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Nontrivial convex covers of trees

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Abstract. We establish conditions for the existence of nontrivial convex covers and nontrivial convex partitions of trees. We prove that a tree G on $n \geq 4$ vertices has a nontrivial convex p-cover for every p, $2 \leq p \leq \varphi_{cn}^{max}(G)$. Also, we prove that it can be decided in polynomial time whether a tree on $n \geq 6$ vertices has a nontrivial convex p-partition, for a fixed p, $2 \leq p \leq \lfloor \frac{n}{3} \rfloor$.

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1 Introduction

We denote by G a connected tree with vertex set $X(G)$, $|X(G)| = n$, and edge set $U(G)$, $|U(G)| = m$. We denote by $d(x, y)$ the *distance* between two vertices x and y of G [3]. The *diameter* of G, denoted $diam(G)$, is the length of the shortest path between the most distant vertices of G. The *neighborhood* of a vertex $x \in X$ is the set of all vertices $y \in X$ such that $x \sim y$, and it is denoted by $\Gamma(x)$.

We remind some notions defined in [1, 2]. The *metric segment*, denoted $\langle x, y\rangle$, is the set of all vertices lying on a shortest path between vertices $x, y \in X(G)$. A subset $S \subseteq X(G)$ is called *convex* if $\langle x, y\rangle \subseteq S$, for all $x, y \in S$.

By [6], a family of sets $\mathcal{P}(G)$ is called a *nontrivial convex cover* of a graph G if the following conditions hold:

1) every set of $\mathcal{P}(G)$ is convex in G;
2) every set S of $\mathcal{P}(G)$ satisfies inequalities: $3 \leq |S| \leq |X(G)| - 1$;
3) $X(G) = \bigcup_{Y \in \mathcal{P}(G)} Y$;
4) $Y \not\subseteq \bigcup_{\substack{Z \in \mathcal{P}(G) \\ Z \neq Y}} Z$ for every $Y \in \mathcal{P}(G)$.

If $|\mathcal{P}(G)| = p$, then this family is called a *nontrivial convex p-cover* of G. In particular, $\mathcal{P}(G)$ is called a *nontrivial convex partition* of G if it is a nontrivial convex cover of G and any two sets of $\mathcal{P}(G)$ are disjoint [6]. A nontrivial convex p-cover of G is called a *nontrivial convex p-partition* if it is a nontrivial convex partition of G.

Generally, convex p-covers and convex p-partitions of graphs are examined in [4–8]. Particularly, nontrivial convex p-cover and nontrivial convex p-partition are defined in [6], where it is proved that it is NP-complete to decide whether a graph has a nontrivial convex p-partition or a nontrivial convex p-cover for a fixed $p \geq 2$. Also, in [8] it is proved that it is NP-complete to decide whether a graph has any nontrivial convex partition. Further, there is specific interest in studying nontrivial convex p-covers and nontrivial convex p-partitions for different classes of graphs. In this paper we study nontrivial convex cover problem of trees.

The greatest $p \geq 2$ for which a graph G has a nontrivial convex p-cover is said to be the *maximum nontrivial convex cover number* $\varphi_{cn}^{max}(G)$. Similarly, we define the *maximum nontrivial convex partition number* $\theta_{cn}^{max}(G)$. A nontrivial convex cover that corresponds to $\varphi_{cn}^{max}(G)$ is denoted by $\boldsymbol{\mathcal{P}}_{\varphi_{cn}^{max}}(G)$. In the same way we denote by $\boldsymbol{\mathcal{P}}_{\theta_{cn}^{max}}(G)$ a nontrivial convex partition that corresponds to $\theta_{cn}^{max}(G)$.

A vertex $x \in X(G)$ is called *resident* in $\boldsymbol{\mathcal{P}}(G)$ if x belongs to only one set of $\boldsymbol{\mathcal{P}}(G)$. Let $L = [x^1, x^2, \ldots, x^k]$ be a vertex path of a tree G. By $R_L(x)$ we denote the set of vertices $v \in X(G)$ for which there is a path $L' = [x, \ldots, v]$ such that L' has no elements of L except x, where $x \in L$.

