \{2(k + 1)d_m + 1\} \overline{N}(r, 0; f) + 2(Q_M + 1)\overline{N}(r, \infty; f) + S(r, f).
\]
i.e.,
\[
\left\{ mnd_M - 1 - 2(Q_M + 1) \right\} \overline{N}(r, \infty; f) \leq \left\{ 2(k + 1)d_M + 1 \right\} \overline{N}(r, 0; f) + S(r, f).
\]

Lemma 12. Let \( \mathcal{F}, \mathcal{G}, \mathcal{F}_1, \mathcal{G}_1 \) be as in Lemma 7 and \( \mathcal{U} \) as in (4.3). Now if \( n > k \) and \( E_p(1, \mathcal{F}) = E_p(1, \mathcal{G}) \) and \( \mathcal{U} \neq 0 \), then the following holds:

1. When \( p \geq 2 \), then
\[
\left\{ (nd_M - Q_M)m - 1 - \frac{1}{p} \right\} \overline{N}(r, 0; f) \leq \left\{ 1 + \frac{1}{p} \right\} \overline{N}(r, \infty; f) + S(r, f). \tag{4.9}
\]

2. When \( p = 0 \), then
\[
\left\{ (nd_M - Q_M)m - (k + 1)d_M - 2 \right\} \overline{N}(r, 0; f) \leq \left\{ Q_M + 3 \right\} \overline{N}(r, \infty; f) + S(r, f). \tag{4.10}
\]

Proof. Let \( p \geq 2 \), then we have
\[
N(r, \infty; \mathcal{U}) \leq \overline{N}(r, \infty; \mathcal{F}) + \overline{N}_{p+1}(r, 1; \mathcal{F}) + S(r, f)
\leq \overline{N}(r, \infty; f) + \left\{ \frac{1}{p} \overline{N}(r, 0; f) + \frac{1}{p} \overline{N}(r, \infty; f) \right\} + S(r, f)
\leq \frac{1}{p} \overline{N}(r, 0; f) + \left( 1 + \frac{1}{p} \right) \overline{N}(r, \infty; f) + S(r, f).
\]

Now by applying Lemma 10 we get
\[
\left\{ (nd_M - Q_M)m - 1 \right\} \overline{N}(r, 0; f) \leq \frac{1}{p} \overline{N}(r, 0; f) + \left( 1 + \frac{1}{p} \right) \overline{N}(r, \infty; f) + S(r, f),
\]
i.e.,
\[
\left\{ (nd_M - Q_M)m - 1 - \frac{1}{p} \right\} \overline{N}(r, 0; f) \leq \left( 1 + \frac{1}{p} \right) \overline{N}(r, \infty; f) + S(r, f).
\]

Let \( p = 0 \), by applying Lemma 10 and Lemma 4 and proceeding in the same way as done in the proof of Lemma 11, we get
\[
N(r, \infty; \mathcal{U}) \leq \overline{N}(r, \infty; \mathcal{F}) + \overline{N}_L(r, 1; \mathcal{F}) + \overline{N}_L(r, 1; \mathcal{G}) + S(r, f)
\leq \overline{N}(r, \infty; f) + \left\{ \overline{N}(r, 0; f) + \overline{N}(r, \infty; f) \right\} + \left\{ (Q_M + 1) \overline{N}(r, \infty; f) \right\}
+(k + 1)d_M \overline{N}(r, 0; f) + S(r, f),
\]
i.e.,
\[
\left\{ (nd_M - Q_M)m - (k + 1)d_M - 2 \right\} \overline{N}(r, 0; f) \leq \left\{ Q_M + 3 \right\} \overline{N}(r, \infty; f) + S(r, f).
\]
Lemma 13. Let $\mathcal{F}$ and $\mathcal{G}$ be two non-constant meromorphic functions such that $E_p(1, \mathcal{F}) = E_p(1, \mathcal{G})$ and $\mathcal{H} \not\equiv 0$ and $p = 0$, then

$$T(r, \mathcal{F}) + T(r, \mathcal{G}) \leq 2N_2(r, 0; \mathcal{F}) + 2N_2(r, 0; \mathcal{G}) + 6\mathcal{N}(r, \infty; \mathcal{F}) + 3\mathcal{N}_L(r, 1; \mathcal{F}) + 3\mathcal{N}_L(r, 1; \mathcal{G}) + S(r, \mathcal{F}).$$

Proof. Noting that $S(r, \mathcal{F}) = S(r, \mathcal{G})$, the lemma can be proved by using Lemma 2.1, Lemma 2.2 and Lemma 2.3 of [4].

