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An investment problem under multicriteriality, uncertainty and risk

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Abstract. The strong stability radius of the multicriteria investment Boolean problem with the Savage risk criteria is investigated. The problem is to find the set of Pareto optimal portfolios. Upper and lower bounds of such a radius are derived for the case where different Hölder metrics are defined in the three problem parameters spaces.

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1 Introduction

Most of business and management decisions are made within uncertain and risky environment. Investment managing problems are as a type of problems with uncertainty of the initial data. Any separate investment asset has higher level of risk and less return than the portfolio of those assets and there is no reason to invest in one particular asset. Creating the portfolio by diversification and mixing variety of investments an investor reduces the riskiness of the portfolio. Following Markowitz's portfolio theory the investor plotting on the graph an efficient frontier depending on various pairs risk and expected return chooses portfolio drawing on individual risk-return preferences. It gives ability to construct a portfolio with the same expected return and less risk.

Based on Markowitz's portfolio optimization concept [1, 2] a multicriteria Boolean discrete variant of portfolio optimization holding constant expected return and minimize risk of portfolios consisting of the investment projects is considered. This problem is viewed as a problem of finding the Pareto optimal portfolios set using Savage's risk criteria. It means that a portfolio is a Pareto optimal one, when its total level of risk, i.e. the sum of all risks of the projects included in the portfolio is minimal in the worst market situation for one type of the risk. Unlike classical modern portfolio theory where a portfolio consisting of percentage of each asset there are several investment projects composes the portfolio. This model can be considered as a discrete variant of Markowitz problem with encoding a portfolio selection where for each project the risk matrix is constructed for several market states related to each type of the risk.

The model formulation requires statistical and expert evaluation of risks (e.g. financial or ecological) [3] to be specified as the initial data. The collected data usually contain computational errors and inaccuracies. It leads to the situation when the initial data representing risk values are inaccurate and uncertain. One of the key questions while analyzing an uncertain data is about the limiting level of the initial data changes (perturbations) which do not violate the optima. The quantitative measure of the data perturbation level is known as the stability radius, which concept is widely presented and analyzed in the recent literature focusing on finding analytical expressions and bounds (see e.g. [4–8]). Similar approaches were also developed in parallel in scheduling theory (see [9]). Analytical formulas are pairwise comparisons of solutions that reflect the specific of the selected principle of optimality, the structure of global perturbation of this problem and the structure of the solution set, namely Boolean portfolios. The evaluation of the stability properties is a global property itself. The particular definition of the stability radius concept depends on chosen optimality principles (if the problem is multicriteria), uncertain data and a type of distance metric used to measure the closeness in problem parameter spaces. Various types of metrics allow to consider a specific of problem parameters perturbation. So in the case of Chebyshev metric l_∞ the maximum changes in the initial data take into account only that allow perturbations to be independent. In the case of Manhattan metric l_1 every change of the initial data can be monitored in total. Hölder metric $l_p, 1 \leq p \leq \infty$, is the metric with a parameter and includes such extreme cases as Chebyshev metric l_∞, Manhattan metrics l_1 and also Euclidean metric l_2. Thus, using Hölder metric l_p for obtaining the stability radius depending on the properties of the initial data the control of perturbations can be varied.

Along with a quantitative approach to analyzing admissible level of the initial data perturbations, a qualitative approach is developed in parallel. This approach concentrates on specifying analytical conditions which will guarantee some certain pre-specified behavior of the optimal solutions set. Within this approach authors focus on finding necessary and sufficient conditions of different types of the problem stability (see the monograph [10], the reviews [11, 12], and the articles [13–17]), on revealing relations between different types of stability [18, 19], and also on finding and describing the stability region of an optimal solution [20].

This work continues started in [21–29] researches of different types of stability of vector nonlinear investment problems. Thus the work follows the approach of obtaining qualitative characteristics of stability. One of such characteristics, called commonly a stability radius of a problem, is defined as a limit level of problem parameters perturbations in the metric space such that pre-specified property of the problem solution set is preserved. Perturbing parameters usually are coefficients of the scalar or vector criteria.

Stability of a multicriteria discrete optimization problem of finding the Pareto set is commonly understood (see e.g. [10]) as discrete analog of the Hausdorff upper semicontinuity property of the point-to-set mapping that defines the Pareto choice function. Thus, the stability property means that there exists a neighborhood of the initial problem parameters in which appearance of a new Pareto optimum is impossible. In other words, the Pareto set inside this neighborhood can only narrow in the result of the problem parameters perturbations. Relaxation of this requirements leads to a new stability type. It is understood as existence of such neighborhood of the initial problem parameters in which appearance of new Pareto optimums is possible; but at least one Pareto optimal solution (not necessarily one and the same) preserves its optimality for any perturbation. Following the terminology of [30–33], we call such a stability strong.

Strong stability was first time investigated in [34] for a one-criterion (scalar) linear trajectorial problem. Later in [32, 35, 36] the lower and upper bounds of this type stability radius were derived for the multicriteria linear Boolean integer programming problem. The article [37] is devoted to obtaining similar bounds for the vector investment problem with the Wald criteria. We also point out the work [30] where necessary and sufficient conditions of the strong stability are found for the multicriteria problem of threshold functions minimization. The mentioned results were obtained in the case of the Chebyshev metric l_∞ in the problem parameter spaces.