2 Existence of nontrivial convex covers

Recall that a *terminal vertex* of a tree G is a vertex of degree 1.

Lemma 1. *A tree G with $diam(G) \geq 3$ has a nontrivial convex cover.*

Proof. We know from [7] that a tree on $n \geq 4$ vertices has a nontrivial convex 2-cover. Since a tree with $diam(G) \geq 3$ has at least $n \geq 4$ vertices, we obtain that G with $diam(G) \geq 3$ has a nontrivial convex cover. □

Theorem 1. *Let G be a tree with $diam(G) \geq 3$. There exists a maximum nontrivial convex cover $\boldsymbol{\mathcal{P}}_{\varphi_{cn}^{max}}(G)$ such that every terminal vertex of G is resident in $\boldsymbol{\mathcal{P}}_{\varphi_{cn}^{max}}(G)$ and any two terminal vertices do not belong to the same set of $\boldsymbol{\mathcal{P}}_{\varphi_{cn}^{max}}(G)$.*

Proof. From Lemma 1 we know that G has a nontrivial convex cover. Let $\boldsymbol{\mathcal{P}}_{\varphi_{cn}^{max}}(G)$ be a maximum nontrivial convex cover of G, where there is at least one terminal vertex x that is not resident in $\boldsymbol{\mathcal{P}}_{\varphi_{cn}^{max}}(G)$. Since x is a terminal vertex of G and $diam(G) \geq 3$, we see that there is a vertex y adjacent to x that is adjacent to the set of nonterminal vertices S and to the set of terminal vertices S' of G such that $S \neq \varnothing$ and $S' \neq \varnothing$.

We consider two cases.

1) Suppose that S contains a vertex z that is not resident in $\boldsymbol{\mathcal{P}}_{\varphi_{cn}^{max}}(G)$. Firstly, we replace vertex x by vertex z in every set of $\boldsymbol{\mathcal{P}}_{\varphi_{cn}^{max}}(G)$ that contains x. Secondly, we add a convex set $\{x, y, z\}$ to $\boldsymbol{\mathcal{P}}_{\varphi_{cn}^{max}}(G)$. Further, we obtain a new nontrivial convex cover $\boldsymbol{\mathcal{P}}(G)$ in which x is resident, where $|\boldsymbol{\mathcal{P}}(G)| > |\boldsymbol{\mathcal{P}}_{\varphi_{cn}^{max}}(G)|$. Hence, we get a contradiction.

2) Now suppose that every vertex of S is resident in $\boldsymbol{\mathcal{P}}_{\varphi_{cn}^{max}}(G)$. Firstly, we choose a vertex z of S and a set Z of $\boldsymbol{\mathcal{P}}_{\varphi_{cn}^{max}}(G)$ that contains z. Secondly, we replace vertex x by vertex z in every set of $\boldsymbol{\mathcal{P}}_{\varphi_{cn}^{max}}(G) \backslash \{Z\}$ which contains x. After, we add x and y to set Z. Finally, we get a new nontrivial convex cover $\boldsymbol{\mathcal{P}}(G)$ in which x is resident, where $|\boldsymbol{\mathcal{P}}(G)| = |\boldsymbol{\mathcal{P}}_{\varphi_{cn}^{max}}(G)|$. On the other hand, if now set S' contains one more vertex that is not resident in $\boldsymbol{\mathcal{P}}(G)$, then taking into account case 1) we obtain a contradiction.

Consequently, there exists a maximum nontrivial convex cover $\mathcal{P}_{\varphi_{cn}^{max}}(G)$ such that every terminal vertex of G is resident in $\mathcal{P}_{\varphi_{cn}^{max}}(G)$.

Now suppose that there are at least two terminal vertices x and y which belong to the same set S of $\mathcal{P}_{\varphi_{cn}^{max}}(G)$.

Let us consider two cases.

1) Assume that $|S| \geq 4$. In this case, we replace set S in $\mathcal{P}_{\varphi_{cn}^{max}}(G)$ by two convex sets $S' = S\backslash\{x\}$, $|S'| \geq 3$, and $S'' = S\backslash\{y\}$, $|S''| \geq 3$. Further, we obtain a new nontrivial convex cover $\mathcal{P}(G)$ in which x and y belong to different sets, where $|\mathcal{P}(G)| > |\mathcal{P}_{\varphi_{cn}^{max}}(G)|$. Whence, we have a contradiction.