Lemma 14. Let $\mathcal{F}$ and $\mathcal{G}$ be two non-constant meromorphic functions such that $E_p(1, \mathcal{F}) = E_p(1, \mathcal{G})$ and $\mathcal{H} \not\equiv 0$ and $p \geq 2$, then

$$T(r, \mathcal{F}) + T(r, \mathcal{G}) \leq 2N_2(r, 0; \mathcal{F}) + 2N_2(r, 0; \mathcal{G}) + 6\mathcal{N}(r, \infty; \mathcal{F}) + S(r, \mathcal{F}).$$

Proof. Since $\mathcal{F}$ and $\mathcal{G}$ share $(1, p)$ where $p \geq 2$, so it is clear that $\mathcal{F}$ and $\mathcal{G}$ share $(1, 2)$. Then the lemma can be obtained from Lemma 13 of [2].

Lemma 15. Let $\mathcal{H}$ be given by (4.1) and $\mathcal{F}, \mathcal{G}, \mathcal{F}_1$ and $\mathcal{G}_1$ be given by Lemma 7. If $n, m$ and $k$ are positive integers such that $n \gg k$ and

$$\mathcal{N}(r, \infty; f) = N(r, 0; f) = S(r, f)$$

and $\mathcal{H} \equiv 0$, then

$$f^{nd_M} \equiv tM[f^n],$$

where $t^m = 1$ and $f$ assumes the form

$$f(z) = ce^{\lambda z},$$

where $c$ is a non-zero constant and $\lambda^{nd_M} = 1$.

Proof. Since $\mathcal{H} \equiv 0$, by integration we obtain

$$\frac{1}{\mathcal{F} - 1} \equiv \frac{\mathcal{C}}{\mathcal{G} - 1} + \mathcal{D},$$

(4.11)

where $\mathcal{C}(\not\equiv 0)$ and $\mathcal{D}$ are constants. Now from (4.11) we have

$$\mathcal{G} = \frac{(D - C)\mathcal{F} + (C - D - 1)}{D\mathcal{F} - (D + 1)},$$

i.e.,

$$\mathcal{G}_1^m = \frac{(D - C)\mathcal{F}_1^m + (C - D - 1)}{D\mathcal{F}_1^m - (D + 1)}.$$  (4.12)

Now we discuss the following cases.

Case 1. Let $\mathcal{D} \not\equiv 0, -1$. Therefore from (4.12) we have

$$\mathcal{N}\left(r, \frac{D + 1}{\mathcal{D}}; \mathcal{F}_1^m\right) = \mathcal{N}(r, \infty; \mathcal{G}_1^m).$$
By applying the Second Fundamental Theorem with \( S(r, \mathcal{F}) = S(r, f) \), we get
\[
mnd_m T(r, f) = T(r, \mathcal{F}^m_1) + S(r, f)
\]
\[
\leq \mathcal{N}(r, \infty; \mathcal{F}^m_1) + \mathcal{N}(r, 0; \mathcal{F}^m_1) + \mathcal{N}\left(r, \frac{D + 1}{D}; \mathcal{F}^m_1\right) + S(r, f)
\]
\[
\leq \mathcal{N}(r, \infty; f) + \mathcal{N}(r, 0; f) + \mathcal{N}(r, \infty; \mathcal{G}^m_1) + S(r, f)
\]
\[
\leq S(r, f),
\]
which is not possible.

**Case 2.** Suppose \( D = 0 \). Then from (4.12), we have
\[
\mathcal{N}\left(r, \frac{C - 1}{C}; \mathcal{F}^m_1\right) = \mathcal{N}(r, 0; \mathcal{G}^m_1).
\]

**Subcase 2.1.** Let \( C \neq 1 \). Now by the Second Fundamental Theorem and using Lemma 3, we get
\[
mnd_m T(r, f) = T(r, \mathcal{F}^m_1) + S(r, f)
\]
\[
\leq \mathcal{N}(r, \infty; \mathcal{F}^m_1) + \mathcal{N}(r, 0; \mathcal{F}^m_1) + \mathcal{N}\left(r, \frac{C - 1}{C}; \mathcal{F}^m_1\right) + S(r, f)
\]
\[
\leq \mathcal{N}(r, \infty; f) + \mathcal{N}(r, 0; f) + \mathcal{N}(r, 0; \mathcal{G}^m_1) + S(r, f)
\]
\[
\leq (nd_m + 1)N(r, 0; f) + (Q_M + 1)\mathcal{N}(r, \infty; f) + S(r, f)
\]
\[
\leq S(r, f),
\]
which is not possible.