In this paper the lower and upper bounds of the strong stability radius are found for the multicriteria investment problem with the Savage risk criteria in the case of different Hölder metrics in the three problem parameter spaces. Separately we investigate a particular case of the investment problem with the linear criteria, i.e. the case when the state of the financial market does not doubt the investor.

2 Problem formulation and basic definitions

Consider a mutlicriteria discrete analogue of the Markowitz portfolio management problem [1], which is based on diversification as a tool of risk minimization. Let

$N_n = \{1, 2, \dots, n\}$ be a variety of alternative investment projects (assets);

N_m be a set of possible financial market states (market situations, scenarios);

N_s be a set of possible risks;

r_{ijk} be a numerical measure of economic risk of type $k \in N_s$, which the investor may face if (s)he chooses project $j \in N_n$ assuming that the market state is $i \in N_m$;

$R = [r_{ijk}] \in \mathbf{R}^{m\times n\times s}$;

$x = (x_1, x_2, \dots, x_n)^T \in \mathbf{E}^n$ be an investment portfolio, where $\mathbf{E} = \{0, 1\}$,

$$x_j = \begin{cases} 1, & \text{if investor chooses project } j, \\ 0 & \text{otherwise;} \end{cases}$$

$X \subset \mathbf{E}^n$ be a set of all admissible investment portfolios, i.e. those realizations which provide expected total income and do not exceed the budget;

$\mathbf{R}^m$ be a financial market state space;

$\mathbf{R}^n$ be a project space;

$\mathbf{R}^s$ be a risk space.

The presence of a risk factor is integral feature of financial market functioning. One can find information about risk measurement methods and their classification in [38]. The last trend is to quantify risks using five R: robustness, redundancy, resourcefulness, response and recovery. The natural target of any investor is to minimize different types of risks. It creates a motivation for multicriteria analysis within risk modelling. It leads to the usage of multicriteria decision making tools [39, 40].

Assume that the efficiency of a chosen portfolio (Boolean vector) $x = (x_1, x_2, ..., x_n)^T \in X,\ |X| \geq 2$, is evaluated by a vector objective function

$$f(x, R) = (f_1(x, R_1), f_2(x, R_2) \dots, f_s(x, R_s))^T,$$

each partial objective represents minmax Savage's risk criterion (extreme pessimism) [41]

$$f_k(x, R_k) = \max_{i \in N_m} r_{ik} x = \max_{i \in N_m} \sum_{j \in N_n} r_{ijk} x_j \to \min_{x \in X}, \quad k \in N_s,$$

where $R_k \in \mathbf{R}^{m \times n}$ is the k-th cut $R = [r_{ijk}] \in \mathbf{R}^{m \times n \times s}$ with rows $r_{ik} = (r_{i1k}, r_{i2k}, \dots, r_{ink}) \in \mathbf{R}^n,\ i \in N_m$.

Thus, if an investor chooses the Savage risk (bottleneck) criterion [42, 43], then (s)he optimizes the total profit of the selected portfolio in the worst (maximum risk) case. This approach takes place when the decision maker has pessimistic expectations and wants to achieve the guaranteed result. In other words, the investor adhere to the wise rule that suggests to expect the worst case.

A problem of finding the Pareto optimal (efficient) portfolios is referred to as a multicriteria investment Boolean problem with the Savage risk criteria and is denoted $Z^s_m(R), s \in \mathbf{N}$. The set of Pareto optimal portfolios is defined as follows

$$P^s(R) = \{x \in X \ : \ X(x, R) = \emptyset\},$$

where

$$X(x, R) = \{x' \in X \ : \ f(x, R) \geq f(x', R) \ \& \ f(x, R) \neq f(x', R).$$

It is evident that $P^s(R) \neq \emptyset$ for any matrix $R \in \mathbf{R}^{m \times n \times s}$. Let us note that the problem $Z^s_m(R)$ can be interpreted as "the worst case optimization".

Let the Hölder metrics (generally speaking different) l_p, l_q, and $l_r,\ p, q, r \in [1, \infty]$, be defined in the spaces $\mathbf{R}^n, \mathbf{R}^m$, and $\mathbf{R}^s$ correspondingly. It means that the norm of the matrix $R \in \mathbf{R}^{m \times n \times s}$ is the number

$$\|R\|_{pqr} = \left\| (\|R_1\|_{pq},\ \|R_2\|_{pq}, \dots,\ \|R_s\|_{pq}) \right\|_r,$$

$$\|R_k\|_{pq} = \left\| (\|r_{1k}\|_p,\ \|r_{2k}\|_p, \dots,\ \|r_{mk}\|_p) \right\|_q,\ k \in N_s.$$

Recall that the Hölder norm l_p in the space $\mathbf{R}^n$ is defined as follows

$$||a||_p = \begin{cases} \left(\sum_{j \in N_n} |a_j|^p\right)^{1/p}, & \text{if } 1 \leq p < \infty, \\ \max\{|a_j| \ : \ j \in N_n\}, & \text{if } p = \infty, \end{cases}$$

where $a = (a_1, a_2, \dots, a_n)^T \in \mathbf{R}^n$.