2) Assume now that $|S| = 3$. In our case $S = \{x, y, z\}$, where $\Gamma(x) = \Gamma(y) = \{z\}$. As above, note that set $\Gamma(z)\backslash\{x, y\}$ contains at least one nonterminal vertex h.

If h is not resident in $\mathcal{P}_{\varphi_{cn}^{max}}(G)$, then we replace S by two convex sets $\{x, z, h\}$ and $\{y, z, h\}$. Further, we obtain a new nontrivial convex cover $\mathcal{P}(G)$ in which x and y belong to different sets, where $|\mathcal{P}(G)| > |\mathcal{P}_{\varphi_{cn}^{max}}(G)|$. Whence, we have a contradiction.

If all nonterminal vertices of $\Gamma(z)\backslash\{x, y\}$ are resident in $\mathcal{P}_{\varphi_{cn}^{max}}(G)$, then we choose a set H that contains h. Further, we subtract x from S and add it to H. Also, we add h to S and z to H. Consequently, we obtain a new nontrivial convex cover $\mathcal{P}(G)$ in which x and y belong to different sets, where $|\mathcal{P}(G)| = |\mathcal{P}_{\varphi_{cn}^{max}}(G)|$.

It follows that any two terminal vertices do not belong to the same set of $\mathcal{P}_{\varphi_{cn}^{max}}(G)$. □

As a consequence of Theorem 1, we obtain 3 corollaries.

Corollary 1. *Let G be a tree with $diam(G) \geq 3$ and p terminal vertices. Then, $\varphi_{cn}^{max}(G) \geq p$.*

Corollary 2. *Let G be a tree with $diam(G) \geq 3$ and p terminal vertices, where every nonterminal vertex of G is adjacent to at least one terminal vertex. Then, $\varphi_{cn}^{max}(G) = p$.*

Corollary 3. *Let G be a tree with $3 \leq diam(G) \leq 5$ and p terminal vertices. Then, $\varphi_{cn}^{max}(G) = p$.*

Theorem 2. *A tree G on $n \geq 4$ vertices has a nontrivial convex p-cover, for every p, $2 \leq p \leq \varphi_{cn}^{max}(G)$.*

Proof. It is know that a tree on $n \geq 4$ vertices has a nontrivial convex cover [7]. Let G be a tree on $n \geq 4$ vertices and let $\mathcal{P}_{\varphi_{cn}^{max}}(G)$ be a maximum nontrivial convex cover of G. If $\varphi_{cn}^{max}(G) = 2$, then the theorem is proved. Let us analyze case $\varphi_{cn}^{max}(G) \geq 3$. We use the following procedure. We select two sets X_1 and X_2 of $\mathcal{P}_{\varphi_{cn}^{max}}(G)$ such that $x_1 \in X_1$ and $x_2 \in X_2$, where x_1 is adjacent to x_2. Since union of sets X_1 and X_2 is convex in G, excluding from $\mathcal{P}_{\varphi_{cn}^{max}}(G)$ sets X_1, X_2 and adding set $X_1 \cup X_2$, we obtain a new family $\mathcal{P}(G)$ that covers G by $p = \varphi_{cn}^{max}(G) - 1$ nontrivial convex sets. If $p = 2$, then the theorem is correct. Conversely, if $p \geq 3$, then repeating $\varphi_{cn}^{max}(G) - 3$ times this procedure for $\mathcal{P}(G)$ we obtain a nontrivial convex 2-cover of G. Consequently, the theorem is proved. □

Next, we analyze nontrivial convex partitions of trees. The following two families of trees $\mathcal{A}$ and $\mathcal{B}$ are needed for the sequel.

$\mathcal{A}$ is a family of trees G which satisfy the following conditions:

1) $X(G) = \{x, y, x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_{k'}\}$, where $k, k' \geq 2$;
2) $U(G) = \{(x, y)\} \cup \bigcup_{i=1}^{k}\{(x, x_i)\} \cup \bigcup_{i=1}^{k'}\{(y, y_i)\}$.

