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Invariant integrability conditions for ternary differential systems with quadratic nonlinearities of the Darboux form

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Abstract. The general integral for ternary differential system with quadratic nonlinearities of the Darboux form was constructed by using the Lie theorem on integrating factor. The case is achieved when the comitant of the linear part of differential system, which is a $GL(3,\mathbb{R})$-invariant particular integral, describes an invariant variety.

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1 Preliminaries

Consider the ternary differential system with quadratic nonlinearities

$$\frac{dx^j}{dt} = a^j_\alpha x^\alpha + a^j_{\alpha\beta}x^\alpha x^\beta \equiv P^j(x) \quad (j,\alpha,\beta=\overline{1,3}), \tag{1}$$

where $a^j_{\alpha\beta}$ is a symmetric tensor in lower indices, in which the complete convolution is done and $x=(x^1,x^2,x^3)$ is the vector of phase variables. The expressions $a^j_\alpha x^\alpha$ represent the linear part of the system (1) and $a^j_{\alpha\beta}x^\alpha x^\beta$ represent the quadratic part of this system. The coefficients and the variables take values from the field of real numbers $\mathbb{R}$. We will use the center-affine group $GL(3,\mathbb{R})$ given by substitutions

$$\bar{x}^j = q^j_\alpha x^\alpha (det(q^j_\alpha)\neq 0) \;\; (j,\alpha=\overline{1,3}).$$

It is well known that $F(x)=C$ is *a first integral* of system (1) if and only if $\Lambda(F)=0$, where

$$\Lambda = P^j\frac{\partial}{\partial x^j} \quad (j=\overline{1,3}), \tag{2}$$

and in index j the complete convolution is done.

The system (1) has two functional-independent first integrals, which form *the general integral* of this system.

Suppose system (1) admits a two-dimensional commutative Lie algebra of operators [1]

$$X_\alpha = \xi^i_\alpha\frac{\partial}{\partial x^i} \quad (\alpha=1,2;\; j=\overline{1,3}), \tag{3}$$

where $\xi^i_\alpha(x)$ $(j = \overline{1,3})$ are polynomials in the coordinates of the vector $x = (x^1, x^2, x^3)$. This means that the coordinates of the operators (3) satisfy the determinant equations

$$\begin{aligned}(\xi^1_\alpha)_{x^1}P^1 + (\xi^1_\alpha)_{x^2}P^2 + (\xi^1_\alpha)_{x^3}P^3 &= \xi^1_\alpha P^1_{x^1} + \xi^2_\alpha P^1_{x^2} + \xi^3_\alpha P^1_{x^3},\\ (\xi^2_\alpha)_{x^1}P^1 + (\xi^2_\alpha)_{x^2}P^2 + (\xi^2_\alpha)_{x^3}P^3 &= \xi^1_\alpha P^2_{x^1} + \xi^2_\alpha P^2_{x^2} + \xi^3_\alpha P^2_{x^3},\\ (\xi^3_\alpha)_{x^1}P^1 + (\xi^3_\alpha)_{x^2}P^2 + (\xi^3_\alpha)_{x^3}P^3 &= \xi^1_\alpha P^3_{x^1} + \xi^2_\alpha P^3_{x^2} + \xi^3_\alpha P^3_{x^3} \ \ (\alpha = 1,2).\end{aligned} \tag{4}$$

Denote by

$$\Delta = \begin{vmatrix} \xi^1_1 & \xi^2_1 & \xi^3_1 \\ \xi^1_2 & \xi^2_2 & \xi^3_2 \\ P^1 & P^2 & P^3 \end{vmatrix} \tag{5}$$

the determinant of coordinates of the operators (2) and (3). From [1] the following assertion follows for system (1).

Theorem 1. *Suppose the ternary polynomial system* (1) *admits the two-dimensional commutative Lie algebra with operators* (3)*. Then the function* $\mu = \Delta^{-1}$ *is the Lie integrating factor for the Pfaff's equations*

$$(\xi^3_\alpha P^2 - \xi^2_\alpha P^3)dx^1 + (\xi^1_\alpha P^3 - \xi^3_\alpha P^1)dx^2 + (\xi^2_\alpha P^1 - \xi^1_\alpha P^2)dx^3 = 0 \quad (\alpha = 1,2),$$

which define the general integral of the system (1)*, where* $\Delta \neq 0$ *has the form* (5)*.*

Consider the comitant of system (1) from [2] with respect to the center-affine group. It depends on two cogradient vectors $x = (x^1, x^2, x^3)$ and $y = (y^1, y^2, y^3)$ defined in [3], whose tensorial form is

$$\eta = a^\alpha_{\beta\gamma} x^\beta x^\gamma x^\delta y^\mu \varepsilon_{\alpha\delta\mu},$$

where $\varepsilon_{\alpha\delta\mu}$ is the unit trivector with coordinates $\varepsilon_{123} = -\varepsilon_{132} = \varepsilon_{312} = -\varepsilon_{321} = \varepsilon_{231} = -\varepsilon_{213} = 1$ and $\varepsilon_{\alpha\delta\mu} = 0$ $(\alpha, \delta, \mu = \overline{1,3})$ in the other cases.

In [2] the following assertions were proved:

Theorem 2. *The system* (1) *with* $\eta \equiv 0$ *can be written in the form*

$$\frac{dx^j}{dt} = \alpha^j_\alpha x^\alpha + 2x^j(gx^1 + hx^2 + kx^3) \equiv P^j(x) \ \ (j = \overline{1,3}) \tag{6}$$

and will be called the ternary differential system with quadratic nonlinearities of the Darboux form.

Theorem 3. *The system* (6) *has the* $GL(3,\mathbb{R})$*-invariant particular integral*

$$\sigma_1 = a^\alpha_\mu a^\beta_\delta a^\gamma_\alpha x^\delta x^\mu x^\nu \varepsilon_{\beta\gamma\nu}, \tag{7}$$

where σ_1 *is the comitant of* (1) *with respect to the center-affine group* $GL(3,\mathbb{R})$*.*

Remark 1. Let $\varkappa_2$ be the mixt comitant from [4] of system (6) with respect to the center-affine group

$$\varkappa_2 = a^{\alpha}_{\beta} x^{\beta} u_{\alpha}, \tag{8}$$

which depends on coordinates of the contravariant vector $x = (x^1, x^2, x^3)$ and of the covariant vector $u = (u_1, u_2, u_3)$ defined in [3]. If $\varkappa_2 \neq 0$, then at least one coefficient of the linear part of system (6) is not equal to zero. Otherwise, from $\varkappa_2 \equiv 0$ it follows that $a^j_{\alpha} = 0$ $(j, \alpha = \overline{1,3})$ and the system (6) can be reduced to a trivial homogeneous quadratic system.

