BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(82), 2016, Pages 38–56
ISSN 1024–7696

Cubic systems with degenerate infinity and invariant straight lines of total parallel multiplicity five

Alexandru Şubă*, Vadim Repeşco

Abstract. In this paper cubic systems which have degenerate infinity and invariant straight lines of total multiplicity five are classified. It is proved that, modulo affine transformations and time rescaling, there are 24 classes of such systems. For every class the qualitative investigation was carried out in the Poincaré disc.

Mathematics subject classification: 34C05.
Keywords and phrases: Cubic differential system, invariant straight line, phase portrait.

1 Introduction and statement of main results

We consider the real cubic differential system

$$\frac{dx}{dt} = \sum_{r=0}^{3} P_r(x,y) \equiv P(x,y), \ \frac{dy}{dt} = \sum_{r=0}^{3} Q_r(x,y) \equiv Q(x,y), \quad \gcd(P,Q)=1, \quad (1)$$

where P_r, Q_r are homogeneous polynomials of degree r and $|P_3(x,y)| + |Q_3(x,y)| \not\equiv 0$.

A curve $f(x,y)=0,\ f \in \mathbb{C}[x,y]$, is said to be an *invariant algebraic curve* of (1) if there exists a polynomial $K_f \in \mathbb{C}[x,y]$ such that the identity $\frac{\partial f}{\partial x}P(x,y)+\frac{\partial f}{\partial y}Q(x,y) \equiv \equiv f(x,y)K_f(x,y)$ holds. We say that an invariant algebraic curve $f(x,y)=0$ has the *parallel multiplicity* equal to m, if m is the greatest positive integer such that f^{m-1} divides K_f.

The system (1) is called *Darboux integrable* if there exists a non-constant function of the form $F = f_1^{\lambda_1} \cdots f_s^{\lambda_s}$, where f_j is an invariant algebraic curve and $\lambda_j \in \mathbb{C}$, $j=\overline{1,s}$, such that either F is a first integral or is an integrating factor for (1). We will be interested in invariant algebraic curves of degree one, that is invariant straight lines $\alpha x + \beta y + \gamma = 0,\ (\alpha,\beta) \neq (0,0)$.

There are a great number of works dedicated to the investigation of polynomial differential systems with invariant straight lines.

The problem of estimating the number of invariant straight lines which a polynomial differential system can have was considered in [1]; the problem of coexistence of invariant straight lines and limit cycles in [4,5]; the problem of coexistence of invariant straight lines and singular points of center type for cubic systems in [3,10]. The

*Supported by FP7-PEOPLE-2012-IRSES-316338 and 15.817.02.03F

classification of all cubic systems with the maximum number of invariant straight lines, taking into account their multiplicities, is given in [6].

In [1] it was proved that the cubic system (1) can have in the finite part of the phase plane at most eight invariant straight lines. Cubic systems with exactly eight invariant straight lines has been studied in [6, 7] and with total parallel multiplicity of invariant straight lines equal to seven in [11, 13]. A qualitative investigation of systems (1) with six real invariant straight lines along two (three) directions is given in [8] ([9]). In [12] we examined some cubic systems with degenerate infinity that have invariant straight lines of total parallel multiplicity five or six, three of which are parallel. In [14] all canonical forms of the cubic systems with degenerate infinity that have invariant straight line of total parallel multiplicity equal to six were obtained.

In this paper we continue the investigation from [8, 9, 12, 14] and give a full qualitative study of cubic systems (1) with degenerated infinity and invariant straight lines of total multiplicity six.

Theorem 1. *Assume that a cubic system with degenerate infinity possesses invariant straight lines of total parallel multiplicty five. Then via an affine transformation and time rescaling this system can be brought to one of the systems 1)–24). Moreover, up to topological equivalence, its phase portrait on the Poincaré disc corresponds to one of the portraits given in Fig. 1 – Fig. 23. In the table below for each of the systems 1) – 24) the first arrow points to the straight lines and the first integral F (or integrating factor μ) that corresponds to the system.*

1) $\begin{cases} \dot{x} = x\,(x+1)\,(x-a)\,, \;\; a>0, \;\; c\neq 2, \\ \dot{y} = y\left(-a+c\,x-y+x^2\right), \; a+c>1; \\ \text{Configuration } (3r, 1r, 1r) \end{cases}$ $\to$ (2) $\to$ Fig. 1;

2) $\begin{cases} \dot{x} = x\,(x+1)\,(x-a)\,, \; a>0, \; b>0, \\ \dot{y} = y\left(b+(b-a)\,x-y+x^2\right), \; b-a\neq 0; \\ \text{Configuration } (3r, 1r, 1r) \end{cases}$ $\to$ (3) $\to$ Fig. 2;

3) $\begin{cases} \dot{x} = x(x+1)(x-a), \;\; a>0, \\ \dot{y} = y(x+1)(x-a)+x^2+y^2; \\ \text{Configuration } (3r, 1c_1, 1c_1) \end{cases}$ $\to$ (4) $\to$ Fig. 3;

4) $\begin{cases} \dot{x} = x(x+1)(x-a), \;\; a>0, \\ \dot{y} = (x+1)^2+xy(x-a)+by^2, \;\; b>0; \\ \text{Configuration } (3r, 1c_1, 1c_1) \end{cases}$ $\to$ (5) $\to$ Fig. 4;

5) $\begin{cases} \dot{x} = (x-a)(x^2+1), \;\; a\in\mathbb{R}, \\ \dot{y} = y(1-ac+cx-y+x^2), \;\; c\neq 0; \\ \text{Configuration } (1r+2c_0, 1r, 1r) \end{cases}$ $\to$ (6) $\to$ Fig. 5;

6) $\begin{cases} \dot{x} = (x-a)(x^2+1), \;\; a\in\mathbb{R}, \\ \dot{y} = (x-a)^2+y+\frac{1}{b}y^2+x^2y, \;\; b>0; \\ \text{Configuration } (1r+2c_0, 1c_1, 1c_1) \end{cases}$ $\to$ (7) $\to$ Fig. 6;

7) $\begin{cases} \dot{x} = x^2(x+1), \ a > 0, \\ \dot{y} = y\left((a+1)x - y + x^2\right); \\ \text{Configuration } (3(2)r, 1r, 1r) \end{cases}$ $\to$ (8) $\to$ Fig. 7;

8) $\begin{cases} \dot{x} = x^2(x+1), \\ \dot{y} = y(a + ax - y + x^2), \ a \neq 0; \\ \text{Configuration } (3(2)r, 1r, 1r) \end{cases}$ $\to$ (9) $\to$ Fig. 8;

9) $\begin{cases} \dot{x} = x^2(x+1), \ a > 0, \\ \dot{y} = ax^2 + xy + ay^2 + x^2y; \\ \text{Configuration } (3r, 1c_1, 1c_1) \end{cases}$ $\to$ (10) $\to$ Fig. 9;

10) $\begin{cases} \dot{x} = x^2(x+1), \ a \neq 0, \\ \dot{y} = a(x+1)^2 + ay^2 + x^2y; \\ \text{Configuration } (3(2)r, 1c_1, 1c_1) \end{cases}$ $\to$ (11) $\to$ Fig. 10;

11) $\begin{cases} \dot{x} = x^3, \ a > 0, \\ \dot{y} = y(ax - y + x^2); \\ \text{Configuration } (3(3)r, 1r, 1r) \end{cases}$ $\to$ (12) $\to$ Fig. 11;

12) $\begin{cases} \dot{x} = x^3, \ a > 0, \\ \dot{y} = ax^2 + ay^2 + x^2y; \\ \text{Configuration } (3(3)r, 1c_1, 1c_1) \end{cases}$ $\to$ (13) $\to$ Fig. 12;

13) $\begin{cases} \dot{x} = x(x-1)(y+a), \\ \dot{y} = y(y-1)(x+a), \ a \notin \{-1; -1/2; 0\}; \\ \text{Configuration } (2r, 2r, 1r) \end{cases}$ $\to$ (14) $\to$ Fig. 13;