$\mathcal{B}$ is a family of trees G which are constructed as follows:

1) We choose $k \geq 0$, $k' \geq 2$, $k_1 \geq 2$ and for every i, $2 \leq i \leq k'$, we select $k_i \geq 1$;
2) If $k \geq 1$, then we get $X = \{x_0\} \cup \bigcup_{i=1}^{k}\{x_i\}$ and $U = \bigcup_{i=1}^{k}\{(x_0, x_i)\}$, otherwise we get $X = \{x_0\}$ and $U = \varnothing$;
3) We obtain sets $X(G) = X \cup \bigcup_{i=1}^{k'}\bigcup_{j=0}^{k_i}\{x_i^j\}$ and $U(G) = U \cup \bigcup_{i=1}^{k'}\{(x_0, x_i^0)\} \cup \bigcup_{i=1}^{k'}\bigcup_{j=1}^{k_i}\{(x_i^0, x_i^j)\}$.

It can easily be checked that diameter of all trees of $\mathcal{A}$ is 3, and diameter of all trees of $\mathcal{B}$ is 4. Moreover, every tree of $\mathcal{A}$ and every tree of $\mathcal{B}$ has at least 6 vertices.

Algorithm 1. Determines whether a tree belongs to one of families: $\mathcal{A}$, $\mathcal{B}$.

Input: *A tree G.*

Output: *YES-$\mathcal{A}$: G belongs to $\mathcal{A}$, or YES-$\mathcal{B}$: G belongs to $\mathcal{B}$, or NO: G does not belong to any of the families.*

Step 1) If $|X(G)| \leq 5$, then return NO.

Step 2) Compute $diam(G)$. If $diam(G) \leq 2$ or $diam(G) \geq 5$, then return NO; otherwise, if $diam(G) = 4$, then go to Step 4).

Step 3) Choose two different vertices $x, y \in X(G)$ such that $|\Gamma(x)| \geq 2$ and $|\Gamma(y)| \geq 2$. Next, if $|\Gamma(x)| \geq 3$ and $|\Gamma(y)| \geq 3$, then return YES–$\mathcal{A}$; otherwise return NO.

Step 4) Check whether there exist two different terminal vertices $x, y \in X(G)$ such that $\Gamma(x) \cap \Gamma(y) \neq \varnothing$ and there is a terminal vertex $z \in X(G)$, where $d(x, z) = diam(G)$. If there exist such vertices $x, y \in X(G)$, then return YES–$\mathcal{B}$; otherwise return NO.

Theorem 3. *Algorithm 1 determines in time $O(n^3)$ whether a tree G belongs to one of families: $\mathcal{A}$, $\mathcal{B}$.*

Proof. Correctness of the algorithm results from structure of trees of families $\mathcal{A}$ and $\mathcal{B}$. Step 1) runs in constant time. If we use Floyd–Warshall algorithm for finding the diameter of a graph, then the complexity of step 2) is $O(n^3)$. It is clear that step 3) is executed in $O(n)$ time. Since Floyd–Warshall algorithm is executed in the step 2), we know all pairs of vertices for which distance is equal to $diam(G)$. Further, step 4) runs in $O(n^2)$ time. Based on the mentioned facts, the execution time of the algorithm is $O(n^3)$. $\square$

Theorem 4. *A tree G has a nontrivial convex 2-partition if and only if one of the following conditions holds:*

1) *$diam(G) \geq 5$;*

2) *$G \in \mathcal{A}$;*

3) *$G \in \mathcal{B}$.*

Proof. It is clear that if a tree G has a nontrivial convex 2-partition, then inequality $n \geq 6$ holds. Let us analyze nontrivial convex 2-partition of G in dependency on its diameter.

Suppose $diam(G) = 2$. Here G is a star graph. It can simply be verified that a star graph has no nontrivial convex 2-partition.