**Subcase 2.2.** Let \( C = 1 \). Then we have \( \mathcal{F}^m_1 \equiv \mathcal{G}^m_1 \), i.e.,
\[
f^{nd_M} = tM[f^n].
\]
Then \( f \) assumes the form
\[
f(z) = ce^{\frac{\lambda}{m}z},
\]
where \( c \) is a non-zero constant and \( \lambda = mQ_M \).

**Case 3.** Let \( D = -1 \), then from (4.12), we get
\[
\mathcal{G}^m_1 \equiv \frac{(C + 1)\mathcal{F}^m_1 - \mathcal{C}}{\mathcal{F}^m_1}
\]
Now proceeding exactly the same way as in Case 2, we get \( \mathcal{F}^m_1 \mathcal{G}^m_1 \equiv 1 \), i.e.,
\[
f^{nd_M}M[f^n] \equiv ta^2, \text{ where } t^n = 1.
\]
Again since \( \mathcal{N}(r, \infty; f) = S(r, f) = N(r, 0; f) \), so
\[
2T\left(r, \frac{f^{nd_M}}{a}\right) = T\left(r, \frac{ta^2}{f^{2nd_M}}\right) + O(1)
\]
\[
\leq T\left(r, \frac{M[f^n]}{f^{nd_M}}\right) + O(1)
\]
\[ \leq m \left( r, \frac{M[f^n]}{f^{nd_M}} \right) + N \left( r, \frac{M[f^n]}{f^{nd_M}} \right) + O(1) \]
\[ \leq N(r, \infty; M[f^n]) + N \left( r, 0; f^{nd_M} \right) + O(1) \]
\[ \leq (nd_M + Q_M)N(r, \infty; f) + nd_M N(r, 0; f) + O(1) \]
\[ \leq S(r, f), \]

which is not possible. \[\square\]

5 Proofs of the Theorems

Proof of Theorem 1.9. Let \( F_1 = \frac{f^{nd_M}}{a} \) and \( G_1 = \frac{M[f^n]}{a} \) and \( F = F_1^n, G = G_1^n \), where \( f \) is a non-constant meromorphic function. Now we discuss the following cases.

Case 1. If \( UV \equiv 0 \), then by using Lemma 7 and Lemma 9, we get the conclusions of the Theorem 1.9.

Case 2. Let \( UV \not\equiv 0 \), then from the assumption of Theorem 1.9, we see that \( E_p(1, F) = E_p(1, G) \).

Subcase 2.1. When \( p \geq 2 \), then by using Lemma 11 and Lemma 12, we get

\[ \left\{ mnd_M - 1 - Q_M - \frac{1}{p} \right\} \left\{ (nd_M - Q_M) m - 1 - \frac{1}{p} \right\} N(r, \infty; f) \]
\[ \leq \left\{ (k + 1)d_M + \frac{1}{p} \right\} \left\{ 1 + \frac{1}{p} \right\} N(r, \infty; f) + S(r, f) \]  
\[ \text{(5.1)} \]

and

\[ \left\{ mnd_M - 1 - Q_M - \frac{1}{p} \right\} \left\{ (nd_M - Q_M) m - 1 - \frac{1}{p} \right\} N(r, 0; f) \]
\[ \leq \left\{ (k + 1)d_M + \frac{1}{p} \right\} \left\{ 1 + \frac{1}{p} \right\} N(r, 0; f) + S(r, f). \]
\[ \text{(5.2)} \]

Now from the equations (5.1) and (5.2), we get

\[ \left\{ (mnd_M - \gamma_M^p) (mnd_M - \gamma_m^p) - C \right\} N(r, \infty; f) \leq S(r, f) \]  
\[ \text{(5.3)} \]

and

\[ \left\{ (mnd_M - \gamma_M^p) (mnd_M - \gamma_m^p) - C \right\} N(r, 0; f) \leq S(r, f), \]  
\[ \text{(5.4)} \]

where \( \gamma_M^p = mQ_M + 1 + \frac{1}{p} \) and \( C = \left\{ (k + 1)d_M + \frac{1}{p} \right\} \left\{ 1 + \frac{1}{p} \right\} \).