14) $\begin{cases} \dot{x} = x^2(y+a), a > 0, b > 0, \\ \dot{y} = y^2(x+b), \ ab \neq 0; \\ \text{Configuration } (2(2)r, 2(2)r, 1r) \end{cases}$ $\to$ (15) $\to$ Fig. 14;

15) $\begin{cases} \dot{x} = (x^2+1)(y+a), \\ \dot{y} = (y^2+1)(x+a), \ \ a \neq 0; \\ \text{Configuration } (2c_0, 2c_0, 1r) \end{cases}$ $\to$ (16) $\to$ Fig. 15;

16) $\begin{cases} \dot{x} = x(a - 2ay + x^2 + y^2), \ a \notin \{0; 1/2; 1\}, \\ \dot{y} = ay + (a-1)x^2 - (a+1)y^2 + x^2y + y^3; \\ \text{Configuration } (2c_1, 2c_1, 1r) \end{cases}$ $\to$ (17) $\to$ Fig. 16;

17) $\begin{cases} \dot{x} = 2(\frac{x}{2} + by + bx^2 - xy - by^2 + x^3 + xy^2), \\ \dot{y} = (2y-1)(2bx - y + x^2 + y^2), \ b \neq 0; \\ \text{Configuration } (2c_1, 2c_1, 1r) \end{cases}$ $\to$ (18) $\to$ Fig. 17;

18) $\begin{cases} \dot{x} = ax^2 + 2bxy - ay^2 + x^3 + xy^2, \\ \dot{y} = -bx^2 + 2axy + by^2 + x^2y + y^3, \\ \quad |a| + |b| \neq 0, \ a \geq 0; \\ \text{Configuration } (2(2)c_1, 2(2)c_1, 1r) \end{cases}$ $\to$ (19) $\to$ Fig. 18;

19) $\begin{cases} \dot{x} = x(x-1)(1 + (a-1)x + (b-1)y), \\ \dot{y} = y(-1 + 2x + y + (a-1)x^2 + (b-1)xy), \\ \quad ab(b-1)(b+1)(a-b) \neq 0; \\ \text{Configuration } (2r, 1r, 1r, 1r) \end{cases}$ $\to$ (20) $\to$ Fig. 19;

20) $\begin{cases} \dot{x} = (1+(x-a)^2)(x+by),\ b \neq 0, \\ \dot{y} = (a^2+1)(y-bx)+(ab-1)x^2-2axy- \\ \quad -(ab+1)y^2+x^2y+bxy^2; \\ \text{Configuration } (2c_0, 1r, 1c_1, 1c_1) \end{cases}$ $\rightarrow$ (21) $\rightarrow$ Fig. 20;

21) $\begin{cases} \dot{x} = x+cy+(2a+c)x^2+2(-1+ac)xy-cy^2+ \\ \qquad +(a^2+b^2-b+ac)x^3+(-2a-c+a^2c+ \\ \qquad +b^2c)x^2y-(b-1+ac)xy^2, \\ \dot{y} = -cx+y+(b-ac)x^2+2(a+c)xy+(b-2+ \\ \qquad +ac)y^2+(a^2+b^2-b+ac)x^2y+(-2a-c+ \\ \qquad +a^2c+b^2c)xy^2-(b-1+ac)y^3, \\ \quad bc(|a|+|b^2-1|) \neq 0; \\ \text{Configuration } (1r, 1c_1, 1c_1, 1c_1, 1c_1) \end{cases}$ $\rightarrow$ (22) $\rightarrow$ Fig. 17, Fig. 21;

22) $\begin{cases} \dot{x} = x(1+2ax-2y+(a^2+b^2-c)x^2- \\ \qquad -2axy-(c-1)y^2), \\ \dot{y} = y+cx^2+2axy+(c-2)y^2+(a^2+b^2- \\ \qquad -c)x^2y-2axy^2-(c-1)y^3, \\ \quad bc(b^2-c^2)(|a|+|b^2-1|) \neq 0; \\ \text{Configuration } (1r, 1c_1, 1c_1, 1c_1, 1c_1) \end{cases}$ $\rightarrow$ (23) $\rightarrow$ Fig. 16;

23) $\begin{cases} \dot{x} = x(1+(a+b)x-2y+(ab-c)x^2- \\ \qquad -(a+b)xy+(1-c)y^2) \\ \dot{y} = y+cx^2+(a+b)xy+(c-2)y^2+(ab- \\ \qquad -c)x^2y-(a+b)xy^2+(1-c)y^3, \\ \quad c(b-a) \neq 0; \\ \text{Configuration } (1r, 1r, 1r, , 1c_1, 1c_1) \end{cases}$ $\rightarrow$ (24) $\rightarrow$ Fig. 22;

24) $\begin{cases} \dot{x} = x(1+(a+b)x-2y+abx^2+(1-a- \\ \qquad -b-c)xy+cy^2), \\ \dot{y} = y(1+\alpha x-(c+1)y+abx^2-\alpha xy+cy^2), \\ \quad \alpha = a+b+c-1, \\ \quad ab(a-1)(b-1)(c-1) \neq 0,\ a > b; \\ \text{Configuration } (1r, 1r, 1r, 1r, 1r) \end{cases}$ $\rightarrow$ (25) $\rightarrow$ Fig. 23.

$$l_1 = x+1,\ l_2 = x,\ l_3 = x-a,\ l_4 = y,\ l_5 = (a+c-1)x-y; \\ F = (l_1/l_3)^{a+c-1}(l_4/l_5)^{a+1}; \tag{2}$$

$$l_1 = x+1,\ l_2 = x,\ l_3 = x-a,\ l_4 = y,\ l_5 = b(x+1)-y; \\ F = l_2^b l_3^{-b} l_4^a l_5^{-a}; \tag{3}$$

$$l_1 = x+1,\ l_2 = x,\ l_3 = x-a,\ l_{4,5} = y \pm ibx; \\ \mu(x,y) = 1/(l_1 l_3 l_4 l_5); \tag{4}$$

$$l_1 = x+1,\ l_2 = x,\ l_3 = x-a,\ l_{4,5} = y \pm i\sqrt{b}\,(x+1); \\ \mu(x,y) = 1/(l_2 l_3 l_4 l_5); \tag{5}$$

$$l_1 = x-i,\ l_2 = x-a,\ l_3 = x+i,\ l_4 = y,\ l_5 = cx-y-ac; \\ \mu(x,y) = 1/(l_1 l_3 l_4 l_5),\ \ F = y\exp(-c \cdot arctan(x))/l_5; \tag{6}$$

$$l_1 = x - i,\ l_2 = x - a,\ l_3 = x + i,\ l_{4,5} = y \pm i(x - a); \quad \mu(x, y) = 1/(l_1 l_3 l_4 l_5); \tag{7}$$

$$l_1 = x + 1,\ l_2 \equiv l_3 = x,\ l_4 = y,\ l_5 = ax - y; \quad F = l_l^{1-a} l_2^{a-1} l_4^{-1} l_5; \tag{8}$$

$$l_1 = x + 1,\ l_2 \equiv l_3 = x,\ l_4 = y,\ l_5 = a + ax - y; \quad F = y \exp(a/x)/(a + ax - y). \tag{9}$$

$$l_1 = x + 1,\ l_2 \equiv l_3 = x,\ l_{4,5} = y \pm ix; \quad \mu(x, y) = 1/(l_1 l_2 l_4 l_5); \tag{10}$$

$$l_1 = x + 1,\ l_2 \equiv l_3 = x,\ l_{4,5} = y \pm i(x + 1); \quad \mu(x, y) = 1/(l_2^2 l_4 l_5); \tag{11}$$

$$l_{1,2,3} = x,\ l_4 = y, l_5 = ax - y; \quad F = y \exp(a/x)/(ax - y); \tag{12}$$