It is easy to see that for any $p, q, r \in [1, \infty]$, the following inequalities hold

$$||r_{ik}||_p \leq ||R_k||_{pq} \leq ||R||_{pqr}, \quad i \in N_m, \quad k \in N_s. \tag{1}$$

Following [30–37, 44, 45], the strong stability radius (in terminology of [10] – T_1-stability radius) of the problem $Z^s_m(R)$, $s \in \mathbf{N}$, with the Hölder norms l_p, l_q, and l_r in the spaces $\mathbf{R}^m, \mathbf{R}^n$, and $\mathbf{R}^s$ correspondingly is the number

$$\rho = \rho^s_m(p, q, r) = \begin{cases} \sup \Xi_{pqr}, & \text{if } \Xi_{pqr} \neq \emptyset, \\ 0, & \text{if } \Xi_{pqr} = \emptyset, \end{cases}$$

where

$$\Xi_{pqr} = \{\varepsilon > 0 \ : \ \forall R' \in \Omega_{pqr}(\varepsilon) \quad (P^s(R) \cap P^s(R + R') \neq \emptyset)\},$$

$$\Omega_{pqr}(\varepsilon) = \{R' \in \mathbf{R}^{m \times n \times s} \ : \ ||R'||_{pqr} < \varepsilon\}.$$

Here $\Omega_{pqr}(\varepsilon)$ is the set of perturbing matrixes R' with cuts $R'_k \in \mathbf{R}^{m \times n}$, $k \in N_s$; $P^s(R + R')$ is the Pareto set of the perturbed problem $Z^s(R + R')$; $||R'||_{pqr}$ is the norm of the matrix $R' = [r'_{ijk}]$.

Thus, the strong stability radius of the problem $Z^s_m(R)$ is a limit level of the matrix R elements perturbations in the metric space $\mathbf{R}^{m \times n \times s}$ such that for each of those perturbations at least one (not necessary one and the same) optimal portfolio of the problem $Z^s_m(R)$ preserves its optimality in the perturbed problem $Z^s_m(R+R')$.

It is obvious that if $P^s(R) = X$, then the set $P^s(R) \cap P^s(R + R')$ is not empty for any perturbing matrix $R \in \Omega_{pqr}(\varepsilon)$ and any number $\varepsilon > 0$. That is why the strong stability radius of such problem is not upper limited. Hereafter, a problem with $P^s(R) \neq X$ is called non-trivial.

3 Auxiliary statements

Let u be any of the numbers p, q, r introduced earlier. For the number u, define a conjugate number u' by the following relations

$$1/u + 1/u' = 1, \quad 1 < u < \infty.$$

Moreover, let $u' = 1$ when $u = \infty$; and $u' = \infty$ when $u = 1$. Thus, the acceptable range of the numbers u and u' is the interval $[1, \infty]$; and the numbers are tied by the relations above. Also we assume $1/u = 0$ if $u = \infty$.

Further we use the known Hölder inequality

$$|a^T b| \leq ||a||_u ||b||_{u'}, \tag{2}$$

valid for any vectors a and b of the same dimension.

Lemma. *For any portfolios $x, x^0 \in X$, indexes $i, i' \in N_n$, $k \in N_s$, and numbers $p, q \in [1, \infty]$, the following inequality is valid*

$$r_{i'k}x^0 - r_{ik}x \geq -||R_k||_{pq}||(||x^0||_{p'}, ||x||_{p'})||_v,$$

where

$$v = \min\{p', q'\}.$$

Indeed, if $i \neq i'$ then applying the Hölder inequality (2), get

$$r_{i'k}x^0 - r_{ik}x \geq -(||r_{i'k}||_p||x^0||_{p'} + ||r_{ik}||_p||x||_{p'}) \geq$$

$$\geq -||(||r_{i'k}||_p, ||r_{ik}||_p)||_q \; ||(||x^0||_{p'}, ||x||_{p'})||_{q'} \geq$$

$$\geq -||R_k||_{pq} \; ||(||x^0||_{p'}, ||x||_{p'})||_{q'} \geq -||R_k||_{pq} \; ||(||x^0||_{p'}, ||x||_{p'})||_v.$$

If $i = i'$ then we apply (1), the Hölder inequality (2) and derive

$$r_{i'k}x^0 - r_{ik}x \geq -||r_{ik}||_p \; ||x^0 - x||_{p'} \geq -||R_k||_{pq} \; ||x^0 - x||_{p'} \geq$$

$$\geq -||R_k||_{pq} \; ||(||x^0||_{p'}, ||x||_{p'})||_{q'} \geq -||R_k||_{pq} \; ||(||x^0||_{p'}, ||x||_{p'})||_v.$$