Remark 2. Let q_1 be the mixt comitant from [2] of system (1) with respect to the center-affine group

$$q_1 = a^{\alpha}_{\beta\gamma} x^{\beta} x^{\gamma} u_{\alpha}, \tag{9}$$

which depends on coordinates of the contravariant vector $x = (x^1, x^2, x^3)$ and of the covariant vector $u = (u_1, u_2, u_3)$ defined in [3]. If $q_1 \neq 0$, then at least one coefficient of the quadratic part of system (1) and hence of system (6) is not equal to zero. Otherwise, from $q_1 \equiv 0$ it follows that $a^j_{\alpha\beta} = 0$ $(j, \alpha, \beta = \overline{1,3})$ and the system (1) and hence the system (6) can be reduced to a linear system.

As it follows from [2], the following assertions hold

Lemma 1. *Assume in* (7) *that* $\sigma_1 \equiv 0$. *Then under the center-affine transformation*

$$\bar{x}^1 = x^2, \quad \bar{x}^2 = x^1 + \frac{a^3_2}{a^3_1} x^2, \quad \bar{x}^3 = x^3$$

where $a^3_1 \neq 0$, *the quadratic part of system* (6) *preserves the form and the coefficients from the linear part of the system obey one of the following conditions:*

$$a^1_2 = a^1_3 = a^2_1 = a^2_3 = a^3_1 = a^3_2 = 0; \;\; a^3_3 = a^2_2; \tag{10}$$

$$a^1_2 = a^1_3 = a^2_1 = a^2_3 = a^3_1 = a^3_2 = 0; \;\; a^3_3 = a^1_1; \tag{11}$$

$$a^1_2 = a^1_3 = a^2_1 = a^2_3 = a^3_1 = a^3_2 = 0; \;\; a^2_2 = a^1_1; \tag{12}$$

$$a^1_2 = a^1_3 = a^2_1 = a^3_1 = a^3_2 = 0; \;\; a^2_3 \neq 0; \;\; a^3_3 = a^1_1; \tag{13}$$

$$a^1_2 = a^1_3 = a^2_1 = a^3_1 = a^3_2 = 0; \;\; a^2_2 = a^1_1; \;\; a^2_3 \neq 0; \tag{14}$$

$$a^1_2 = a^2_1 = a^2_3 = a^3_1 = a^3_2 = 0; \;\; a^1_3 \neq 0; \;\; a^3_3 = a^2_2; \tag{15}$$

$$a^1_2 = a^2_1 = a^3_1 = a^3_2 = 0; \;\; a^1_3 \neq 0; \;\; a^2_2 = a^1_1; \tag{16}$$

$$a_1^2 = a_3^2 = a_1^3 = a_2^3 = 0;\ \ a_2^1 \neq 0;\ \ a_3^3 = a_2^2; \tag{17}$$

$$a_1^2 = a_1^3 = a_2^3 = 0;\ \ a_2^1 \neq 0;\ \ a_3^2 = \frac{a_3^1(a_2^2 - a_1^1)}{a_2^1};\ \ a_3^3 = a_1^1; \tag{18}$$

$$a_1^3 = a_2^3 = 0;\ \ a_1^2 \neq 0;\ \ a_2^1 = \frac{(a_1^1 - a_3^3)(a_2^2 - a_3^3)}{a_1^2};\ \ a_3^1 = \frac{a_3^2(a_1^1 - a_3^3)}{a_1^2}; \tag{19}$$

$$a_1^2 = a_1^3 = 0;\ \ a_2^3 \neq 0;\ \ a_3^1 = \frac{a_2^1(a_3^3 - a_1^1)}{a_2^3};\ \ a_3^2 = \frac{(a_1^1 - a_2^2)(a_1^1 - a_3^3)}{a_2^3}. \tag{20}$$

Lemma 2. *Suppose for linear part of system* (1) *or* (6) *we have* $\sigma_1 \equiv 0$*, where* σ_1 *is from* (7)*. Then the characteristic equation of these systems has real roots.*

Proof. The characteristic equation of the systems (1) and (6) looks

$$\lambda^3 - n\lambda^2 - m\lambda - l = 0, \tag{21}$$

where l, m and n are the center-affine invariants of these systems

$$l = \frac{1}{6}(\theta_1^3 - 3\theta_1\theta_2 + 2\theta_3), \qquad m = \frac{1}{2}(\theta_2 - \theta_1^2), \qquad n = \theta_1 \tag{22}$$

with

$$\theta_1 = a_\alpha^\alpha, \qquad \theta_2 = a_\beta^\alpha a_\alpha^\beta, \qquad \theta_3 = a_\gamma^\alpha a_\alpha^\beta a_\beta^\gamma. \tag{23}$$

According to [5], the discriminant of the equation (21) can be written

$$D = -27l^2 - 18lmn + 4m^3 - 4ln^3 + m^2n^2 \tag{24}$$

and it is a center-affine invariant of the systems (1) and (6).

By Lemma 1, from $\sigma_1 \equiv 0$, without considering the center-affine transformation (1), we have the conditions (10)–(20). Then for each of them, calculating the expressions (22)–(24), we get $D = 0$. □

2 Lie's integrating factor and the general integral of system (6) with $\boldsymbol{\sigma_1 \equiv 0}$ and $\boldsymbol{\varkappa_2 q_1 \not\equiv 0}$

Theorem 4. *Suppose the coefficients of the linear part of system* (6) *satisfy conditions* (10) *with* $\varkappa_2 q_1 \not\equiv 0$ *from* (8)–(9)*. Then the general integral of this system with notations* $x = x^1,\ y = x^2,\ z = x^3$ *consists of the following two first integrals:*

$$F_1 \equiv yz^{-1} = C_1,\ \ F_2 \equiv x^{a_2^2} y^{-a_1^1} \Phi^{a_1^1 - a_2^2} = C_2, \tag{25}$$

where

$$\Phi = a_1^1 a_2^2 + 2[a_2^2 gx + a_1^1(hy + kz)]. \tag{26}$$

Proof. Assume that the coordinates of the operator (3) have the form

$$\xi^i_\alpha = A^i_{\alpha\beta}x^\beta + A^i_{\alpha\beta\gamma}x^\beta x^\gamma \;\; (\alpha \geq 1;\; \beta,\gamma = \overline{1,3}), \tag{27}$$

and satisfy the determinant equations (4). Solving (4) under the conditions $\varkappa_2 q_1 \not\equiv 0$ from (8)–(9) and the expressions (27), we obtain for differential system (6) the following operators ($x = x^1,\; y = x^2,\; z = x^3$):

$$\begin{aligned}
Y_1 &= (a^1_1 + 2gx)x\frac{\partial}{\partial x} + 2gxy\frac{\partial}{\partial y} + 2gxz\frac{\partial}{\partial z},\\
Y_2 &= 2hxy\frac{\partial}{\partial x} + (a^2_2 + 2hy)y\frac{\partial}{\partial y} + 2hyz\frac{\partial}{\partial z},\\
Y_3 &= 2hxz\frac{\partial}{\partial x} + (a^2_2 + 2hy)z\frac{\partial}{\partial y} + 2hz^2\frac{\partial}{\partial z},\\
Y_4 &= 2kxy\frac{\partial}{\partial x} + 2ky^2\frac{\partial}{\partial y} + (a^2_2 + 2kz)y\frac{\partial}{\partial z},\\
Y_5 &= 2kxz\frac{\partial}{\partial x} + 2kyz\frac{\partial}{\partial y} + (a^2_2 + 2kz)z\frac{\partial}{\partial z}.
\end{aligned} \tag{28}$$

These operators compose the Lie algebra L_5 with the structure equations

$$\begin{gathered}
[Y_1, Y_i] = 0 \;(i = \overline{2,5}),\; [Y_2, Y_3] = -a^2_2 Y_3,\; [Y_2, Y_4] = a^2_2 Y_4,\; [Y_2, Y_5] = 0,\\
[Y_3, Y_4] = a^2_2(Y_5 - Y_2),\; [Y_3, Y_5] = -a^2_2 Y_3,\; [Y_4, Y_5] = a^2_2 Y_4.
\end{gathered} \tag{29}$$

Using the operators Y_1 and Y_2, which form by (28) and (29) a two-dimensional commutative Lie algebra, we obtain from (5) (making abstraction of a constant) the Lie integrating factor $\mu^{-1} = xyz\Phi$, where Φ is given in (26).