Suppose $diam(G) = 3$. We choose two vertices $x, x' \in X(G)$ such that there is a path $L = [x, y, z, x']$ and length of L is equal to diameter of G. Evidently, L is a unique path between vertices x and x' and vertices x, x' are terminal, i.e., $\Gamma(x) = y$ and $\Gamma(x') = z$. From relation $n \geq 6$, it follows that G contains at least two vertices different from x, y, z, x'. Assume that $v \in X(G)$ is different from vertices x, y, z, x', and $v \in R_L(y)$ such that $d(y, v) \geq 2$, or $v \in R_L(z)$ and $d(z, v) \geq 2$. Further, we obtain a contradiction, because $d(y, x') = d(z, x) = 2$ and length of paths $L^1 = [x', z, y, \dots, v]$, $L^2 = [x, y, z, \dots, v]$ is greater then or equal to 4. Consequently, all vertices of G different from x, y, z, x' are adjacent only to y or to z. It can easily be checked that if y is adjacent only to x and z, or z is adjacent only to x' and y, then G has no nontrivial convex 2-partition. In the converse case G has a nontrivial convex 2-partition:

$$\mathcal{P}(G) = \{\{x, y\} \cup R_L(y), \{z, x'\} \cup R_L(z)\}.$$

In other words, if $diam(G) = 3$, then G has a nontrivial convex 2-partition if and only if $G \in \mathcal{A}$.

Suppose $diam(G) = 4$. We choose two vertices $x, x' \in X(G)$ such that there is a path $L = [x, y, z, h, x']$. Length of the L is equal to diameter of G and vertices x and x' are terminal. Since $n \geq 6$, tree G contains at least one vertex v different from x, y, z, h, x'. If v is adjacent to y or to h, then G has a nontrivial convex 2-partition:

$$\mathcal{P}(G) = \{\{x, y\} \cup R_L(y), \{z, h, x'\} \cup R_L(z) \cup R_L(h)\} \text{ or}$$

$$\mathcal{P}(G) = \{\{x, y, z\} \cup R_L(y) \cup R_L(z), \{h, x'\} \cup R_L(h)\}, \text{ respectively.}$$

Assume that there are no vertices different from x, y, z, h, x' which are adjacent to y or to h. Then, there exist vertices z' different from y and h which are adjacent to z. If we have $|\Gamma(z')| = 1$ or $|\Gamma(z')| = 2$, for all such z', then it is not hard to check that G has no nontrivial convex 2-partition. Now assume that there are at least two vertices z'' and z''' different from z and adjacent to z', i.e., $|\Gamma(z')| \geq 3$. In this case, we obtain a path $L = [z'', z', z, y, x]$. As mentioned above, it follows that G has a nontrivial convex 2-partition. Equivalently, if $diam(G) = 4$, then G has a nontrivial convex 2-partition if and only if $G \in \boldsymbol{\mathcal{B}}$.

Suppose $diam(G) \geq 5$. There are two vertices x and x' in G such that $d(x, x') = diam(G)$. Let $L = [x, x^1, x^2, \ldots, x^k, x']$, $k \geq 4$, be a path between x and x'. L contains at least 6 vertices. Moreover, L is a unique path between x and x'. Hence, paths $[x, x^1, x^2]$ and $[x^3, \ldots, x^k, x']$ generate a nontrivial convex 2-partition of G:

$$\boldsymbol{\mathcal{P}}(G) = \{\{x\} \cup \bigcup_{i=1}^{2} R_L(x^i), \{x'\} \cup \bigcup_{i=3}^{k} R_L(x^i)\}.$$

The theorem is proved. □

Theorem 5. *If a tree G on $n \geq 6$ vertices has a nontrivial convex partition, then G has a nontrivial convex p-partition, for every p, $2 \leq p \leq \theta_{cn}^{max}(G)$.*

Proof. If a tree G has a nontrivial convex partition, then there is a maximum nontrivial convex partition $\boldsymbol{\mathcal{P}}_{\theta_{cn}^{max}}(G)$. If $\theta_{cn}^{max}(G) = 2$, then the theorem is proved. If $\theta_{cn}^{max}(G) \geq 3$, then repeating $\theta_{cn}^{max}(G) - 2$ times the procedure described in proof of Theorem 2 we obtain a nontrivial convex 2-partition of G. Hence, G has a nontrivial convex p-partition, for every p, $2 \leq p \leq \theta_{cn}^{max}(G)$. □

The following corollaries are true.