Moreover, for a vector $a = (a_1, a_2, \dots, a_n)^T \in \mathbf{R}^n$ with $|a_j| = \alpha$, $j \in N_n$, and any number $p \in [1, \infty]$, easily obtain

$$||a||_p = \alpha n^{1/p}. \tag{3}$$

4 The strong stability radius bounds

For a non-trivial problem $Z^s_m(R)$ we denote

$$\varphi = \varphi^s(p, q) = \min_{x \notin P^s(R)} \; \max_{x' \in P(x,R)} \; \min_{k \in N_s} \frac{g_k(x.x', R_k)}{||(||x||_{p'}, ||x'||_{p'})||_v},$$

$$\psi = \psi^s(p, q, r) = \max_{x' \in P^s(R)} \; \min_{x \notin P^s(R)} \frac{||[g(x, x', R)]^+||_r}{||(||x||_{p'}, ||x'||_{p'})||_v},$$

$$\chi = \chi^s(p, q, r) = n^{1/p}m^{1/q}s^{1/r} \min_{x \notin P^s(R)} \; \max_{x' \in P^s(R)} \; \max_{k \in N_s} \frac{g_k(x, x', R_k)}{||x - x'||_1}.$$

Here

$$P(x, R) = X(x, R) \cap P^s(R),$$

$$g(x, x', R) = (g_1(x, x', R_1), g_2(x, x', R_2), \dots, g_s(x, x', R_s))^T,$$

$$g_k(x, x', R_k) = f_k(x, R_k) - f_k(x', R_k), \quad k \in N_s,$$

$$v = \min\{p', q'\},$$

$$[y]^+ = (y^+_1, y^+_2, \dots, y^+_s)^T$$

is a positive cutoff of a vector $y = (y_1, y_2, \dots, y_s)^T \in \mathbf{R}^s$, i.e. $y^+_k = \max\{0, y_k\}$, $k \in N_s$.

Theorem 1. *For any* $s, m \in \mathbf{N}$ *and* $p, q, r \in [1, \infty]$, *for the strong stability radius* $\rho^s_m(p, q, r)$ *of the non-trivial problem* $Z^s_m(R)$ *the following bounds are valid*

$$0 < \max\{\varphi^s(p, q),\ \psi^s(p, q, r)\} \leq \rho^s_m(p, q, r) \leq \min\{\chi^s(p, q, r),\ ||R||_{pqr}\}.$$

Proof. From the evident formula

$$\forall x' \in P^s(R) \quad \forall x \notin P^s(R) \quad \exists k \in N_s \quad (f_k(x, R_k) > f_k(x', R_k)),$$

we easily get the inequality

$$\psi = \psi^s(p, q, r) > 0,$$

which shows that lower bound of the radius $\rho^s_m(p, q, r)$ and the radius itself are positive numbers.

Now let us show validity of the lower bound

$$\rho = \rho^s_m(p, q, r) \geq \varphi^s(p, q) = \varphi. \tag{4}$$

Suppose that $\varphi > 0$ (otherwise the inequality is evident).

Let $R' = [r'_{ijk}] \in \mathbf{R}^{m\times n\times s}$ be a perturbing matrix with cuts R'_k, $k \in N_s$, from the set $\Omega_{pqr}(\varphi)$. By the definition of φ and inequality (1), we get the formula

$$\forall x \notin P^s(R) \quad \exists x^0 \in P(x, R) \quad \forall k \in N_s$$

$$\left(\frac{f_k(x, R_k) - f_k(x^0, R_k)}{||(||x||_{p'}, ||x^0||_{p'})||_v} \geq \varphi > ||R'||_{pqr} \geq ||R'_k||_{pq} \right).$$

Using the lemma, for any $k \in N_s$ derive

$$f_k(x, R_k + R'_k) - f_k(x^0, R_k + R'_k) = \max_{i\in N_m}(r_{ik} + r'_{ik})x - \max_{i\in N_m}(r_{ik} + r'_{ik})x^0 =$$

$$= \min_{i\in N_m} \max_{i'\in N_m} (r_{i'k}x + r'_{i'k}x - r_{ik}x^0 - r'_{ik}x^0) =$$

$$= f_k(x, R_k) - f_k(x^0, R_k) - ||R'_k||_{pq}\ ||(||x||_{p'}, ||x^0||_{p'})||_v > 0,$$

where r'_{ik} is the i-th row of the k-th cut R'_k of R'. This means that $x \notin P^s(R + R')$. Resuming, we conclude that any non-efficient portfolio of the problem $Z^s_m(R)$ preserves optimality in the perturbed problem $Z^s_m(R + R')$. Therefore, the following relations are valid

$$\emptyset \neq P^s(R + R') \subseteq P^s(R).$$

Hence, $P^s(R) \cap P^s(R+R') \neq \emptyset$ for any perturbing matrix $R' \in \Omega_{pqr}(\varphi)$, i.e. inequality (4) is true.

Now we pass to the proof of the lower bound

$$\rho = \rho^s_m(p, q, r) \geq \psi^s(p, q, r) = \psi.$$

As in the previous case, let $R' = [r'_{ijk}] \in \mathbf{R}^{m\times n\times s}$ be a perturbing matrix from the set $\Omega_{pqr}(\psi)$. As it was established earlier, ψ is a positive number. To prove inequality $\rho > \psi$ it is sufficient to show that there exists portfolio x^* that belongs to the set $P^s(R) \cap P^s(R + R')$.