Taking into account this expression and Theorem 1, we obtain the functional-independent integrals (25)–(26) of system (6). The conditions (10) and $\varkappa_2 q_1 \not\equiv 0$ from (8)–(9) imply that not all coefficients in this system are equal to zero. □

Theorem 5. *Assume the coefficients of the linear part of system* (6) *satisfy the conditions* (11) *with* $\varkappa_2 q_1 \not\equiv 0$ *from* (8)–(9). *Then the general integral of this system with notations* $x = x^1,\; y = x^2,\; z = x^3$ *is composed from the following two first integrals:*

$$F_1 \equiv xz^{-1} = C_1;\;\; F_2 \equiv x^{-a^2_2} y^{a^1_1} \Phi^{a^2_2 - a^1_1} = C_2, \tag{30}$$

where

$$\Phi = a^1_1 a^2_2 + 2[a^2_2(gx + kz) + a^1_1 hy]. \tag{31}$$

Proof. We make the substitutions $\bar{x}^1 = x^2,\; \bar{x}^2 = x^1,\; \bar{x}^3 = x^3$ in (6) under the conditions (11). Then we obtain the system (6) with conditions (10) for which the general integral is determined in Theorem 4. Using this result and the above-mentioned notations, we obtain for system (6) the integrals (30)–(31) on the conditions (11). □

Theorem 6. *If the coefficients of the linear part of system* (6) *satisfy the conditions* (12) *with* $\varkappa_2 q_1 \not\equiv 0$ *from* (8)–(9), *then the general integral of this system with notations* $x = x^1,\ y = x^2,\ z = x^3$ *consists of two first integrals:*

$$F_1 \equiv x^{-1}y = C_1;\ \ F_2 \equiv y^{-a_3^3} z^{a_1^1} \Phi^{a_3^3 - a_1^1} = C_2,$$

where

$$\Phi = a_1^1 a_3^3 + 2[a_3^3(gx + hy) + a_1^1 kz].$$

The proof of Theorem 6 is similar to Theorem 5 if we make the substitutions $\bar{x}^1 = x^3,\ \bar{x}^2 = x^2,\ \bar{x}^3 = x^1$ in (6) and take into account the conditions (10).

Theorem 7. *Suppose the coefficients of the linear part of system* (6) *satisfy the conditions* (13) *with* $\varkappa_2 q_1 \not\equiv 0$ *from* (8)–(9). *Then the general integral of this system with notations* $x = x^1,\ y = x^2,\ z = x^3$ *consists of the following two first integrals:*

$$F_1 \equiv xz^{-1} = C_1;\ \ F_2 \equiv x^{-a_2^2}[(a_1^1 - a_2^2)y - a_3^2 z]^{a_1^1} \Phi^{a_2^2 - a_1^1} = C_2, \tag{32}$$

where

$$\Phi = a_1^1 a_2^2 + 2[a_2^2 gx + a_1^1 hy + (a_2^2 k - a_3^2 h)z]. \tag{33}$$

Proof. Assume the coordinates of the operator (3) have the form (27). Solving the system (4) we obtain the following operators ($x = x^1,\ y = x^2,\ z = x^3$) for the differential system (6):

$$\begin{gathered}
Y_1 = (a_1^1 + 2gx)x\frac{\partial}{\partial x} + 2gxy\frac{\partial}{\partial y} + 2gxz\frac{\partial}{\partial z}, \\
Y_2 = (a_1^1 + 2gx)z\frac{\partial}{\partial x} + 2gyz\frac{\partial}{\partial y} + 2gz^2\frac{\partial}{\partial z}, \\
Y_3 = 2[a_3^2 h + (a_1^1 - a_2^2)k]x^2\frac{\partial}{\partial x} + [a_1^1 a_3^2 + 2(a_3^2 h + \\
+(a_1^1 - a_2^2)k)y]x\frac{\partial}{\partial y} + [(a_1^1 - a_2^2)(a_1^1 + 2kz) + 2a_3^2 hz]x\frac{\partial}{\partial z}, \\
Y_4 = 2[a_3^2 h + (a_1^1 - a_2^2)k]xz\frac{\partial}{\partial x} + [a_3^2(a_1^1 + 2hy) + \\
+2(a_1^1 - a_2^2)ky]z\frac{\partial}{\partial y} + [(a_1^1 - a_2^2)(a_1^1 + 2kz) + 2a_3^2 hz]z\frac{\partial}{\partial z}, \\
Y_5 = 2[a_1^1 hy - (a_3^2 h - a_2^2 k)z]x\frac{\partial}{\partial x} + \{a_1^1 a_2^2 + 2[a_1^1 hy - \\
-(a_3^2 h - a_2^2 k)z]\}y\frac{\partial}{\partial y} + [a_1^1 a_2^2 + 2(a_1^1 hy - (a_3^2 h - a_2^2 k)z)]z\frac{\partial}{\partial z}.
\end{gathered} \tag{34}$$

These operators form the Lie algebra L_5 with the structure equations

$$\begin{gathered}
[Y_1, Y_2] = -a_1^1 Y_2,\ \ [Y_1, Y_3] = a_1^1 Y_3,\ \ [Y_1, Y_4] = [Y_1, Y_5] = [Y_4, Y_5] = 0, \\
[Y_2, Y_3] = a_1^1[(a_2^2 - a_1^1)Y_1 + Y_4],\ \ [Y_2, Y_4] = a_1^1(a_2^2 - a_1^1)Y_2, \\
[Y_2, Y_5] = -a_1^1 a_2^2 Y_2,\ \ [Y_3, Y_4] = a_1^1(a_1^1 - a_2^2)Y_3,\ \ [Y_3, Y_5] = a_1^1 a_2^2 Y_3.
\end{gathered} \tag{35}$$

Using the operators Y_1 and Y_4, which form by (34) and (35) a two-dimensional commutative Lie algebra, we obtain from (5) the Lie integrating factor of the form

$$\mu^{-1} = xz[(a^1_1 - a^2_2)y - a^2_3 z]\Phi,$$

where Φ is from (33).