$$l_{1,2,3} = x,\ l_{4,5} = y \pm ix; \quad \mu(x, y) = 1/(l_1^2 l_4 l_5). \tag{13}$$

$$l_1 = x,\ \ l_2 = x - 1,\ \ l_3 = y,\ \ l_4 = y - 1,\ \ l_5 = x - y; \quad F = (l_1/l_3)^a (l_4/l_2)^{a+1}; \tag{14}$$

$$l_1 \equiv l_2 = x,\ \ l_3 \equiv l_4 = y,\ \ l_5 = ax - by; \quad F = l_1 l_3^{-1} exp((ax - by)/(xy)); \tag{15}$$

$$l_{1,2} = x \pm i,\ l_{3,4} = y \pm i,\ l_5 = x - y; \quad \mu(x, y) = 1/(l_1 l_2 l_3 l_4); \tag{16}$$

$$l_{1,2} = y \mp ix,\ l_{3,4} = y \mp ix - 1,\ l_5 = x; \quad \mu(x, y) = 1/(l_1 l_2 l_3 l_4); \tag{17}$$

$$l_{1,2} = y \mp ix,\ l_{3,4} = y \mp ix - 1,\ l_5 = 2y - 1; \quad \mu(x, y) = 1/(l_1 l_2 l_3 l_4); \tag{18}$$

$$l_{1,3} = y - ix,\ l_{2,4} = y + ix,\ l_5 = bx - ay; \quad \mu(x, y) = 1/(l_1 l_2)^2. \tag{19}$$

$$l_1 = x,\ l_2 = x - 1,\ l_3 = y,\ l_4 = x + y - 1,\ l_5 = ax + by; \quad F = l_1 l_2^{-b} l_4^b l_5^{-1}; \tag{20}$$

$$l_{1,2} = x - a \pm i,\ l_{3,4} = y \pm ix,\ l_5 = ax + y - a^2 - 1; \quad \mu(x, y) = 1/(l_1 l_2 l_3 l_4. \tag{21}$$

$$l_{1,2} = y \mp xi,\ l_{3,4} = y - (a \pm bi)x - 1,\ l_5 = 1 + ax - y + by; \quad \mu(x, y) = 1/(l_1 l_2 l_3 l_4); \tag{22}$$

$$l_{1,2} = y \mp xi,\ l_{3,4} = y - (a \pm bi)x - 1,\ l_5 = x; \quad \mu(x, y) = 1/(l_1 l_2 l_3 l_4); \tag{23}$$

$$l_{1,2} = y \mp ix,\ l_3 = y - ax - 1,\ l_4 = y - bx - 1,\ l_5 = x; \quad \mu(x, y) = 1/(l_1 l_2 l_3 l_4); \tag{24}$$

$$l_1 = x,\ l_2 = y,\ l_3 = y - x,\ l_4 = y - ax - 1,\ l_5 = y - bx - 1; \quad F = l_2^{(b-a)} l_3^{(a-b)} l_4^{(c-1)} l_5^{(1-c)}. \tag{25}$$

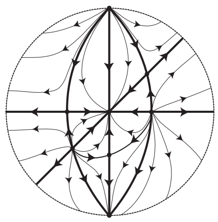


Fig. 1

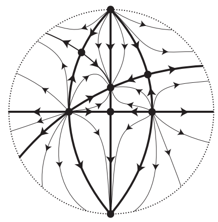
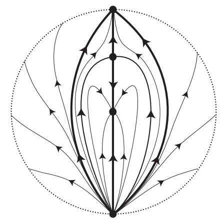


Fig. 2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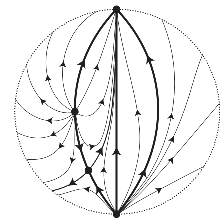
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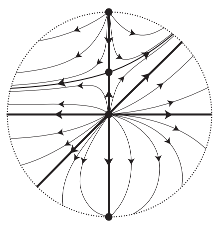
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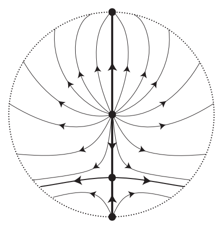
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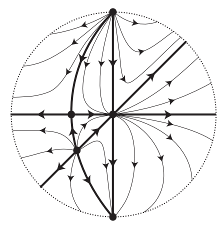
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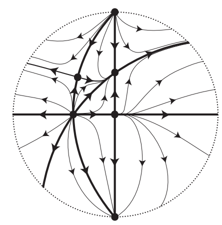
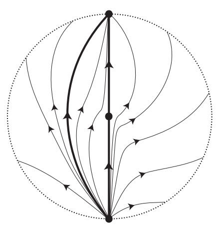


Fig. 8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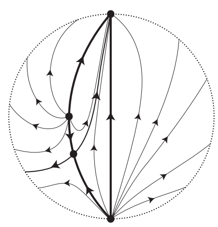
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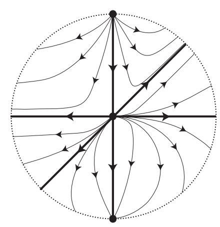
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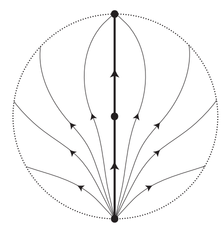
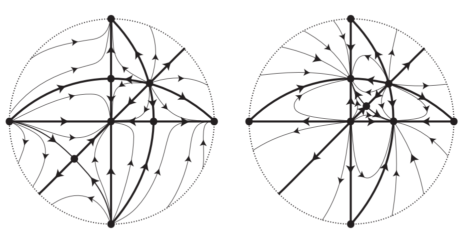
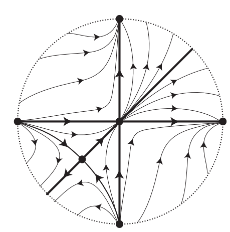


Fig. 12


a) b)
Fig. 13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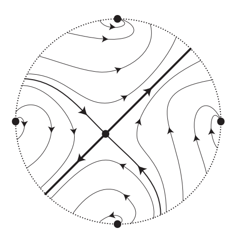
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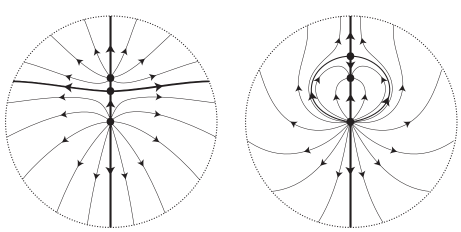

a) b)
Fig. 16

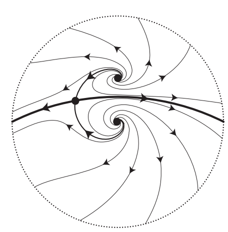
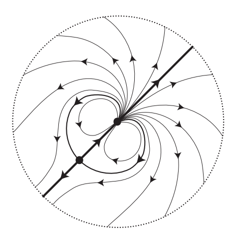


Fig. 17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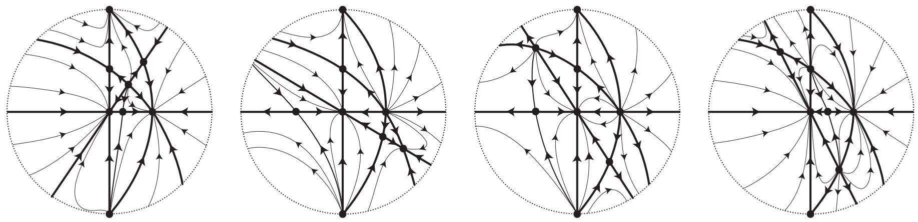
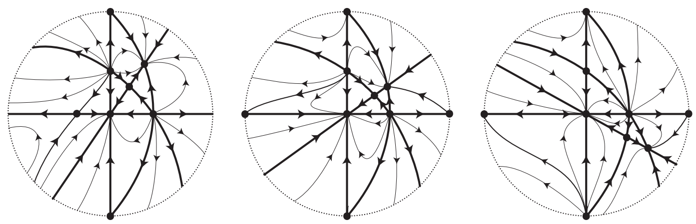