By the definition of ψ, there exists a portfolio $x^0 \in P^s(R)$ such that for any portfolio $x \notin P^s(R)$ the following inequalities hold

$$0 < \psi ||(||x||_{p'}, ||x^0||_{p'})||_v \leq ||[g(x, x^0, R)]^+||_r. \tag{5}$$

Let us now prove the formula

$$\forall x \notin P^s(R) \quad \forall R' \in \Omega_{pqr}(\psi) \quad \left(x \notin X(x^0, R + R')\right). \tag{6}$$

We prove it by contradiction. Supposing to the contrary, obtain the formula

$$\exists \tilde{x} \notin P^s(R) \quad \exists \tilde{R} \in \Omega_{pqr}(\psi) \quad \left(\tilde{x} \in X(x^0, R + \tilde{R})\right).$$

It implies that for any index $k \in N_s$ we get the inequality

$$g_k(\tilde{x}, x^0, R_k + \tilde{R}_k) \leq 0,$$

where $\tilde{R}_k$ is the k-th cut of the matrix $\tilde{R} = [\tilde{r}_{ijk}]$. Hence, taking into account the lemma and inequality (1), we get relations

$$\begin{aligned} 0 \geq g_k(\tilde{x}, x^0, R_k + \tilde{R}_k) = f_k(\tilde{x}, R_k + \tilde{R}_k) - f_k(x^0, R_k + \tilde{R}_k) = \\ = \max_{i \in N_m} (r_{ik} + \tilde{r}_{ik})\tilde{x} - \max_{i \in N_m} (r_{ik} + \tilde{r}_{ik})x^0 = \\ = \min_{i \in N_m} \max_{i' \in N_m} \left(r_{ik}\tilde{x} - r_{i'k}x^0 + \tilde{r}_{ik}\tilde{x} - \tilde{r}_{i'k}x^0\right) \geq \\ \geq g_k(\tilde{x}, x^0, R_k) - ||\tilde{R}_k||_{pq} \; ||(||\tilde{x}||_{p'}, ||x^0||_{p'})||_v. \end{aligned}$$

Having them, we derive

$$g_k(\tilde{x}, x^0, R_k) \leq ||\tilde{R}_k||_{pq} \; ||(||\tilde{x}||_{p'}, ||x^0||_{p'})||_v$$

and then conclude that

$$[g_k(\tilde{x}, x^0, R_k)]^+ \leq ||\tilde{R}_k||_{pq} \; ||(||\tilde{x}||_{p'}, ||x^0||_{p'})||_v.$$

As a result, we get the following contradiction with inequality (5)

$$||[g_k(\tilde{x}, x^0, R_k)]^+||_r \leq ||\tilde{R}||_{rpq} \; ||(||\tilde{x}||_{p'}, ||x^0||_{p'})||_v < \psi \; ||(||\tilde{x}||_{p'}, ||x^0||_{p'})||_v.$$

Hence, formula (6) is proved.

Now we show the way of choosing the required portfolio

$$x^* \in P^s(R) \cap P^s(R + R'),$$

where $R' \in \Omega_{pqr}(\psi)$. If $x^0 \in P^s(R+R')$ then $x^* = x^0$. Suppose $x^0 \notin P^s(R+R')$. Due to the external stability property of the Pareto set $P^s(R + R')$ (see e.g. [46], p. 39) we can choose a portfolio $x^* \in P^s(R+R')$ such that $x^* \in X(x^0, R+R')$. Using the proved formula (6), we easily find out that $x^* \in P^s(R)$. Thus, the inequality $\rho \geq \psi$ is proved.

Further we show correctness of the upper bound

$$\rho_m^s(p,q,r) \leq \chi^s(p,q,r) = \chi. \tag{7}$$

By definition of χ, there exists a portfolio $x^0 \notin P^s(R)$ such that for any efficient portfolio $x \in P^s(R)$ and any index $k \in N_s$ the following inequality holds

$$\chi||x^0 - x||_1 \geq n^{1/p} m^{1/q} s^{1/r} g_k(x^0, x, R_k). \tag{8}$$

Let $\varepsilon > \chi$. We set the elements of the perturbing matrix $R^0 = [r^0_{ijk}] \in \mathbf{R}^{m\times n\times s}$ with cuts $R^0_k,\ k \in N_s$, by the rule

$$r_{ijk} = \begin{cases} -\delta, & \text{if} \quad i \in N_m, \quad x^0_j = 1, \quad k \in N_s, \\ \delta, & \text{if} \quad i \in N_m, \quad x^0_j = 0, \quad k \in N_s. \end{cases}$$

Here the number δ is chosen to satisfy the inequality

$$\chi < \delta n^{1/p} m^{1/q} s^{1/r} < \varepsilon. \tag{9}$$

Therefore, with proved (3) we derive

$$||r^0_{ik}||_p = \delta n^{1/p}, \quad i \in N_m, \quad k \in N_s,$$

$$||R^0_k||_{pq} = \delta n^{1/p} m^{1/q}, \quad k \in N_s,$$

$$||R^0||_{pqr} = \delta n^{1/p} m^{1/q} s^{1/r}.$$

This means that $R^0 \in \Omega_{pqr}(\varepsilon)$. Moreover, all the rows $r^0_{ik},\ i \in N_m$, of any k-th cut $R^0_k,\ k \in N_s$, are equal and consist of the components δ and $-\delta$. So, denoting $c = r^0_{ik},\ i \in N_m,\ k \in N_s$, we obtain the relations