Taking into account this expression and Theorem 1, we obtain the functional-independent integrals (32)–(33) of system (6). The conditions (13) and $\varkappa_2 q_1 \not\equiv 0$ from (8)–(9) imply that not all coefficients in this system are equal to zero. □

Theorem 8. *Assume the coefficients of the linear part of system* (6) *satisfy the conditions* (14) *with* $\varkappa_2 q_1 \not\equiv 0$ *from* (8)–(9)*. Then the general integral of this system with notations* $x = x^1,\ y = x^2,\ z = x^3$ *consists of two first integrals:*

$$\begin{aligned} F_1 &\equiv [(a^1_1 - a^3_3)y + a^2_3 z]x^{-1} = C_1;\\ F_2 &\equiv z^{a^1_1}[(a^1_1 - a^3_3)y + a^2_3 z]^{-a^3_3}\Phi^{a^3_3 - a^1_1} = C_2, \end{aligned} \tag{36}$$

where

$$\Phi = a^1_1 a^3_3 + 2[a^3_3(gx + hy) + (a^1_1 k - a^2_3 h)z]. \tag{37}$$

Proof. Let the coordinates of the operator (3) have the form (27). Solving (4) we obtain for differential system (6) the following operators ($x = x^1,\ y = x^2,\ z = x^3$):

$$\begin{aligned} Y_1 &= (a^1_1 + 2gx)x\frac{\partial}{\partial x} + 2gxy\frac{\partial}{\partial y} + 2gxz\frac{\partial}{\partial z},\\ Y_2 &= (a^1_1 + 2gx)[(a^1_1 - a^3_3)y + a^2_3 z]\frac{\partial}{\partial x} + 2g[(a^1_1 - a^3_3)y + a^2_3 z]y\frac{\partial}{\partial y} +\\ &\quad + 2g[(a^1_1 - a^3_3)y + a^2_3 z]z\frac{\partial}{\partial z},\\ Y_3 &= 2hx^2\frac{\partial}{\partial x} + (a^1_1 + 2hy)x\frac{\partial}{\partial y} + 2hxz\frac{\partial}{\partial z},\\ Y_4 &= 2h[(a^1_1 - a^3_3)y + a^2_3 z]x\frac{\partial}{\partial x} + (a^1_1 + 2hy)[(a^1_1 - a^3_3)y + a^2_3 z]\frac{\partial}{\partial y} +\\ &\quad + 2h[(a^1_1 - a^3_3)y + a^2_3 z]z\frac{\partial}{\partial z},\\ Y_5 &= 2[a^3_3 hy + (a^1_1 k - a^2_3 h)z]x\frac{\partial}{\partial x} + \{a^1_1 a^3_3 + 2[a^3_3 hy +\\ &\quad + (a^1_1 k - a^2_3 h)z]\}y\frac{\partial}{\partial y} + [a^1_1 a^3_3 + 2(a^3_3 hy + (a^1_1 k - a^2_3 h)z)]z\frac{\partial}{\partial z}. \end{aligned} \tag{38}$$

These operators form the Lie algebra L_5 with the structure equations

$$\begin{aligned} &[Y_1, Y_2] = -a^1_1 Y_2,\ \ [Y_1, Y_3] = a^1_1 Y_3,\ \ [Y_1, Y_4] = [Y_1, Y_5] = [Y_4, Y_5] = 0,\\ &[Y_2, Y_3] = a^1_1[(a^3_3 - a^1_1)Y_1 + Y_4],\ \ [Y_2, Y_4] = a^1_1(a^3_3 - a^1_1)Y_2,\\ &[Y_2, Y_5] = -a^1_1 a^3_3 Y_2,\ \ [Y_3, Y_4] = a^1_1(a^1_1 - a^3_3)Y_3,\ \ [Y_3, Y_5] = a^1_1 a^3_3 Y_3. \end{aligned} \tag{39}$$

Using the operators Y_1 and Y_5, which form by (38) and (39) a two-dimensional commutative Lie algebra, we obtain from (5) the Lie integrating factor of the form

$$\mu^{-1} = xz[(a_1^1 - a_3^3)y + a_3^2 z]\Phi,$$

where Φ is from (37).

Taking into account this expression and Theorem 1, we obtain the functional-independent integrals (36)–(37) of system (6). The conditions (14) and $\varkappa_2 q_1 \not\equiv 0$ from (8)–(9) imply that not all coefficients in this system are equal to zero. □

Theorem 9. *Suppose the coefficients of the linear part of system* (6) *satisfy the conditions* (15) *with* $\varkappa_2 q_1 \not\equiv 0$ *from* (8)*–*(9)*. Then the general integral of this system with notations* $x = x^1,\ y = x^2,\ z = x^3$ *consists of two first integrals:*

$$F_1 \equiv yz^{-1} = C_1;$$
$$F_2 \equiv y^{-a_1^1}[(a_2^2 - a_1^1)x - a_3^1 z]^{a_2^2}\Phi^{a_1^1 - a_2^2} = C_2, \tag{40}$$

where

$$\Phi = a_1^1 a_2^2 + 2[a_2^2 gx + a_1^1 hy + (a_1^1 k - a_3^1 g)z]. \tag{41}$$

Proof. Let us make the substitutions $\bar{x}^1 = x^2,\ \bar{x}^2 = x^1,\ \bar{x}^3 = x^3$ in (6) taking into account (15). We obtain the system (6) under the conditions (13) for which the general integral is determined in Theorem 7. Using this result and the above-mentioned notations, we obtain for (6) the integrals (40)–(41) with conditions (15). □

Theorem 10. *Assume the coefficients of the linear part of system* (6) *satisfy the conditions* (16) *with* $\varkappa_2 q_1 \not\equiv 0$ *from* (8)*–*(9)*. Then the general integral of this system with notations* $x = x^1,\ y = x^2,\ z = x^3$ *is composed from the following two first integrals:*

$$F_1 \equiv (-a_3^2 x + a_3^1 y)[(a_1^1 - a_3^3)y + a_3^2 z]^{-1} = C_1;$$
$$F_2 \equiv z^{-a_1^1}[(a_1^1 - a_3^3)y + a_3^2 z]^{a_3^3}\Phi^{a_1^1 - a_3^3} = C_2, \tag{42}$$

where

$$\Phi = a_1^1 a_3^3 + 2[a_3^3(gx + hy) + (a_1^1 k - a_3^1 g - a_3^2 h)z]. \tag{43}$$