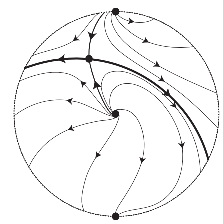
a) b) c) d)

Fig. 19



e) f) g)

Fig. 19



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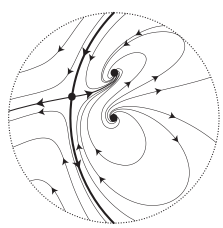
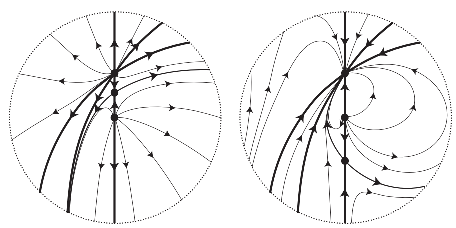
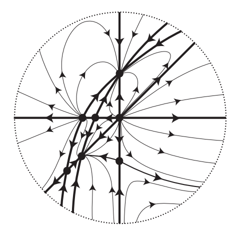


Fig. 21



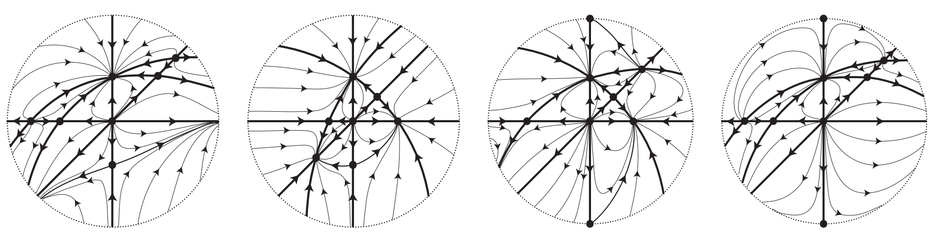
a) b)

Fig. 22



a)

Fig. 23



b) c) d) e)

Fig. 23

2 Some properties of cubic systems with straight lines

By *a configuration of straight lines* we understand the $\mathbb{R}^2$ plane with a certain number of straight lines.

To each two-dimensional differential system (with invariant straight lines) we can associate a configuration consisting of invariant straight lines of this system. It is easy to show that the converse is not always true.

The problem arises to determine for invariant straight lines such properties that allow to construct all realizable configurations of invariant straight lines for (1). Below we shall enumerate these properties. Their proofs are rather easy and we omit them.

Proposition 1. *The system* (1) *has at most nine singular points in the finite part of the phase plane.*

Proposition 2. *There are at most 3 singular points of system* (1) *on any invariant straight line in the finite part of the phase plane.*

A straight line l will be called *complex* if $l \in \mathbb{C}[x, y] \setminus \mathbb{R}[x, y]$.

Proposition 3. *Complex invariant straight lines of system* (1) *occur in complex conjugate pairs (l and $\bar{l}$).*

Proposition 4. *The intersection point* (x_0, y_0) *of two invariant straight lines* l_1 *and* l_2 *of system* (1) *is a singular point. Moreover, if* $l_1, l_2 \in \mathbb{R}[x, y]$ *or* $l_2 \equiv \bar{l_1}$*, then* $x_0, y_0 \in \mathbb{R}$*.*

Proposition 5. *A complex straight line l can pass through at most one point with real coordinates.*

Proposition 6. *If a straight line passes through two distinct real points or through two complex conjugate points, then this straight line is real.*

A complex straight line passing through a real point will be called *a relative complex straight line* and a complex straight line not passing through any real point – *a purely imaginary straight line.*

Proposition 7. *Through any point of a purely imaginary straight line at most one real straight line can pass.*

Proposition 8. *A complex invariant straight line of system* (1) *is purely imaginary iff this straight line is parallel to its conjugate one ($l \parallel \bar{l}$).*

Proposition 9. *Let l_1 and l_2 be two parallel invariant straight lines of the system* (1)*, then only one of the following properties occurs:*
1. $l_1, l_2 \in \mathbb{R}[x, y]$*;* *2. l_1 is real and l_2 is purely imaginary;*
3. l_1 and l_2 are purely imaginary; *4. l_1 and l_2 are relative complex.*

We say that the cubic system (1) has *degenerate infinity* if the following identity

$$yP_3(x, y) - xQ_3(x, y) \equiv 0 \tag{26}$$

holds. In such a case the infinity consists only of singular points.

Proposition 10. *The identity* (26) *is invariant under any affine transformation of the system* (1)*.*

Proposition 11. *Invariant straight lines of the cubic system* (1) *with degenerate infinity passing through the same point* $M_0\,(x_0, y_0)$*,* $x_0, y_0 \in \mathbb{C}$*, have at most three slopes.*

Proposition 12. *Through any point of a complex invariant straight line of the cubic system with degenerate infinity at most one real straight lines can pass.*

Proposition 13. *A straight line passing through three distinct singular points of system (1) with degenerate infinity is invariant for (1).*

Proposition 14. *The maximum number of invariant straight lines for a differential cubic system with degenerate infinity is equal to six.*

Proposition 15. *Let the cubic system* (1) *have two concurrent invariant straight lines* l_1, l_2. *If* l_1 *has the parallel multiplicity equal to* m, $1 \leq m \leq 3$, *then this system cannot have more than* $3 - m$ *singular points on* $l_2 \setminus l_1$.

We say that three straight lines are in *generic position* if all lines have different slopes and no more than two lines pass through a point.

Proposition 16. *Let the cubic system* (1) *have 3 invariant straight lines in generic position, then their total parallel multiplicity is at most four.*

Proposition 17. *The cubic system* (1) *with degenerate infinity can have at most one triplet of parallel invariant straight lines.*

Proposition 18. *The cubic system* (1) *with degenerate infinity can have at most two pair of parallel invariant straight lines.*

3 The proof of Theorem 1

Using Propositions 17 and 18, the family of cubic systems [(1)][(26)] with six invariant straight lines can be divided in four classes:

A) Systems with a triplet of parallel invariant straight lines;

B) Systems with two pairs of parallel invariant straight lines;

C) Systems with only a pair of parallel invariant straight lines;

D) Systems with invariant straight lines of different slopes.

The class A) was studied in [8,12] and is characterized by the systems 1)–12) of Theorem 1.

3.1 Class B): two pairs of parallel invariant straight lines

For cubic systems in class B) the following 8 configurations of invariant straight lines are possible:

B1) $(\mathbf{2r}, \mathbf{2r}, \mathbf{1r})$	B2) $(2(2)r, 2r, 1r,)$	**B3)** $(\mathbf{2(2)r}, \mathbf{2(2)r}, \mathbf{1r})$
B4) $(2r, 2c_0, 1r)$	B5) $(2(2)r, 2c_0, 1r)$	**B6)** $(\mathbf{2c_0}, \mathbf{2c_0}, \mathbf{1r})$
B7) $(\mathbf{2c_1}, \mathbf{2c_1}, \mathbf{1r})$	**B8)** $(\mathbf{2(2)c_1}, \mathbf{2(2)c_1}, \mathbf{1r})$	

By $(2r, 2r, 1r)$ we denoted the configuration which consists of five distinct real straight lines $l_1, \ldots, l_5 \in \mathbb{R}[x, y]$, of which l_1, l_2 and l_3, l_4 form two pairs of parallel straight lines, i.e. $l_1 \parallel l_2$, $l_3 \parallel l_4$, $l_1 \nparallel l_3$ and $l_j \nparallel l_5$, $j = 1, \ldots, 4$. In the case of configuration $(2c_0, 2c_0, 1r)$ we have five straight lines $l_1, \ldots, l_5$, where l_1, l_2, l_3 and l_4 are purely imaginary, l_5 is real, l_1, l_2 and l_3, l_4 form two pairs of parallel straight lines. The configuration $(2(2)r, 2r, 1r)$ consists of five real straight lines, where $l_1 \equiv l_2$, $l_3 \parallel l_4$, $l_1 \nparallel l_3$, $l_j \nparallel l_5$, $j = 1, \ldots, 4$, and the straight line l_1 (or l_2) has parallel multiplicity equal to two.