$$c(x^0 - x) = -\delta||x^0 - x||_1 < 0$$

valid for any portfolio $x \neq x^0$. Therefore, taking into account (8) and (9), for any portfolio $x \in P^s(R)$ and any index $k \in N_s$, we derive

$$g_k(x^0, x, R_k + R^0_k) = \min_{i \in N_m} (r_{ik} + c)x^0 - \min_{i \in N_m} (r_{ik} + c)x =$$

$$= \min_{i \in N_m} r_{ik}x^0 - \min_{i \in N_m} r_{ik}x + c(x^0 - x) = g_k(x^0, x, R_k) + c(x^0 - x) \leq$$

$$\leq (\chi(n^{1/p} m^{1/q} s^{1/r})^{-1} - \delta)||x^0 - x||_1 < 0.$$

Thus, any portfolio $x \in P^s(R)$ of the problem $Z^s_m(R)$ does not belong to the Pareto set of the pertubed problem $Z^s_m(R + R^0)$. In other words, for any number $\varepsilon > \chi$, there exists a matrix $R^0 \in \Omega_{pqr}(\varepsilon)$ such that $P^s(R+R^0) \cap P^s(R) = \emptyset$, i.e. $\rho < \varepsilon$ for any $\varepsilon > \chi$. Inequality (7) is proved.

Now we must only verify the inequality $\rho \leq ||R||_{pqr}$. Suppose $x^0 \notin P^s(R)$ and $\varepsilon > ||R||_{pqr}$. Choose a number δ such that

$$0 < \delta n^{1/p} m^{1/q} < \varepsilon - ||R||_{pqr}. \tag{10}$$

We build an auxiliary matrix $V = [v_{ij}] \in \mathbf{R}^{m\times n}$ with components

$$v_{ij} = \begin{cases} -\delta, & \text{if} \quad i \in N_m, \quad x_j^0 = 1, \\ \delta, & \text{if} \quad i \in N_m, \quad x_j^0 = 0. \end{cases}$$

Using (3), calculate

$$||V||_{pq} = \delta n^{1/p} m^{1/q}. \tag{11}$$

It is evident that all the rows $v_i,\ i \in N_m$, of the matrix V are the same and consist of the components δ and $-\delta$. Denoting $d = v_i,\ i \in N_m$, we get the relation

$$d(x^0 - x) = -\delta ||x^0 - x||_1 < 0 \tag{12}$$

valid for any portfolio $x \neq x^0$ and, in particular, for the efficient portfolio $x \in P^s_m(R)$.

Let $R^0 \in \mathbf{R}^{m\times n\times s}$ be a perturbing matrix with cuts $R^0_k,\ k \in N_s$, set by the rule

$$R^0_k = \begin{cases} V - R_1, & \text{if} \quad k = 1, \\ -R_k, & \text{if} \quad k \neq 1. \end{cases}$$

Applying (10) and (11), get

$$||R^0||_{pqr} \leq ||V||_{pq} + ||R||_{pqr} = \delta n^{1/p} m^{1/q} + ||R||_{pqr} < \varepsilon.$$

Furthermore, taking into account the structure of the matrix V we derive

$$f_1(x^0, V) - f_1(x, V) = d(x^0 - x),$$

what with (12) gives

$$g_1(x^0, x, R_1 + R^0_1) = f_1(x^0, R_1 + R^0_1) - f_1(x, R_1 + R^0_1) =$$
$$= f_1(x^0, V) - f_1(x, V) = d(x^0 - x) = -\delta ||x^0 - x||_1 < 0.$$

Additionally, it is evident that

$$g_k(x^0, R_k + R^0_k) = 0, \quad k \in N_s \backslash \{1\}$$

Finally, we conclude that

$$x^0 \in X(x, R + R^0).$$

Hence, $x \notin P^s(R+R^0)$ if $x \in P^s(R)$. That is the set $P^s(R) \cap P^s(R+R^0)$ is empty. Resuming, we have $\rho^s_m(p, q, r) < \varepsilon$ for any number $\varepsilon > ||R||_{pqr}$. Consequently, $\rho^s_m(p, q, r) \leq ||R||_{pqr}$. □

From Theorem 1 the known result follows.

Corollary 1 [37]. *If $p = q = r = \infty$ then, for any $s, m \in \mathbf{N}$, the following bounds of the strong stability radius of the problem $Z^s_m(R)$ hold*

$$0 < \max_{x' \in P^s(R)} \min_{x \notin P^s(R)} \max_{k \in N_s} \frac{g_k(x, x', R_k)}{||x + x'||_1} \leq$$

$$\leq \rho^s_m(\infty, \infty, \infty) \leq \min_{x \notin P^s(R)} \max_{x' \in P^s(R)} \max_{k \in N_s} \frac{g_k(x, x', R_k)}{||x - x'||_1}.$$

5 Case of linear criteria ($m - 1$)

When $m = 1$ our investment problem becomes a vector (s-criteria) linear Boolean programming problem. We rewrite the problem in more convenient form

$$Z^s_1(R): \quad r_k x \to \max_{x \in X}, \quad k \in N_s,$$

where $x = (x_1, x_2, ..., x_n)^T \in X \subset \mathbf{R}^n$; $r_k \in \mathbf{R}^n$ is the k-th row of the matrix $R = [r_{kj}] \in \mathbf{R}^{s \times n}$. Such a case can be interpreted as a situation when the financial market state does not doubt the investor. As previously, we assume that the Hölder norms l_p and l_r, $p, r \in [1, \infty]$, are defined correspondingly in the project space $\mathbf{R}^n$ and in the criterial risk space $\mathbf{R}^s$. For the problem $Z^s_1(R)$ we will use the previous notations $P^s(R), P(x, R)$ etc.