Proof. Let the coordinates of the operator (3) have the form (27). Then solving the system (4) we obtain the operators ($x = x^1,\ y = x^2,\ z = x^3$):

$$Y_1 = (a_1^1 + 2gx)(a_3^2 x - a_3^1 y)\frac{\partial}{\partial x} + 2g(a_3^2 x - a_3^1 y)y\frac{\partial}{\partial y} + 2g(a_3^2 x - a_3^1 y)z\frac{\partial}{\partial z},$$

$$Y_2 = (a_1^1 + 2gx)[(a_1^1 - a_3^3)y + a_3^2 z]\frac{\partial}{\partial x} + 2g[(a_1^1 - a_3^3)y + a_3^2 z]y\frac{\partial}{\partial y} +$$
$$+ 2g[(a_1^1 - a_3^3)y + a_3^2 z]z\frac{\partial}{\partial z},$$

$$Y_3 = \{a^3_3[a^1_1a^1_3 + 2(a^1_3g + a^2_3h)x]y + Wxz\}\frac{\partial}{\partial x}+$$
$$+\{a^3_3[a^1_1a^2_3 + 2(a^1_3g + a^2_3h)y] + Wz\}y\frac{\partial}{\partial y}+$$
$$+\{a^3_3[a^1_1a^2_3 + 2(a^1_3g + a^2_3h)y] + Wz\}z\frac{\partial}{\partial z},$$
$$Y_4 = \{a^3_3[2(a^2_3)^2hx^2 + (a^1_3)^2(a^1_1 + 2gx)y] + a^1_3Wxz\}\frac{\partial}{\partial x}+$$
$$+\{a^3_3[(a^2_3)^2(a^1_1 + 2hy)x + 2(a^1_3)^2gy^2] + a^1_3Wyz\}\frac{\partial}{\partial y}+ \qquad (44)$$
$$+\{a^3_3[a^1_3(a^1_1a^2_3 + 2a^1_3gy) + 2(a^2_3)^2hx] + a^1_3Wz\}z\frac{\partial}{\partial z},$$
$$Y_5 = \{a^3_3[2a^1_1a^2_3kxz - a^1_3[(a^1_1 - a^3_3)(a^1_1 + 2gx)y + 2a^2_3gxz]] - a^1_1Wxz\}\frac{\partial}{\partial x}+$$
$$+\{a^3_3[a^1_1a^2_3(a^2_3 + 2ky)z - 2a^1_3[(a^1_1 - a^3_3)gy + a^2_3gz]y] - a^1_1Wyz\}\frac{\partial}{\partial y}+$$
$$+\{a^3_3[a^3_3(a^1_1a^2_3 + 2a^1_3gy) - a^1_1a^2_3(a^1_1 - 2kz) - 2a^1_3g(a^1_1y + a^2_3z)] - a^1_1Wz\}z\frac{\partial}{\partial z},$$

where $W = 2a^2_3(a^1_1k - a^2_3h - a^1_3g)$.

These operators form the Lie algebra L_5 with the structure equations

$$\begin{gathered}[Y_3, Y_5] = a^1_3(a^3_3)^2[Y_1, Y_2] = -a^3_3[Y_1, Y_5] = a^1_3a^3_3[Y_2, Y_3] = -a^1_1a^1_3a^2_3(a^3_3)^2Y_2,\\ [Y_3, Y_4] = -a^3_3[Y_1, Y_4] = -a^1_1a^2_3a^3_3[a^1_3(a^3_3Y_1 - Y_3) + Y_4],\\ [Y_1, Y_3] = [Y_2, Y_5] = 0, \quad [Y_4, Y_5] = a^1_3a^3_3[Y_2, Y_4] =\\ = a^1_1a^1_3a^2_3a^3_3[a^3_3(a^3_3 - a^1_1)Y_1 - a^1_3a^3_3Y_2 + (a^1_1 - a^3_3)Y_3 + Y_5].\end{gathered} \qquad (45)$$

We use the operators Y_1 and Y_3, which form by (44) and (45) a two-dimensional commutative Lie algebra. Then from (5) we obtain the Lie integrating factor of the form

$$\mu^{-1} = (-a^2_3x + a^1_3y)z[(a^1_1 - a^3_3)y + a^2_3z]\Phi,$$

where Φ is from (43).

Taking into account this expression and Theorem 1, we obtain the functional-independent integrals (42)–(43) of system (6). The conditions (16) and $\varkappa_2q_1 \not\equiv 0$ from (8)–(9) imply that not all coefficients in this system are equal to zero. □

Theorem 11. *Let the coefficients of the linear part of system* (6) *satisfy the conditions* (17) *with* $\varkappa_2q_1 \not\equiv 0$ *from* (8)–(9). *Then the general integral of this system with notations* $x = x^1,\ y = x^2,\ z = x^3$ *consists of two first integrals:*

$$\begin{gathered}F_1 \equiv yz^{-1} = C_1;\\ F_2 \equiv z^{a^1_1}[(a^1_1 - a^2_2)x + a^1_2y + a^1_3z]^{-a^2_2}\Phi^{a^2_2 - a^1_1} = C_2,\end{gathered} \qquad (46)$$

where

$$\Phi = a^2_2(a^1_1 + 2gx) + 2[(a^1_1h - a^1_2g)y + (a^1_1k - a^1_3g)z]. \qquad (47)$$

Proof. Suppose the coordinates of the operator (3) have the form (27). Then from system (4) we obtain the following operators ($x = x^1,\ y = x^2,\ z = x^3$) for differential system (6):

$$\begin{gathered}
Y_1 = \{a_2^1 a_2^2 + 2[a_2^1 g + (a_2^2 - a_1^1)h]x\}y\frac{\partial}{\partial x} + \\
+[2a_2^1 gy + (a_2^2 + 2hy)(a_2^2 - a_1^1)]y\frac{\partial}{\partial y} + 2[a_2^1 g + (a_2^2 - a_1^1)h]yz\frac{\partial}{\partial z}, \\
Y_2 = [a_2^1 a_2^2 + 2(a_2^1 g + (a_2^2 - a_1^1)h)x]z\frac{\partial}{\partial x} + \\
+[2a_2^1 gy + (a_2^2 + 2hy)(a_2^2 - a_1^1)]z\frac{\partial}{\partial y} + 2[a_2^1 g + (a_2^2 - a_1^1)h]z^2\frac{\partial}{\partial z}, \\
Y_3 = 2(a_2^1 k - a_3^1 h)xy\frac{\partial}{\partial x} + [-a_3^1 a_2^2 + 2(a_2^1 k - a_3^1 h)y]y\frac{\partial}{\partial y} + \\
+[a_2^1 a_2^2 + 2(a_2^1 k - a_3^1 h)z]y\frac{\partial}{\partial z}, \\
Y_4 = 2(a_2^1 k - a_3^1 h)xz\frac{\partial}{\partial x} + [-a_3^1 a_2^2 + 2(a_2^1 k - a_3^1 h)y]z\frac{\partial}{\partial y} + \\
+[a_2^1 a_2^2 + 2(a_2^1 k - a_3^1 h)z]z\frac{\partial}{\partial z}, \\
Y_5 = [a_2^1 a_2^2(a_1^1 + 2gx) + 2(a_2^1 y + a_3^1 z)(a_1^1 h - a_2^1 g)]x\frac{\partial}{\partial x} + \\
+\{[a_2^1 a_2^2(a_1^1 + 2gx) + 2(a_2^1 y + a_3^1 z)(a_1^1 h - a_2^1 g)]y + a_1^1 a_3^1 a_2^2 z\}\frac{\partial}{\partial y} + \\
+2[a_2^1 a_2^2 gx + (a_2^1 y + a_3^1 z)(a_1^1 h - a_2^1 g)]z\frac{\partial}{\partial z}.
\end{gathered} \tag{48}$$