Corollary 4. *If a tree G on $n \geq 6$ vertices has a nontrivial convex partition, then G has a nontrivial convex 2-partition.*

Corollary 5. *A tree G has a nontrivial convex p-partition, for every p, $2 \leq p \leq \theta_{c}^{max}(G)$, if and only if one of the following conditions holds:*

1) $diam(G) \geq 5$;

2) $G \in \boldsymbol{\mathcal{A}}$;

3) $G \in \boldsymbol{\mathcal{B}}$.

3 Determination of nontrivial convex partitions

Let C be the set of all terminal vertices of G. Let x be a vertex of G for which $|\Gamma(x) \cap C| \geq 2$ or there is another vertex $y \in \Gamma(x)$ such that $\Gamma(y) = \{x, z\}$, $z \in C$.

For x that satisfies the announced properties we define the set:

$$S_x = \{x\} \cup \{v \in X(G) : v \in \Gamma(x) \cap C\} \cup \{v_1, v_2 \in X(G) : \Gamma(v_1) = \{x, v_2\}, v_2 \in C\}.$$

The set S_x is called a *nontrivial terminal set* of G. Note that S_x is a nontrivial convex set of G. We say that a terminal vertex z of a tree G corresponds to a nontrivial terminal set S_x of G if S_x contains z.

Let $\boldsymbol{\mathcal{S}}(G)$ be the family of all nontrivial terminal sets of G.

Lemma 2. *All nontrivial terminal sets of* $\boldsymbol{\mathcal{S}}(G)$ *are disjoint.*

Proof. Suppose that there are at least two different nontrivial terminal sets S_x and S_y of $\boldsymbol{\mathcal{S}}(G)$ such that $S_x \cap S_y \neq \varnothing$. By the definition of nontrivial terminal set, we have $x = y$ and consequently $S_x = S_y$. Whence, we obtain a contradiction. □

Lemma 3. $\boldsymbol{\mathcal{S}}(G)$ *is unique for* G.

Proof. Correctness of the lemma results from the definition of nontrivial terminal set and Lemma 2. □

Lemma 4. *Every set of* $\boldsymbol{\mathcal{S}}(G)$ *belongs to exactly one set of* $\boldsymbol{\mathcal{P}}_{\theta_{cn}^{max}}(G)$ *such that any two nontrivial terminal sets of* $\boldsymbol{\mathcal{S}}(G)$ *do no belong to the same set of* $\boldsymbol{\mathcal{P}}_{\theta_{cn}^{max}}(G)$.

Proof. From the definition of nontrivial terminal set and definition of nontrivial convex partition, it follows that every set of $\boldsymbol{\mathcal{S}}(G)$ belongs to exactly one set of $\boldsymbol{\mathcal{P}}_{\theta_{cn}^{max}}(G)$. Suppose that there is a set C of $\boldsymbol{\mathcal{P}}_{\theta_{cn}^{max}}(G)$ that contains at least two different nontrivial terminal sets of G. Let $\boldsymbol{\mathcal{S}}_C$ be the family of all nontrivial terminal sets which are in C and $k = |\boldsymbol{\mathcal{S}}_C| \geq 2$. By Lemmas 2 and 3, we know that $\boldsymbol{\mathcal{S}}(G)$ is unique for G and all nontrivial terminal sets are disjoint. Further, we separate C into disjoint nontrivial convex sets $S_1, S_1, ..., S_k$, where every set contains exactly one nontrivial terminal set of $\boldsymbol{\mathcal{S}}_C$. We select a vertex x from all vertices of C which remain uncovered by new nontrivial convex sets such that x is adjacent to a vertex y, $y \in S$, $S \in \{S_1, S_1, ..., S_k\}$, and further add x to S. If some uncovered vertices remain, then we repeat the above procedure. Since $k \geq 2$, we get a new convex cover $\boldsymbol{\mathcal{P}}(G)$ of G such that $|\boldsymbol{\mathcal{P}}(G)| > |\boldsymbol{\mathcal{P}}_{\theta_{cn}^{max}}(G)|$. Hence, we have a contradiction. □

Lemma 5. *A tree* G *on* $n \geq 3$ *vertices with* $2 \leq diam(G) \leq 4$ *has at least one nontrivial terminal set.*