In this linear case the lower bound of the problem $Z^s_1(R)$ strong stability radius $\rho^s_1(p, r)$ can be improved.

Theorem 2. *For any $p, r \in [1, \infty]$ and $s \in \mathbf{N}$, for the strong stability radius $\rho^s_1(p, r)$ of the non-trivial problem $Z^s_1(R)$ the following bounds are valid*

$$0 < \max\{\varphi^*, \psi^*\} \leq \rho^s_1(p, r) \leq \min\{\chi^*, ||R||_{pr}\},$$

where

$$\varphi^* = \varphi^*(p) = \min_{x \notin P^s(R)} \max_{x' \in P^s(x, R)} \min_{k \in N_s} \frac{r_k(x - x')}{||x - x'||_{p'}},$$

$$\psi^* = \psi^*(p, r) = \max_{x' \in P^s(R)} \min_{x \notin P^s(R)} \frac{||[R(x - x')]^+||_r}{||x - x'||_{p'}},$$

$$\chi^* = \chi^*(p, r) = n^{1/p} s^{1/r} \min_{x \notin P^s(R)} \max_{x' \in P^s(R)} \max_{k \in N_s} \frac{r_k(x - x')}{||x - x'||_1},$$

$$||R||_{pr} = ||(||r_1||_p, ||r_2||_p, ..., ||r_s||_p)||_r.$$

Proof. The upper bounds follow directly from Theorem 1.

From the evident formula

$$\forall x' \in P^s(R) \quad \forall x \notin P^s(R) \quad \exists k \in N_s \quad (r_k(x - x') > 0),$$

we conclude that

$$\psi^* = \psi^*(p, r) > 0.$$

Thus, the lower bound of the strong stability radius and the radius itself are positive numbers.

Now let us show that $\rho_1^s(p,r) \geq \varphi^*$. Suppose $\varphi^* > 0$ (otherwise the inequality is evident).

Let $R' \in \mathbf{R}^{s\times n}$ be a perturbing matrix with rows $r'_k \in \mathbf{R}^n,\ k \in N_s$ and the norm

$$||R'||_{pr} = ||(||r'_1||_p, ||r'_2||_p, ..., ||r'_s||_p)||_r < \varphi^*,$$

i.e. $R' \in \Omega_{pr}(\varphi^*)$. By the definition of φ^*, for any portfolio $x \notin P^s(R)$ there exists a portfolio $x^0 \in P(x, R)$ such that

$$\frac{r_k(x - x^0)}{||x - x^0||_{p'}} \geq \varphi^* > ||R'||_{pr} \geq ||r'_k||_p, \quad k \in N_s.$$

Having these inequalities and Hölder's inequality (2), derive

$$(r_k + r'_k)(x - x^0) \geq r_k(x - x^0) - ||r'_k||_p ||x - x^0||_{p'} > 0, \quad k \in N_s,$$

and, as a result, deduce

$$x \notin P(x, R + R').$$

Therefore, any non-efficient portfolio of the problem $Z_1^s(R)$ retains this non-efficiency in any perturbed problem $Z_1^s(R + R')$ with $R' \in \Omega_{pq}(\varphi^*)$ or, strictly, $\emptyset \neq P^s(R+R') \subseteq P^s(R)$. Thus, $P^s(R) \cap P^s(R+R') \neq \emptyset$ for any perturbing matrix $R' \in \Omega_{pr}(\varphi)$, i.e. $\rho_1^s(p,r) \geq \varphi^*$.

Further, remembering that $\psi^* > 0$, we show the inequality $\rho_1^s(p,r) \geq \psi^*$.

As earlier, let $R' \in \mathbf{R}^{s\times n}$ be a perturbing matrix with rows $r'_k \in \mathbf{R}^n,\ k \in N_s$ and the norm $||R'||_{pr} < \psi^*$, i.e $R' \in \Omega_{pq}(\psi^*)$.

By the definition of ψ^*, there exists a portfolio $x^0 \in P^s(R)$ such that for any portfolio $x \notin P^s(R)$

$$0 < \psi^* ||x - x^0||_{p'} \leq ||[R(x - x^0)]^+||_r. \tag{13}$$

First, let us show that

$$\forall x \notin P^s(R) \quad \forall R' \in \Omega_{pr}(\psi^*) \quad (x \notin X(x^0, R + R')). \tag{14}$$