Proposition 19. *Cubic systems with degenerate infinity possessing invariant straight lines of the configuration* $(2(2)r, 2r)$ *can not have other invariant straight lines.*

Indeed, a system of this configuration can be brought to the form:

$$\dot{x} = x^2(y + a), \quad \dot{y} = y(y - 1)(x + b).$$

Since this system has only the following singular points: $(0, 0)$, $(0, 1)$, $(-b, -a)$ and $a(a + 1)b \neq 0$, the above proposition follows.

Remark 1. Propositions 2, 7 and 15 (Proposition 19) do not allow the realization of configurations B4) and B5) (configuration B2)) in the class of cubic systems with degenerate infinity.

Configuration B1) $(\mathbf{2r}, \mathbf{2r}, \mathbf{1r})$. Via an affine transformation and time rescaling the system [(1)][(26)] with two pairs of real invariant straight lines can be written in the form:

$$\dot{x} = x(x - 1)(y + a), \quad \dot{y} = y(y - 1)(x + b), \quad a, b \notin \{-1; 0\}. \tag{27}$$

The system (27) has the invariant straight lines $l_1 = x$, $l_2 = x - 1$, $l_3 = y$, $l_4 = y - 1$ and the singular points $(0, 0)$, $(1, 0), (0, 1)$, $(1, 1)$, $(-b, -a)$. Therefore, any other invariant straight line of (27) must pass through the singular points $(0, 0)$ and $(1, 1)$ or through the singular points $(1, 0)$ and $(0, 1)$. When $(0, 0), (1, 1) \in l_5$ and l_5 is invariant for (27), we get $b = a$, i.e. the system 13) of Theorem 1. The case $(1, 0), (0, 1) \in l_5$ provides an affine equivalent system with the system 13).

Configuration B3) $(\mathbf{2(2)r}, \mathbf{2(2)r}, \mathbf{1r})$. The cubic system with degenerate infinity possessing real invariant straight lines with the configuration $(2(2)r, 2(2)r)$ can be written as:

$$\dot{x} = x^2(y + a), \quad \dot{y} = y^2(x + b), \tag{28}$$

This system has the invariant straight lines $l_{1,2} = x$, $l_{3,4} = y$, $l_5 = ax - by$, i.e. we obtained the system 14) of Theorem 1.

Configuration B6) $(\mathbf{2c_0}, \mathbf{2c_0}, \mathbf{1r})$ In this case the pairs of parallel invariant straight lines can be brought to the form $l_{1,2} = x \pm i$ and $l_{3,4} = y \pm i$. The system [(1)][(26)] with these invariant straight lines has the form

$$\dot{x} = (x^2 + 1)(y + a), \quad \dot{y} = (y^2 + 1)(x + b), \tag{29}$$

with the following singular points: $(-i, -i)$, $(-i, i)$, (i, i), $(i, -i)$, $(-b, -a)$. Any other invariant straight line of system (29) can pass only through the pairs of reciprocally conjugate singular points $(-i, -i)$, (i, i) or $(-i, i)$, $(i, -i)$, therefore it is described by equation $l_5 = x + y$ or $l_5 = x - y$, respectively. The invariance for (29) is conditioned by $b = a$ or $b = -a$. When $b = a$ we have the system 15) of Theorem 1. The case $b = -a$ is affine equivalent with the system 15).

Configuration B7) $(\mathbf{2c_1}, \mathbf{2c_1}, \mathbf{1r})$ Via an affine change of coordinates, the straight lines $l_1, \ldots, l_4$ can be brought to the form $l_{1,2} = y \pm ix$, $l_{3,4} = y \pm ix - 1$. The cubic system [(1)][(26)] with these invariant straight lines has the form:

$$\begin{cases} \dot{x} = ax + by + bx^2 - 2axy - by^2 + x^3 + xy^2, \\ \dot{y} = -bx + ay + (a-1)x^2 + 2bxy - (a+1)y^2 + x^2y + y^3. \end{cases} \tag{30}$$

The obtained system has the following singular points: $(0,0)$, $(-i/2, 1/2)$, $(0,1)$, $(i/2, 1/2)$, $(-b, a)$. Any other real invariant straight line l_5 can pass only through the singular points $(0,0)$, $(0,1)$ or $(-i/2, 1/2)$, $(i/2, 1/2)$, therefore it is described by equation $l_5 = x$ or $l_5 = 2y - 1$, respectively. This straight line is invariant for system (30) iff $b = 0$ or $a = 1/2$. Thus, was obtained the systems 16) and 17) of Theorem 1.

Configuration B8) $(\mathbf{2(2)c_1}, \mathbf{2(2)c_1}, \mathbf{1r})$ Via an affine transformation and time rescaling, we can bring the pair of conjugate complex invariant straight lines to the form $l_{1,2} = y \pm ix$. The cubic system [(1)][(26)] with these invariant straight lines has the form:

$$\begin{cases} \dot{x} = a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + (a_{20} - b_{11})y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2, \\ \dot{y} = -a_{01}x + a_{10}y + (b_{02} - a_{11})x^2 + b_{11}xy + b_{02}y^2 + a_{30}x^2y + a_{21}xy^2 + a_{12}y^3. \end{cases} \tag{31}$$

Each of straight lines $l_{1,2} = y \pm ix$ has parallel multiplicity equal to two iff $a_{01} = a_{10} = a_{21} = 0$, $a_{11} = 2b_{02}$, $b_{11} = 2a_{20}$, $a_{30} = a_{12}$. Via a time rescaling, we can make $a_{12} = 1$. Denoting by $a_{20} = a$ and $b_{02} = b$, we obtain the system 18) of Theorem 1.

3.2 Class C): one pair of parallel invariant straight lines

For cubic systems in class C) the following 6 configurations of invariant straight lines are possible:

C1) $(\mathbf{2r}, \mathbf{1r}, \mathbf{1r}, \mathbf{1r})$	C2) $(2(2)r, 2r, 1r, 1r)$	C3) $(2r, 1r, 1c_1, 1c_1)$
C4) $(2(2)r, 1r, 1c_1, 1c_1)$	C5) $(2c_0, 1r, 1r, 1r)$	**C6)** $(\mathbf{2c_0}, \mathbf{1r}, \mathbf{1c_1}, \mathbf{1c_1})$

Remark 2. Propositions 2, 7 and 15 do not allow the realization of configurations C2) and C4) in the class of cubic systems with degenerate infinity.

Proposition 20. *The configurations C3) and C5) do not realize in the class of cubic systems with degenerate infinity.*

Proof. Let the cubic system [(1)][(26)] has only two distinct parallel invariant straight lines l_1 and l_2. If these straight lines are real, then [(1)][(26)] can be written in the following form:

$$\begin{cases} \dot{x} = x(x-a)(a_{20} + a_{30}x + a_{21}y), \\ \dot{y} = b_{00} + b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{02}y^2 + a_{30}x^2y + a_{21}xy^2, \end{cases} \tag{32}$$

and if these straight lines are complex, then we have the system

$$\begin{cases} \dot{x} = (x^2+1)(a_{20}+a_{30}x+a_{21}y), \\ \dot{y} = b_{00}+b_{10}x+b_{01}y+b_{20}x^2+b_{11}xy+b_{02}y^2+a_{30}x^2y+a_{21}xy^2. \end{cases} \tag{33}$$

The invariant straight lines of the system (32) (respectively, (33)) are $l_1 = x$ and $l_2 = x - a$ (respectively, $l_{1,2} = x \pm i$). Taking into account that the right-hand sides of these systems have no common factors, it is easy to see that, for both systems, each straight line l_1 and l_2 can pass through at most two singular points.