Proof. From the definition of nontrivial terminal set, we get that every tree G of order $n \geq 3$ with $diam(G) = 2$ contains exactly one nontrivial terminal set $S_x = X(G)$. It can easily be checked that a tree $G \in \boldsymbol{\mathcal{A}}$ has exactly two nontrivial terminal sets, and a tree $G \in \boldsymbol{\mathcal{B}}$ has at least two nontrivial terminal sets. Similarly, if a tree G with $diam(G) = 3$ does not belong to $\boldsymbol{\mathcal{A}}$, or $diam(G) = 4$ and $G \notin \boldsymbol{\mathcal{B}}$, then G has exactly one nontrivial terminal set $S_x = X(G)$. □

Lemma 6. *A tree* G *with* $diam(G) \geq 5$ *has at least two nontrivial terminal sets.*

Proof. Let G be a tree with $diam(G) \geq 5$. Let x and y be two terminal vertices such that $d(x, y) = diam(G)$. Assume that x does not correspond to any nontrivial terminal set. By the definition of nontrivial terminal set, we see that x is adjacent to a vertex z that is adjacent to at least two vertices different from x and all of them are nonterminal. Let z^1, z^2, $\ldots$, z^k, where $k \geq 2$, be vertices different from x and adjacent to z. Path between x and y contains exactly one vertex $z' \in \{z^1, z^2, \ldots, z^k\}$. Since z^1, z^2, $\ldots$, z^k are nonterminal vertices, to every vertex $z'' \in \{z^1, z^2, \ldots, z^k\} \backslash \{z'\}$ corresponds a vertex z^* different from z such that z^* is adjacent to z''. Since for every two vertices of G there is only one path that connects them, this yields that for every z^* we get $d(z^*, y) > diam(G)$. Consequently, we obtain a contradiction. Similarly, we get a contradiction if assume that y does not correspond to any nontrivial terminal set. Since $diam(G) \geq 5$, vertices x and y correspond to different nontrivial terminal sets. Hence, a connected tree G with $diam(G) \geq 5$ has at least two nontrivial terminal sets. □

Algorithm 2. Determines $\boldsymbol{\mathcal{S}}(G)$ for a tree G.

Input: *A tree G.*

Output: $\boldsymbol{\mathcal{S}}(G)$.

Step 1) Fix set $\boldsymbol{\mathcal{S}}(G) = \varnothing$.

Step 2) Determine all terminal vertices C of G.

Step 3) Go through all vertices $x \in X(G) \backslash C$. If for a vertex x of G we have $|\Gamma(x) \cap C| \geq 2$ or there is another vertex $y \in \Gamma(x)$ such that $\Gamma(y) = \{x, z\}$, where $z \in C$, then we define the set $S_x = \{x\} \cup \{v \in X(G) : v \in \Gamma(x) \cap C\} \cup \{v_1, v_2 \in X(G) : \Gamma(v_1) = \{x, v_2\}, v_2 \in C\}$ and then add it to $\boldsymbol{\mathcal{S}}(G)$.

Step 4) Return $\boldsymbol{\mathcal{S}}(G)$.

Theorem 6. *Algorithm 2 determines family of nontrivial terminal sets $\boldsymbol{\mathcal{S}}(G)$ of a tree G in time $O(n^2)$.*

Proof. Correctness of the algorithm results from Lemmas 2, 3, 5 and 6. Clearly, steps 1) and 4) run in constant time. The step 2) operates in $O(n)$ and the step 3) is executed in $O(n^2)$ time. Further, the execution time of the algorithm is $O(n^2)$. □

Let $\boldsymbol{\mathcal{F}}(G)$ be a family of subtrees that is obtained after elimination of all nontrivial terminal sets of $\boldsymbol{\mathcal{S}}(G)$ from a tree G.