These operators form the Lie algebra L_5 with the structure equations

$$\begin{gathered}
[Y_1, Y_2] = a_2^2(a_1^1 - a_2^2)Y_2 \quad [Y_1, Y_3] = a_2^2[a_3^1 Y_1 + (a_2^2 - a_1^1)Y_3], \\
a_1^1 a_2^1[Y_1, Y_4] = -a_2^1[Y_1, Y_5] = -a_1^1 a_3^1[Y_2, Y_4] = a_3^1[Y_2, Y_5] = a_1^1 a_2^1 a_3^1 a_2^2 Y_2, \\
[Y_2, Y_3] = -a_2^2[a_2^1 Y_1 + (a_1^1 - a_2^2)Y_4], \\
a_1^1[Y_3, Y_4] = -[Y_3, Y_5] = a_1^1 a_2^2(a_2^1 Y_3 + a_3^1 Y_4), \quad [Y_4, Y_5] = 0.
\end{gathered} \tag{49}$$

If we use the operators Y_4 and Y_5, which form by (48) and (49) a two-dimensional commutative Lie algebra, we obtain from (5) the Lie integrating factor of the form

$$\mu^{-1} = z(a_2^1 y + a_3^1 z)[(a_1^1 - a_2^2)x + a_2^1 y + a_3^1 z]\Phi,$$

where Φ is from (47).

Taking into account this expression and Theorem 1, we obtain the functional-independent integrals (46)–(47) of system (6). The conditions (17) and $\varkappa_2 q_1 \not\equiv 0$ from (8)–(9) imply that not all coefficients in this system are equal to zero. □

Theorem 12. *If the coefficients of the linear part of system* (6) *satisfy the conditions* (18) *with* $\varkappa_2 q_1 \not\equiv 0$ *from* (8)–(9), *then the general integral of this system with notations* $x = x^1,\ y = x^2,\ z = x^3$ *consists of two first integrals:*

$$F_1 \equiv z[(a_1^1 - a_2^2)x + a_2^1 y + a_3^1 z]^{-1} = C_1;$$
$$F_2 \equiv (a_2^1 y + a_3^1 z)^{a_1^1}[(a_1^1 - a_2^2)x + a_2^1 y + a_3^1 z]^{-a_2^2}\Phi^{a_2^2 - a_1^1} = C_2, \tag{50}$$

where

$$\Phi = a_1^1 a_2^1 a_2^2 + 2\{a_2^1[a_2^2 g x + (a_1^1 h - a_2^1 g)y + (a_2^2 k - a_3^1 g)z] + a_3^1(a_1^1 - a_2^2)hz\}. \tag{51}$$

Proof. Assume the coordinates of the operator (3) have the form (27). Then system (4) yields the following operators ($x = x^1,\ y = x^2,\ z = x^3$):

$$\begin{aligned}
Y_1 &= (a_1^1 + 2gx)[(a_1^1 - a_2^2)x + a_2^1 y]\frac{\partial}{\partial x} + 2g[(a_1^1 - a_2^2)x + a_2^1 y]y\frac{\partial}{\partial y} + \\
&\quad + 2g[(a_1^1 - a_2^2)x + a_2^1 y]z\frac{\partial}{\partial z},\\
Y_2 &= (a_1^1 + 2gx)z\frac{\partial}{\partial x} + 2gyz\frac{\partial}{\partial y} + 2gz^2\frac{\partial}{\partial z},\\
Y_3 &= 2(a_2^1 k - a_3^1 h)xz\frac{\partial}{\partial x} + [-a_1^1 a_3^1 + 2(a_2^1 k - a_3^1 h)y]z\frac{\partial}{\partial y} + \\
&\quad + [a_1^1 a_2^1 + 2(a_2^1 k - a_3^1 h)z]z\frac{\partial}{\partial z},\\
Y_4 &= \{a_1^1 a_2^1 a_2^2 + 2[a_2^1 a_2^2 g x + (a_1^1 h - a_2^1 g)(a_2^1 y + a_3^1 z)]\}x\frac{\partial}{\partial x} + \\
&\quad + [a_1^1 a_2^2(a_2^1 y + a_3^1 z) + 2[a_2^1 a_2^2 g x + (a_1^1 h - a_2^1 g)(a_2^1 y + a_3^1 z)]y]\frac{\partial}{\partial y} + \\
&\quad + 2[a_2^1 a_2^2 g x + (a_1^1 h - a_2^1 g)(a_2^1 y + a_3^1 z)]z\frac{\partial}{\partial z},\\
Y_5 &= (a_1^1 a_2^1 a_3^1 a_2^2 + W)x\frac{\partial}{\partial x} + [a_1^1 a_3^1 a_2^2(a_2^2 - a_1^1)x + Wy + a_1^1(a_3^1)^2 a_2^2 z]\frac{\partial}{\partial y} + \\
&\quad + [a_1^1 a_2^1 a_2^2((a_1^1 - a_2^2)x + a_2^1 y) + Wz]\frac{\partial}{\partial z},
\end{aligned} \tag{52}$$

where $W = 2a_2^1 a_3^1 g(a_2^2 x - a_2^1 y - a_3^1 z) - 2a_3^1 h[(a_2^2 x - a_2^1 y)(a_1^1 - a_2^2) - a_1^1 a_3^1 z] - 2a_2^1 a_2^2 k((a_2^2 - a_1^1)x - a_2^1 y)$.

These operators form the Lie algebra L_5 with the structure equations

$$\begin{gathered}
[Y_1, Y_2] = a_1^1(a_2^2 - a_1^1)Y_2,\quad [Y_2, Y_5] = a_1^1 a_2^2[-a_2^1 Y_1 + a_2^1 a_3^1 Y_2 + (a_1^1 - a_2^2)Y_3],\\
a_2^2[Y_1, Y_3] = -[Y_1, Y_4] = -a_3^1 a_2^2[Y_2, Y_3] = a_3^1[Y_2, Y_4] = a_1^1 a_2^1 a_3^1 a_2^2 Y_2,\\
[Y_1, Y_5] = a_1^1[a_2^1 a_3^1 a_2^2 Y_1 - a_2^1(a_3^1)^2 a_2^2 Y_2 - a_3^1(a_1^1 - a_2^2)Y_4 + (a_1^1 - a_2^2)Y_5],\\
[Y_3, Y_4] = 0,\quad a_2^2[Y_3, Y_5] = -[Y_4, Y_5] = a_1^1 a_2^1 a_2^2[-a_3^1 a_2^2 Y_3 + a_3^1 Y_4 - Y_5].
\end{gathered} \tag{53}$$

Using the operators Y_3 and Y_4, which form by (52) and (53) a two-dimensional commutative Lie algebra, we obtain from (5) the Lie integrating factor of the form

$$\mu^{-1} = z(a_2^1 y + a_3^1 z)[(a_1^1 - a_2^2)x + a_2^1 y + a_3^1 z]\Phi,$$

where Φ is from (51).