Let the system (32) have another real invariant straight line, then via an affine transformation, this system can be brought to the form:

$$\begin{cases} \dot{x} = x(x-a)(a_{20}+a_{30}x+a_{21}y), \\ \dot{y} = y(b_{01}+b_{11}x+b_{02}y+a_{30}x^2+a_{21}xy). \end{cases} \tag{34}$$

The invariant straight lines of (34) are: $l_1 = x$, $l_2 = x-a$, $l_3 = y$. All singular points have real coordinates, thus, considering Proposition 6, all other invariant straight lines must be real, i.e. the configuration C3) is not possible.

The system (33) has at most four invariant straight lines, because of Proposition 7 and the fact that on each invariant straight line l_1, l_2 only two singular points lie. Therefore, the configuration C5) is not realizable. □

Configuration C1) $(\mathbf{2r}, \mathbf{1r}, \mathbf{1r}, \mathbf{1r})$. Let the straight lines l_1, l_2, l_3, l_4 with configuration $(2r, 1r, 1r)$ be invariant for system [(1)][(26)]. These straight lines can be brought to the form $l_1 = x$, $l_2 = x - 1$, $l_3 = y$ and $l_4 = x + y - 1$. Therefore, the system [(1)][(26)] has the following form:

$$\begin{cases} \dot{x} = x(x-1)(b_{01}+b_{11}+a_{30}x+a_{21}y), \\ \dot{y} = y(b_{01}+b_{11}x-b_{01}y+a_{30}x^2+a_{21}xy). \end{cases} \tag{35}$$

The intersection points of the straight lines of the system (35) are $(0,0)$, $(0,1)$ and $(1,0)$. Through the singular point $(1,0)$ the invariant straight lines l_2, l_3 and l_4 pass. According to Proposition 11 any other real invariant straight line must pass through the point $(0,0)$ or $(0,1)$.

Let l_5 be a real straight line for system (35) passing through the point $(0,0)$, i.e. it is described by equation $y = Ax$. This straight line is invariant for the system (35) iff $b_{11} = -2b_{01}$, $A = (a_{30} - b_{01})/(b_{01} - a_{21})$. Without loss of generality we consider $b_{01} = -1$. Let $a_{30} = a-1$ and $a_{21} = b-1$, then we obtain the system 19) of Theorem 1. The conditions $ab(b-1)(b+1)(a-b) \neq 0$ will guarantee that the system 19) is not from another class. Similarly, from the system (35), we can obtain a system possessing five invariant straight lines with $(0,1) \in l_5$, but it will be affine equivalent with system 19).

Configuration C6) $(\mathbf{2c_0}, \mathbf{1r}, \mathbf{1c_1}, \mathbf{1c_1})$. Let the system [(1)][(26)] have four invariant straight lines with configuration $(2c_0, 1c_1, 1c_1)$. The straight lines can be written as $l_{1,2} = x - a \pm i$ and $l_{3,4} = y \pm ix$. The system [(1),(26)] with these invariant straight lines looks as

$$\begin{cases} \dot{x} = ((x-a)^2+1)(a_{30}x + a_{21}y), \\ \dot{y} = (a^2+1)(a_{30}y - a_{21}x) + b_{20}x^2 - 2aa_{30}xy + (b_{20} - 2aa_{21})y^2 + \\ \qquad + a_{30}x^2y + a_{21}xy^2 \end{cases} \tag{36}$$

and has the following singular points: $O_1(a-i, 1+ai)$, $O_2(a+i, 1-ai)$, $O_3(a+i, -1+ai)$, $O_4(a-i, -1-ai)$, $O_5(0,0)$, $O_6(a_{21}(1+a^2)/b_{20}, -a_{30}(1+a^2)/b_{20})$, $O_1 = l_1 \cap l_4$, $O_2 = l_2 \cap l_3$, $O_3 = l_2 \cap l_4$, $O_4 = l_1 \cap l_3$. Any other real invariant straight line of the system (36) must pass through one of two pairs of conjugate complex singular points $\{O_1, O_2\}$ or $\{O_3, O_4\}$, therefore, $l_5 = ax + y - a^2 - 1$ or $l_5 = ax - y - a^2 - 1$, respectively. In the first case, $l_5 = ax + y - a^2 - 1$ is invariant for (36) iff $b_{20} = aa_{21} - a_{30}$. Furthermore, if $a_{30} = 0$, then the system (36) has six invariant straight lines. Let $a_{30} \neq 0$ and denote $a_{21} = b \cdot b_{30}$. After rescaling the time $t = 1/a_{30}\tau$, we get the system 20) of Theorem 1. In the second case, $l_5 = ax - y - a^2 - 1$ is invariant for the system (36) iff $b_{20} = aa_{21} + a_{30}$. Moreover, (36) has exactly five invariant straight lines if $a_{30} \neq 0$. The obtained system is affine equivalent with system 20).

3.3 Class D): invariant straight lines with different slopes

For cubic systems in class B) the following three configurations of invariant straight lines are possible:

D1) $(\mathbf{1r}, \mathbf{1c_1}, \mathbf{1c_1}, \mathbf{1c_1}, \mathbf{1c_1})$ **D2)** $(\mathbf{1r}, \mathbf{1r}, \mathbf{1r}, \mathbf{1c_1}, \mathbf{1c_1})$
D3) $(\mathbf{1r}, \mathbf{1r}, \mathbf{1r}, \mathbf{1r}, \mathbf{1r})$

Configuration D1) $(\mathbf{1r}, \mathbf{1c_1}, \mathbf{1c_1}, \mathbf{1c_1}, \mathbf{1c_1})$**.** Let the system [(1)][(26)] have the invariant straight lines $l_j \in \mathbb{C}[x,y] \setminus \mathbb{R}[x,y]$, $j = \overline{1,4}$, $l_j = \overline{l}_{j+1}$, $j = 1, 3$, $l_j \not\parallel l_k$, $j \neq k$. Via an affine transformation and time rescaling we can bring them to the form $l_{1,2} \equiv y \pm ix = 0$, $l_{3,4} = y - (a \pm bi)x - 1 = 0$, $a, b \in \mathbb{R}$, $b(|a| + |b \pm 1|) \neq 0$. There are two affine different systems [(1)][(26)] with these invariant straight lines:

$$\begin{cases} \dot{x} = y + x^2 + 2axy - y^2 + (2a - b_{02})x^3 + (a^2 + b^2 - 1)x^2y - b_{02}xy^2, \\ \dot{y} = -x + (b_{02} - 2a)x^2 + 2xy + b_{02}y^2 + (2a - b_{02})x^2y + \\ \qquad + (a^2 + b^2 - 1)xy^2 - b_{02}y^3; \end{cases} \tag{37}$$

$$\begin{cases} \dot{x} = x + cy + (2a+c)x^2 + 2(-1+ac)xy - cy^2 + (-2 + a^2 + b^2 - b_{02} + \\ \qquad + 2ac)x^3 + (-2a - c + a^2c + b^2c)x^2y - (1 + b_{02})xy^2, \\ \dot{y} = -cx + y + (2 + b_{02} - 2ac)x^2 + 2(a+c)xy + b_{02}y^2 + (-2 + a^2 + \\ \qquad b^2 - b_{02} + 2ac)x^2y + (-2a - c + a^2c + b^2c)xy^2 - (1 + b_{02})y^3. \end{cases} \tag{38}$$

Let $O_{j,k}$ be the intersection point of the invariant straight lines l_j and $l_k, j \neq k$. Then, $O_{1,2} = (0,0)$, $O_{1,3} = (-1/(-i + a + bi), 1/(1 - b + ai))$, $O_{1,4} = (-1/(-i + a - bi), 1/(1 + b + ai))$, $O_{3,4} = (0,1)$, $O_{2,3} \equiv \overline{O_{1,4}}$ and $O_{2,4} \equiv \overline{O_{1,3}}$. The straight line passing through singular points $O_{1,3}$ and $O_{2,4}$ ($O_{1,4}$ and $O_{2,3}$) is described by equation $1 + ax - y + by = 0$ ($1 + ax - y - by = 0$). Using only the information provided by singular points we can state that besides the invariant straight lines

$l_{1,2,3,4}$, the systems (37), (38) can have also the invariant straight lines described by equations $1 + ax - y + by = 0$, $1 + ax - y - by = 0$ and $x = 0$.