Theorem 7. *The following relation holds:*

$$\theta_{cn}^{max}(G) = \begin{cases} |\boldsymbol{\mathcal{S}}(G)| + \sum_{G' \in \boldsymbol{\mathcal{F}}(G)} \theta_{cn}^{max}(G'), & \text{if } |X(G)| \geq 3; \\ 0, & \text{if } 0 \leq |X(G)| \leq 2. \end{cases}$$

Proof. By Lemma 4, we conclude that through the elimination of all nontrivial terminal sets of $\boldsymbol{\mathcal{S}}(G)$ from G, in fact, we eliminate minimal nontrivial convex sets of G which contain nontrivial terminal sets. Besides, after elimination of all nontrivial terminal sets of $\boldsymbol{\mathcal{S}}(G)$ from G we obtain a family of subtrees $\boldsymbol{\mathcal{F}}(G)$ such that some of them also contain nontrivial terminal sets.

If $0 \leq |X(G)| \leq 2$, then evidently $\theta_{cn}^{max}(G) = 0$. In the contrary case, if $|X(G)| \geq 3$, then taking into account Lemmas 2 – 6, we obtain:

$$\theta_{cn}^{max}(G) = |\boldsymbol{\mathcal{S}}(G)| + \sum_{G' \in \boldsymbol{\mathcal{F}}(G)} \theta_{cn}^{max}(G').$$

The theorem is proved.

□

Next, we propose recursive procedure $Max\theta(G)$ that determines the number $\theta_{cn}^{max}(G)$ of a tree G. After, we prove that this procedure executes in polynomial time.

$Max\theta(G)$

Input: *A tree G.*

Output: *$\theta_{cn}^{max}(G)$.*

Step 1) If $0 \leq |X(G)| \leq 2$, then return 0.

Step 2) Apply Algorithm 2, i.e., determine $\boldsymbol{\mathcal{S}}(G)$, remove every nontrivial terminal set of $\boldsymbol{\mathcal{S}}(G)$ from G and obtain $\boldsymbol{\mathcal{F}}(G)$.

Step 3) For every tree G' of $\boldsymbol{\mathcal{F}}(G)$ apply procedure $Max\theta(G')$ and after return the number $\theta_{cn}^{max}(G) = |\boldsymbol{\mathcal{S}}(G)| + \sum_{G' \in \boldsymbol{\mathcal{F}}(G)} Max\theta(G')$.

Theorem 8. *Procedure $Max\theta(G)$ determines the number $\theta_{cn}^{max}(G)$ of a tree G in time $O(n^3)$.*

Proof. From Theorem 7, we know that for a tree G procedure $Max\theta(G)$ returns the number $\theta_{cn}^{max}(G)$. By Theorem 6 we obtain that in general case the processing time of procedure $Max\theta(G)$ is:

$$T(n) = \sum_{i=1}^{k} T(n_i) + O(n^2),$$

where $\sum_{i=1}^{k} n_i \leq n - 6$ and $k \geq 1$.

The worst behavior of procedure $Max\theta(G)$ occurs when in every examined tree there are exactly two nontrivial terminal sets which consist of three elements such that after their elimination a single subtree remains. In this case, processing time of $Max\theta(G)$ is:

$$T(n) = T(n-6) + O(n^2).$$

Using arithmetic progression, we get $T(n) = O(n^3)$. Finally, the procedure $Max\theta(G)$ determines number $\theta_{cn}^{max}(G)$ in time $O(n^3)$. □

Corollary 6. *It can be decided in time $O(n^3)$ whether a tree G on $n \geq 6$ vertices has a nontrivial convex p-partition, for a fixed p, $2 \leq p \leq \lfloor \frac{n}{3} \rfloor$.*

4 Conclusion

In this paper we establish conditions for the existence of nontrivial convex covers and nontrivial convex partitions of trees. We prove that a tree G on $n \geq 4$ vertices has a nontrivial convex p-cover for every p, $2 \leq p \leq \varphi_{cn}^{max}(G)$. In addition, we prove that if a tree G has a nontrivial convex partition, then G has a nontrivial convex p-partition for every p, $2 \leq p \leq \theta_{cn}^{max}(G)$. Also, we propose polynomial algorithm that recognizes whether a tree belongs to one of families $\boldsymbol{\mathcal{A}}$ or $\boldsymbol{\mathcal{B}}$. Finally, we develop polynomial algorithm for determining the number $\theta_{cn}^{max}(G)$ of a tree G. But the general convex cover problem of trees remains the task of further research.

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