Since

\[ \left\{ mnd_M - \gamma_M^p \right\} \left\{ mnd_M - \gamma_m^p \right\} - C \]
\[
= m^2 d_m^2 n^2 - m d_m \left\{ \gamma_1^p + \gamma_m^p \right\} n + \left\{ \gamma_1^p \gamma_m^p - C \right\} d_m \\
= m^2 d_m^2 \left\{ n - \frac{\gamma_m^p + \gamma_1^p + \sqrt{(\gamma_m^p - \gamma_1^p)^2 + 4C}}{2 m d_m} \right\} \left\{ n - \frac{\gamma_m^p + \gamma_1^p - \sqrt{(\gamma_m^p - \gamma_1^p)^2 + 4C}}{2 m d_m} \right\},
\]

in view of the assumptions of Theorem 1.9, we get a contradiction from (5.3) and (5.4).

Thus we obtained from above
\[
N(r, 0; f) = S(r, f) = \overline{N}(r, \infty; f).
\] (5.5)

We now consider the following two cases:

**Case 2.1.1.** Let \( H \neq 0 \). Using Lemma 13 and Lemma 14 and (5.5), we get
\[ T(r, f) = S(r, f), \] which is a contradiction.

**Case 2.1.2.** Let \( H \equiv 0 \). Then from Lemma 15, we get the conclusion of Theorem 1.9.

**Subcase 2.2.** When \( p = 0 \), using Lemma 11 and Lemma 12, we get
\[
\left\{ m n d_m - 1 - 2 (Q_m + 1) \right\} \left\{ (n d_m - Q_m) m - (k + 1) d_m - 2 \right\} N(r, \infty; f) \\
\leq \left\{ 2 (k + 1) d_m + 1 \right\} \left\{ Q_m + 3 \right\} \overline{N}(r, \infty; f) + S(r, f)
\] and
\[
\left\{ m n d_m - 1 - 2 (Q_m + 1) \right\} \left\{ (n d_m - Q_m) m - (k + 1) d_m - 2 \right\} \overline{N}(r, 0; f) \\
\leq \left\{ 2 (k + 1) d_m + 1 \right\} \left\{ Q_m + 3 \right\} \overline{N}(r, 0; f) + S(r, f).
\]

Now using equations (5.6) and (5.7) and proceeding the same way as done in Subcase 2.1, the rest of the proof can be carried out. So we omit the detail.

**Proof of Theorem 1.10.** Since \( f \) is an entire function, we have
\[ N(r, \infty; f) = S(r, f). \] Now if \( U \equiv 0 \), then using Lemma 9, we get the conclusion of Theorem 1.10.

If \( U \neq 0 \), then using Lemma 10 for \( p \geq 2 \) we get from (5.2) that
\[
(m n d_m - \gamma_1^p) (m n d_m - \gamma_m^p) \overline{N}(r, 0; f) \leq S(r, f).
\]
Since \( n > \frac{p m Q_m + p + 1}{p m d_m} \), we get a contradiction.

Again when \( p = 0 \), using Lemma 10 we get from (5.7)
\[
\left\{ m n d_m - (2 Q_m + 3) \right\} \left\{ m n d_m - (m Q_m + (k + 1) d_m + 2) \right\} \overline{N}(r, 0; f) \leq S(r, f),
\]
which is a contradiction since \( n > \frac{m Q_m + (k + 1) d_m + 2}{m d_m} \).

Therefore \( \overline{N}(r, 0; f) = S(r, f) \). Now the rest of the proof follows Case 1 and Case 2 of the proof of Theorem 1.9.
6 Some Open Questions

**Question 6.1.** Can we replace $f^n$ by a general linear expression $P(f)$ in anyway in Theorem 1.9 and Theorem 1.10 to get the same specific form the function?

**Question 6.2.** Can we replace the differential monomial $M[f^n]$ by a differential polynomial $P[f^n]$ in anyway in Theorem 1.9 and Theorem 1.10 to get the same specific form the function?

**Question 6.3.** Can the lower bound of $n$ be further reduced in Theorem 1.9 and Theorem 1.10 to get the same conclusions?

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