The straight line $x = 0$ can't be invariant for (37), because the coefficients of the monomials y, $-y^2$ from right-hand side of first equation from system (37) are constant. The straight lines $l_5 = 1+ax-y+by$ and $l_6 = 1+ax-y-by$ are invariant for (37) only simultaneously, therefore this system can't have exactly five invariant straight lines.

The straight line $l_5 = 1 + ax - y + by$ ($l_5 = 1 + ax - y - by$) is invariant for the system (38) iff $b_{02} = ac - 2 + b$ ($b_{02} = ac - 2 - b$), i.e. we obtained the system 21) of Theorem 1 (a system affine equivalent with 21)).

Also, the straight line $x = 0$ is invariant for the system (38) iff $c = 0$. In (38) we take $c = 0$ and denote $b_{02} = c - 2$, where c is a real parameter. This way we get the system 22) of Theorem 1.

Configuration D2) $(\mathbf{1r}, \mathbf{1r}, \mathbf{1r}, \mathbf{1c_1}, \mathbf{1c_1})$**.** The complex invariant straight lines of the system [(1)][(26)], via an affine transformation, can be brought to the form $l_{1,2} = y \pm ix$. According to Proposition 11, two of real invariant straight lines $l_{3,4,5}$ can't pass through the intersection point $(0, 0)$ of l_1 and l_2. Therefore, via a rotation and a contraction $x \to kx$, $y \to ky$, $k \in \mathbb{R}^*$, we can bring the intersection point of the straight lines l_3 and l_4 in $(0, 1)$, i.e. these straight lines are described by $l_3 = y - ax - 1$ and $l_4 = y - bx - 1$, $a, b \in \mathbb{R}$, $a \neq b$. The fifth invariant straight line must pass through the points $(0, 0)$ and $(0, 1)$, i.e. it is described by $l_5 = x$. Asking that these straight lines to be invariant for the system [(1)][(26)] we get the system 23) of Theorem 1.

Configuration D3) $(\mathbf{1r}, \mathbf{1r}, \mathbf{1r}, \mathbf{1r}, \mathbf{1r})$**.** Let the system [(1)][(26)] have at least five real invariant straight lines with diffefrent slopes l_j, $j = \overline{1, 5}$. Via an affine transformation we can bring these straight lines to be described by equations: $x = 0$, $y = 0$, $y = x$, $y = ax + 1$, $y = bx + 1$, $ab(a - 1)(b - 1) \neq 0$, $a < b$. The cubic system with these invariant straight lines has the form 24) of Theorem 1.

3.4 Qualitative investigation of systems 13)-24)

In this section, the qualitative study of systems $13) - 24)$ of Theorem 1 will be done. For this purpose, in order to determine the topological behavior of trajectories, the singular points will be examined. Using also the information provided by the existence of invariant straight lines, we will construct all phase portraits of systems $3) - 11)$ on Poincaré disk.

We denote by SP singular points; λ_1 and λ_2 the characteristic roots of the SP; TSP $-$ type of SP; S $-$ saddle ($\lambda_1 \lambda_2 < 0$); N^s $-$ stable node ($\lambda_1, \lambda_2 < 0$); N^i $-$ instable node ($\lambda_1, \lambda_2 > 0$); $DN^{s(i)}$ $-$ improper stable (instable) node ($\lambda_1 = \lambda_2 \neq 0$); C – centre, $P^{i(s)}$– instable (stable) parabolic sector, $F^{i(s)}$ – instable (stable) focus.

In the next tables, the first column will indicate the singular points of the systems; the second column - the eigenvalues corresponding to these singular points and the third column - the types of the singularities. All these points are simple and together with the invariant straight lines, fully determine the phase portrait for each of the systems 13)–24).

Table 1. Systems 13), 15), 16), 17), 19), 20), 21), 22) and 23)

SP	$\lambda_1;\lambda_2$	TSP		SP	$\lambda_1;\lambda_2$	TSP	
System 13); Fig. 15							
$J=\{a\mid a\in(-1,0)\}$		$a<-1$	J			$a<-1$	J
$O_1(0,0)$	$-a\,;-a$	DN^i	DN^i	$O_2(1,0)$	$a;\,-a-1$	S	N^s
$O_3(0,1)$	$a\,;-a-1$	S	N^s	$O_4(1,1)$	$a+1;\,a+1$	DN^s	DN^i
$O_5(-a,-a)$	$\pm a(a+1)$	S	S	$X_\infty(1,0,0)$	$-a;\,-a-1$	N^i	S
$Y_\infty(0,1,0)$	$-a;\,-a-1$	N^s	S				
System 15); Fig. 15							
$O(-a,-a)$	$\pm(a^2+1)$	S		$X_\infty(1,0,0)$	$-a\pm i$	F^s	
$Y_\infty(0,1,0)$	$-a\pm i$	F^s					
System 16); Fig. 16							
$G=\{a\mid a\in(0,1)\}$		G	$a>1$			G	$a>1$
$O_1(0,0)$	$a;\,a$	DN^i	DN^i	$O_2(0,1)$	$1-a;\,1-a$	DN^i	DN^s
$O_3(0,a)$	$\pm a(a-1)$	S	S				
System 17); Fig. 17)							
$O_1(0,0)$	$1\pm 2bi$	F^i		$O_2(0,1)$	$1\pm 2bi$	F^i	
$O_3(-b,\frac{1}{2})$	$\pm\frac{1}{2}\left(4b^2+1\right)$	S					
System 20); Fig. 20							
$O_1(0,0)$	$(a^2+1)(1\pm ib)$	F^i		$O_2\left(\frac{b(a^2+1)}{ab-1},\frac{a^2+1}{1-ab}\right)$	$\pm\frac{(a^2+1)(b^2+1)}{ab-1}$	S	
$I_\infty(0,1,0)$	$1\pm bi$	F^i					
System 21; Fig. 17.a), Fig. 21							
		$b<0$	$b>0$				
$O_1(0,0)$	$1\pm ci$	F^i		$O_3(\frac{-c}{ac-b+1},\frac{1}{a-b+1})$	$\pm\frac{b(c^2+1)}{ac-b+1}$	S	
$O_2(0,1)$	$b(-1\pm ci)$	F^i	F^s				
System 22); Fig. 16							
		$c<0$	$c>0$				
$O_1(0,0)$	$1;\,1$	N^i		$O_3(0,\frac{1}{1-c})$	$\pm\frac{1}{1-c}$	S	
$O_2(0,1)$	$-c;\,-c$	N^i	N^s				
System 23); Fig. 22							
		$c<0$	$c>0$				
$O_1(0,0)$	$1;\,1$	N^i		$O_3(0,\frac{1}{1-c})$	$\pm\frac{1}{1-c}$	S	
$O_2(0,1)$	$-c;\,-c$	N^i	N^s				

Systems 13), 15)–17), 19)–23). All these systems have hyperbolic singular points in the finite part of the phase plane and at the infinity. These singular points, their type and the phase portraits corresponding to each system are shown in Table 1.

System 14). This system has two singular points in the finite part of the phase plane and other two at the infinity. Their coordinates, their types and the phase portraits corresponding to each system are shown in Table 2.