Suppose that there exist a portfolio $\widetilde{x} \notin P^s(R)$ and a perturbing matrix $\widetilde{R} \in \Omega_{pr}(\psi^*)$ with rows $\widetilde{r_k},\ k \in N_s$, such that $\widetilde{x} \in X(x^0, R + \widetilde{R})$. Then for any $k \in N_s$ we have

$$(r_k + \widetilde{r_k})\widetilde{x} \leq (r_k + \widetilde{r_k})x^0,$$

and, consequently,

$$r_k(\widetilde{x} - x^0) \leq \widetilde{r_k}(x^0 - \widetilde{x}).$$

Having this, easily get the inequality

$$[r_k(\widetilde{x}-x^0)]^+ \le |\widetilde{r_k}(x^0-\widetilde{x})|,$$

that with Hólder's inequality (2) gives us

$$[r_k(\widetilde{x}-x^0)]^+ \le ||\widetilde{r_k}||_p||\widetilde{x}-x^0||_{p'}.$$

This means that

$$||[R(\widetilde{x}-x^0)]^+||_r \le ||\widetilde{R}||_{pr}||\widetilde{x}-x^0||_{p'} < \psi^*||\widetilde{x}-x^0||_{p'}.$$

This derived contradiction to (13) proves (14).

Next, we show that there exists a portfolio $x^* \in P^s(R) \cap P^s(R+R')$ in the case where $R' \in \Omega_{pr}(\psi^*)$.

If the portfolio $x^0 \in P^s(R)$ from (13) is in the Pareto set $P^s(R+R')$ then $x^* = x^0$. If $x^0 \notin P^s(R+R')$ then due to the external stability property of the Pareto set $P^s(R+R')$ (see, e.g., [46], p. 39) we can choose a portfolio $x^* \in P^s(R+R')$ such that $x^* \in X(x^0, R+R')$. Using the proved formula (14), we easily find out that $x^* \in P^s(R)$. Therefore, the inequality $\rho_1^s(p,r) \ge \psi^*$ is proved. □

From Theorem 2 the two known results follow.

Corollary 2 **[36]** *(see also* **[10]***). If $p = r = \infty$ then for any $s \in \mathbf{N}$ the following bounds of the strong stability radius of the linear non-trivial problem $Z_1^s(R)$ hold*

$$\psi^*(\infty,\infty) = \max_{x' \in P^s(R)} \min_{x \notin P^s(R)} \max_{k \in N_s} \frac{r_k(x-x')}{||x-x'||_1} \le$$

$$\le \rho_1^s(\infty,\infty) \le \chi^*(\infty,\infty) = \min_{x \notin P^s(R)} \max_{x' \in P^s(R)} \max_{k \in N_s} \frac{r_k(x-x')}{||x-x'||_1}.$$

Corollary 3 **[34].** *If $p = \infty$ then for any $r \in [1,\infty]$ the following bounds of the strong stability radius of the linear scalar (single criterion) non-trivial problem $Z_1^1(R)$, $R \in \mathbf{R}^{1\times n}$, hold*

$$\rho_1^1(\infty,r) = \varphi^*(\infty) = \chi^*(\infty,r) = \min_{x \notin P^1(R)} \max_{x' \in P^1(R)} \frac{R(x-x')}{||x-x'||_1}.$$

In another particular case the lower bound takes the following form.

Corollary 4. *If $p = 1$, $r \in [1,\infty]$, and $s \in \mathbf{N}$ then*

$$\rho_1^s(1,r) \ge \max\{\varphi^*(1), \quad \psi^*(1,r),$$

where

$$\varphi^*(1) = \min_{x \notin P^s(R)} \max_{x' \in P(x,R)} \min_{k \in N_s} r_k(x-x'),$$

$$\psi^*(1,r) = \max_{x' \in P^s(R)} \min_{x \notin P^s(R)} ||[R(x-x')]^+||_r.$$

Here is one more case where a formula is valid for the strong stability radius.

Consider a linear problem $Z_1^s(R),\ s \in \mathbf{N}$, with the Hölder norms l_p and l_r in the spaces $\mathbf{R}^n$ and $\mathbf{R}^s$. A stability radius of an efficient portfolio $x^0 \in P^s(R)$ of the problem $Z_1^s(R)$ is the number

$$\rho_1^s(x^0,p,r) = \begin{cases} \sup \Theta_{pr}, & \text{if} \quad \Theta_{pr} \neq \emptyset, \\ 0, & \text{if} \quad \Theta_{pr} = \emptyset. \end{cases}$$

where

$$\Theta_{pr} = \{\varepsilon > 0 :\ \forall R' \in \Omega_{pr}(\varepsilon)\ (x^0 \in P^s(R+R'))\}.$$

For the case $P^s(R) = \{x^0\}$, it is easy to see that

$$\rho_1^s(p,r) = \rho_1^s(x^0,p,r).$$

Therefore, using the known formula (see [47, 48]) for the stability radius of an efficient solution of the linear boolean programming problem with the Hölder norms, we state the following

Corollary 5. *If $P^s(R) = \{x^0\}$ then for any $p,\ r \in [1,\infty]$ and $s \in \boldsymbol{N}$ the strong stability radius of the problem $Z_1^s(R)$ is calculated by the formula*

$$\rho_1^s(p,r) = \min_{x \in X \setminus \{x^0\}} \frac{||[R(x-x^0)]^+||_r}{||(x-x^0)||_{p'}}.$$

The results presented in the work were partially reported at the 28th European Conference on Operational Research (EURO-2016) [49].

In conclusion we remark that in [8] similar bounds of the stability radius are found for the multicriteria linear Boolean problem $Z_1^s(R)$ with the Hölder metrics in the parameter spaces.

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