Taking into account this expression and Theorem 1, we obtain the functional-independent integrals (50)–(51) of system (6). The conditions (18) and $\varkappa_2 q_1 \not\equiv 0$ from (8)–(9) ensure that not all coefficients in this system are equal to zero. □

Theorem 13. *Suppose the coefficients of the linear part of system* (6) *satisfy the conditions* (19) *with* $\varkappa_2 q_1 \not\equiv 0$ *from* (8)–(9)*. Then the general integral of this system with notations* $x = x^1,\ y = x^2,\ z = x^3$ *consists of two first integrals:*

$$\begin{gathered} F_1 \equiv z[-a_1^2 x + (a_1^1 - a_3^3)y - a_3^2 z]^{-1} = C_1; \\ F_2 \equiv z^{-(a_1^1+a_2^2-a_3^3)}[-a_1^2 x - (a_2^2 - a_3^3)y - a_3^2 z]^{a_3^3} \Phi^{a_1^1+a_2^2-2a_3^3} = C_2, \end{gathered} \tag{54}$$

where

$$\begin{gathered} \Phi = a_1^2 a_3^3 (a_1^1 + a_2^2 - a_3^3) + 2\{a_1^2(a_2^2 g - a_1^2 h)x + [a_3^3 g(a_1^1 + a_2^2 - a_3^3) - \\ -a_1^1(a_2^2 g - a_1^2 h)]y + [-a_3^2(g(a_1^1 - a_3^3) + a_1^2 h) + a_1^2 k(a_1^1 + a_2^2 - a_3^3)]z\}. \end{gathered} \tag{55}$$

Proof. Assuming the coordinates of the operator (3) have the form (27), we obtain from (4) the following operators ($x = x^1,\ y = x^2,\ z = x^3$):

$$\begin{gathered} Y_1 = [-2Wx - a_3^3(a_2^2 - a_3^3)]z\frac{\partial}{\partial x} + [a_1^2 a_3^3 - 2Wy]z\frac{\partial}{\partial y} - 2Wz^2\frac{\partial}{\partial z}, \\ Y_2 = (Vx - a_3^2 a_3^3)z\frac{\partial}{\partial x} + Vyz\frac{\partial}{\partial y} + (a_1^2 a_3^3 + Vz)z\frac{\partial}{\partial z}, \\ Y_3 = [a_1^2 a_3^3 Tx + 2a_1^2 Ux^2 + 2(a_3^3 gT - a_1^1 U)xy + a_3^2(a_3^3 T + 2Ux)z]\frac{\partial}{\partial x} + \\ +[a_1^2 a_3^3 T + 2a_1^2 Ux + 2(a_3^3 gT - a_1^1 U)y + 2a_3^2 Uz]y\frac{\partial}{\partial y} + \\ +2[a_1^2 Ux + (a_3^3 gT - a_1^1 U)y + a_3^2 Uz]z\frac{\partial}{\partial z}, \\ Y_4 = \{a_1^2 a_3^3(a_3^3 gT - a_1^1 U)x + a_3^3(-a_1^1 a_2^2 + a_3^3 T)Uy + \\ +2a_3^3[a_3^3 g^2 T - (a_1^1 g + a_1^2 h)U]xy - a_3^2 a_3^3 WTz - 2a_3^2 WUxz\}\frac{\partial}{\partial x} + \\ +[-(a_1^2)^2 a_3^3 Ux + a_1^2 a_3^3(a_3^3 gT - a_2^2 U)y + 2a_3^3[a_3^3 g^2 T - (a_1^1 g + a_1^2 h)U]y^2 - \\ -2a_3^2 UWyz]\frac{\partial}{\partial y} + \{2a_3^3[a_3^3 g^2 T - (a_1^1 g + a_1^2 h)U]yz - 2a_3^2 UWz^2\}\frac{\partial}{\partial z}, \\ Y_5 = \{a_3^3[-a_1^2 a_3^2[(a_1^1 - a_3^3)g + a_1^2 h] + (a_1^2)^2 kT]x - \\ -a_3^2 a_3^3(a_1^1 - a_3^3)Uy + a_3^3[(a_1^1 - a_3^3)g + a_1^2 h]Vxy - a_3^2 a_3^3(a_3^2 g - a_1^2 k)Tz + \\ +a_3^2 UVxz\}\frac{\partial}{\partial x} + \{-a_1^2 a_3^3(a_3^2 g - a_1^2 k)Ty + a_3^3[(a_1^1 - a_3^3)g + \\ +a_1^2 h]Vy^2 + a_3^2 UVyz\}\frac{\partial}{\partial y} + \{-(a_1^2)^2 a_3^3 Ux + a_1^2 a_3^3(a_1^1 - a_3^3)Uy + \\ a_3^3[(a_1^1 - a_3^3)g + a_1^2 h]Vyz + a_3^2 UVz^2\}\frac{\partial}{\partial z}, \end{gathered} \tag{56}$$

where

$$W=(a_2^2-a_3^3)g-a_1^2h, V=-2(a_3^2g-a_1^2k),\\ T=a_1^1+a_2^2-a_3^3, U=a_2^2g-a_1^2h. \tag{57}$$

These operators form the Lie algebra L_5 with the structure equations

$$\begin{gathered}[Y_1,Y_2]=-a_1^2a_3^3Y_1,\ \ [Y_1,Y_3]=a_1^2a_3^3TY_1,\\ [Y_1,Y_4]=a_1^2(a_3^3)^2[(a_1^1-a_3^3)g+a_1^2h]Y_1,\\ [Y_1,Y_5]=-a_1^2a_3^3[-\frac{TV}{2}Y_1-(T-a_3^3)UY_2+WY_3+Y_4],\\ [Y_2,Y_3]=0,\ \ [Y_2,Y_4]=a_1^2a_3^2a_3^3UY_1,\ \ [Y_2,Y_5]=a_1^2a_3^3(a_3^2UY_2+\frac{V}{2}Y_3-Y_5),\\ [Y_3,Y_4]=-a_1^2a_3^2a_3^3TUY_1,\ \ [Y_3,Y_5]=a_1^2a_3^3(-a_3^2TUY_2-\frac{TV}{2}Y_3+TY_5),\\ [Y_4,Y_5]=a_1^2a_3^3\{\frac{a_3^2TUV}{2}Y_1+a_3^2TUWY_2+[a_3^2a_3^3g^2T-a_3^2U^2-\\ -a_1^2a_3^3k[(a_1^1-a_3^3)g+a_1^2h]]Y_3-a_3^2UY_4+a_3^3[(a_1^1-a_3^3)g+a_1^2h]Y_5\},\end{gathered} \tag{58}$$

where T,U and V are from (57).

Using the operators Y_2 and Y_3, which form by (56) and (58) a two-dimensional commutative Lie algebra, we obtain from (5) the Lie integrating factor of the form

$$\mu^{-1}=z(-a_1^2x+(a_1^1-a_3^3)y-a_3^2z)(-a_1^2x-(a_2^2-a_3^3)y-a_3^2z)\Phi,$$

where Φ is from (55).

Taking into account this expression and Theorem 1, we obtain the functional-independent integrals (54)–(55) of system (6). The conditions (19) and $\varkappa_2q_1\not\equiv 0$ from (8)–(9) imply that not all coefficients in this system are equal to zero. □

Theorem 14. *Assume the coefficients of the linear part of system* (6) *satisfy the conditions* (20) *with* $\varkappa_2q_1\not\equiv 0$ *from* (8)–(9)*. Then the general integral of this system with notations* $x=x^1,\ y=x^2,\ z=x^3$ *consists of two first integrals:*

$$F_1\equiv(a_2^3x-a_2^1z)[a_2^3y+(a_1^1-a_2^2)z]^{-1}=C_1;\\ F_2\equiv(a_2^3x-a_2^1z)^{a_1^1-a_2^2-a_3^3}[a_2^3y-(a_1^1-a_3^3)z]^{a_1^1}\Phi^{-2a_1^1+a_2^2+a_3^3}=C_2, \tag{59}$$

where

$$\Phi=a_2^3(a_1^1-a_2^2-a_3^3)(a_1^1+2gx)+2a_2^3(a_2^1g-a_3^3h+a_2^3k)y+\\ +2[-a_2^1(a_1^1-a_3^3)g+a_1^1(a_1^1-a_2^2-a_3^3)h+a_2^2(a_3^3h-a_2^3k)]z. \tag{60}$$

Proof. Let the coordinates of the operator (3) have the form (27). Then solving (4) we obtain for differential system (6) the operators ($x=x^1, y=x^2, z=x^3$):

$$\begin{gathered}
Y_1=(a_1^1+2gx)(a_2^3x-a_2^1z)\frac{\partial}{\partial x}+2g(a_2^3x-a_2^1z)y\frac{\partial}{\partial y}+2g(a_2^3x-a_2^1z)z\frac{\partial}{\partial z},\\
Y_2=(a_1^1+2gx)(a_2^3y+(a_1^1-a_2^2)z)\frac{\partial}{\partial x}+2g(a_2^3y+(a_1^1-a_2^2)z)y\frac{\partial}{\partial y}+\\
+2g(a_2^3y+(a_1^1-a_2^2)z)z\frac{\partial}{\partial z},\\
Y_3=(2a_2^3Wxy+a_1^1a_2^1Tz-2(a_2^2W-a_1^1hT)xz)\frac{\partial}{\partial x}+\\
+[a_1^1a_2^3T+2a_2^3Wy-2(a_2^2W-a_1^1hT)z]y\frac{\partial}{\partial y}+[a_1^1a_2^3T+\\
+2a_2^3Wy-2(a_2^2W-a_1^1hT)z]z\frac{\partial}{\partial z},\\
Y_4=\{-2a_2^3(a_1^1hT+a_3^3(a_3^3h-a_2^3k)-a_2^1(a_1^1-a_2^2)g)xy+\\
+a_1^1a_2^1(a_1^1-a_2^2)Tz+2(a_1^1-a_2^2)[(a_1^1-a_3^3)(a_3^3h-a_2^3k)-a_2^1a_2^2g]xz\}\frac{\partial}{\partial x}+\\
+\{a_1^1a_2^3(a_3^3-a_2^2)Ty-2a_2^3[a_1^1hT+a_3^3(a_3^3h-a_2^3k)-a_2^1g(a_1^1-a_2^2)]y^2-\\
-a_1^1(a_1^1-a_2^2)(a_1^1-a_3^3)Tz+2(a_1^1-a_2^2)[(a_3^3h-a_2^3k)(a_1^1-a_3^3)-\\
-a_2^1a_2^2g]yz\}\frac{\partial}{\partial y}+\{-a_1^1(a_2^3)^2Ty-2a_2^3[a_1^1hT++a_3^3(a_3^3h-a_2^3k)-\\
-a_2^1g(a_1^1-a_2^2)]yz+2(a_1^1-a_2^2)[(a_3^3h-a_2^3k)(a_1^1-a_3^3)-a_2^1a_2^2g]z^2\}\frac{\partial}{\partial z},\\
Y_5=\{-a_1^1a_2^3[(a_1^1-a_3^3)h+a_2^3k]Tx+2a_2^1a_2^3gWxy+\\
+a_1^1a_2^1[a_2^1g+(a_1^1-a_3^3)h+a_2^3k]Tz-2a_2^1g(a_2^2W-a_1^1hT)xz\}\frac{\partial}{\partial x}+\\
+[a_1^1a_2^3(a_1^1-a_3^3)gTx+a_1^1a_2^1a_2^3gTy+2a_2^1a_2^3gWy^2-\\
-a_1^1a_2^1(a_1^1-a_3^3)gTz-2a_2^1g(a_2^2W-a_1^1hT)yz]\frac{\partial}{\partial y}+\\
+[a_1^1(a_2^3)^2gTx+2a_2^1a_2^3gWyz-2a_2^1g(a_2^2W-a_1^1hT)z^2]\frac{\partial}{\partial z},
\end{gathered}\tag{61}$$

where $W=a_2^1g-a_3^3h+a_2^3k,\ \ T=a_1^1-a_2^2-a_3^3$.

These operators form the Lie algebra L_5 with the structure equations

$$\begin{gathered}
a_2^1T[Y_1,Y_2]=[Y_1,Y_4]=a_2^1[Y_2,Y_3]=-T[Y_3,Y_4]=-a_1^1a_2^1a_2^3TY_2,\\
[Y_1,Y_3]=0,\ [Y_1,Y_5]=a_1^1a_2^3[(a_2^1g+(a_1^1-a_3^3)h+a_2^3k)TY_1-a_2^1gY_3+Y_5,\\
[Y_2,Y_5]=a_1^1a_2^3[-(2a_1^1-a_2^2-a_3^3)gTY_1-[a_2^1g+(a_1^1-a_3^3)h+a_2^3k]TY_2+\\
+(a_1^1-a_2^2)gY_3-gY_4],\ \ [Y_2,Y_4]=a_1^1a_2^3(a_1^1-a_3^3)TY_2,\\
[Y_3,Y_5]=a_1^1a_2^3T\{-[a_2^1g+(a_1^1-a_3^3)h+a_2^3k]TY_1+a_2^1gY_3-Y_5\},\\
[Y_4,Y_5]=-a_1^1a_2^3T\{[a_2^1g(a_1^1-a_2^2)-[(a_1^1-a_3^3)h+a_2^3k](a_1^1-a_3^3)]TY_1+\\
+a_2^1(a_2^1g+(a_1^1-a_3^3)h+a_2^3k)TY_2+a_2^1(a_2^2-a_3^3)gY_3+a_2^1gY_4-(a_1^1-a_3^3)Y_5\}.
\end{gathered}\tag{62}$$

If we use the operators Y_1 and Y_3, which form by (61) and (62) a two-dimensional commutative Lie algebra, we obtain from (5) the Lie integrating factor of the form

$$\mu^{-1} = (a_2^3 x - a_2^1 z)[a_2^3 y + (a_1^1 - a_2^2)z][a_2^3 y - (a_1^1 - a_3^3)z]\Phi,$$

where Φ is from (60).

Taking into account this expression and Theorem 1, we obtain the functional-independent integrals (59)–(60) of system (6). The conditions (20) and $\varkappa_2 q_1 \not\equiv 0$ from (8)–(9) imply that not all coefficients in this system are equal to zero. □

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