As we see from Table 2, the origin is a nonhyperbolic singular point. Using polar

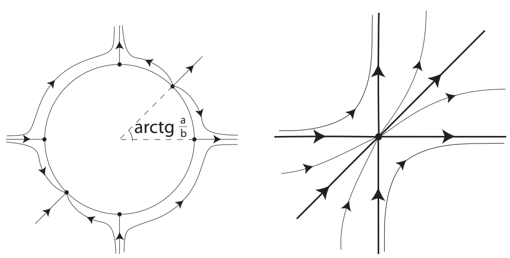
Table 2. System 14); Fig. 14

SP	$\lambda_1;\ \lambda_2$	TSP	SP	$\lambda_1;\ \lambda_2$	TSP
$O_1(0,0)$	$0\ ;0$	HP^sHP^i	$O_2(-b,-a)$	$ab;\ -ab$	S
$X_\infty(1,0,0)$	$-a\ ;-a$	DN^s	$Y_\infty(0,1,0)$	$-b;\ -b$	DN^s
Blow-up of the origin $(0,0)$					
$M_1(0,0)$	$a\ ;-a$	S	$M_2(0,\frac{\pi}{2})$	$b;\ -b$	S
$M_3(0,\pi)$	$a\ ;-a$	S	$M_4(0,\frac{3\pi}{2})$	$b;\ -b$	S
$M_5(0,\mathrm{arctg}\frac{b}{a})$	$\frac{ab}{\sqrt{a^2+b^2}}\ ;0$	DN^i	$M_6(0,\mathrm{arctg}\frac{b}{a}+\pi)$	$\frac{-ab}{\sqrt{a^2+b^2}};\ \frac{-ab}{\sqrt{a^2+b^2}}$	DN^s

coordinates and after rescaling the time $t=\tau/\rho$, this systems takes the form:

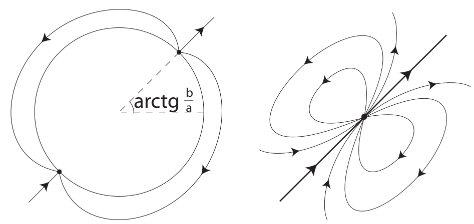
$$\begin{cases} \dot\rho=\rho\left(a\cos^3\theta+\rho\cos^3\theta\sin\theta+b\sin^3\theta+\rho\cos\theta\sin^3\theta\right), \\ \dot\theta=\cos\theta\sin\theta(a\cos\theta-b\sin\theta). \end{cases}$$

We get six singular points of the form $M_i(0,\theta_i)$, their coordinates and types are given in Table 2. Using this information, we get Fig. 24a) and after "compressing" all these points to the origin we obtain Fig. 24b), i.e. the origin can be described as HP^sHP^i singular point.



a) b)

Fig. 24



a) b)

Fig. 25

System 18). The system has only two singular points in the finite part of the phase plane (Table 3). To study neighborhood of the origin of coordinates we will use the blow-up method. In polar coordinates the system has the form:

$$\begin{cases} \dot\rho=\rho\left(a\cos\theta+b\sin\theta+\rho\right) \\ \dot\theta=a\sin\theta-b\cos\theta \end{cases}$$

Table 3. System 18); Fig. 18

SP	$\lambda_1;\ \lambda_2$	TSP	SP	$\lambda_1;\ \lambda_2$	TSP
$O_1(0,0)$	$0\ ;0$	EP^iP^iE	$O_2(-a,-b)$	$\pm\left(a^2+b^2\right)$	S

Solving the equation $a\sin\theta-b\cos\theta=0$ gives us the information that $O_1(0,0)$ consists of two hyperbolic singular points: $M_1(0,\mathrm{arctg}\frac{b}{a})$ – instable improper node and $M_2(0,\mathrm{arctg}\frac{b}{a}+\pi)$ – stable improper node.

Compressing these points to the origin of coordinates we get that the neighborgood of the origin consists of two eliptic sectors separated by a separatrix (Fig. 25).

System 19). If $a \neq 1$, then this system has the singular points $O_1(0,0)$, $O_2(0,1)$, $O_3(1,0)$, $O_4(-1/(a-1),\ 0)$, $O_5(1,-a/b)$, $O_6(b/(b-a),-a/(b-a))$.

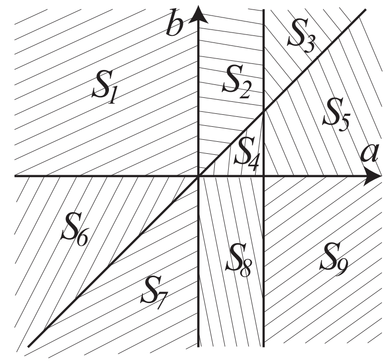


Fig. 26

The straight lines $a=0$, $b=0$, $a=1$ and $a-b=0$ divide the plane of coefficients (a,b) in 9 sectors (Fig. 26). Using relative positions of the singular points and the invariant straight lines, also the qualitative structure of these points, we notice that some systems with coefficients from the different sectors have the same trajectories. In particular, the phase portraits of systems with coefficients from S_6 and S_7 are topologically equivalent, and the phase portraits of systems S_2 (respectively, S_3, S_8) and S_5 (respectively, S_4, S_9) are equivalent. Therefore we obtain Table 4 which contains information about sectors S_1, S_3, S_5, S_6 and S_8.

Table 4. System 19), $a \neq 1$.

<table>
<tr><td>S.P.</td><td>$O_1(0,0)$</td><td>$O_2(0,1)$</td><td>$O_3(1,0)$</td><td>$O_4(-\frac{1}{a-1},0)$</td><td>$O_5(1,-\frac{a}{b})$</td><td>$O_6(\frac{b}{b-a},-\frac{a}{b-a})$</td><td>$I_\infty(0,1,0)$</td><td rowspan="3">Fig.</td></tr>
<tr><td>λ_1</td><td>-1</td><td>1</td><td>a</td><td>$-\frac{a}{a-1}$</td><td>$-a$</td><td>$\frac{a}{a-b}$</td><td>-1</td></tr>
<tr><td>λ_2</td><td>-1</td><td>$-b$</td><td>a</td><td>$\frac{a}{a-1}$</td><td>$\frac{a}{b}$</td><td>$\frac{ab}{a-b}$</td><td>$-b$</td></tr>
<tr><td>S_1</td><td rowspan="5">ND^s</td><td rowspan="2">S</td><td>ND^s</td><td rowspan="5">S</td><td rowspan="2">S</td><td>N^i</td><td rowspan="2">N^s</td><td>19a)</td></tr>
<tr><td>S_3</td><td rowspan="2">ND^i</td><td>N^s</td><td>19b)</td></tr>
<tr><td>S_5</td><td rowspan="3">N^i</td><td>N^s</td><td rowspan="3">S</td><td rowspan="3">S</td><td>19c)</td></tr>
<tr><td>S_6</td><td>ND^s</td><td>N^i</td><td>19d)</td></tr>
<tr><td>S_8</td><td>ND^i</td><td>N^s</td><td>19e)</td></tr>
</table>

If $a=1$, then the singular point $O_4(-1/(a-1),0)$ goes to the infinity. We note that the cases $b \in (0,1)$ and $b \in (1,+\infty)$ are topologically equivalent, therefore we have Table 5.

Table 5. System 19), $a=1$.

<table>
<tr><td>S.P.</td><td>$O_1(0,0)$</td><td>$O_2(0,1)$</td><td>$O_3(1,0)$</td><td>$O_5(1,-\frac{1}{b})$</td><td>$O_6(\frac{b}{b-1},-\frac{1}{b-1})$</td><td>$I_\infty(1,0,0)$</td><td>$I_\infty(0,1,0)$</td><td rowspan="3">Fig.</td></tr>
<tr><td>λ_1</td><td>-1</td><td>1</td><td>1</td><td>-1</td><td>$\frac{a}{a-b}$</td><td>-1</td><td>-1</td></tr>
<tr><td>λ_2</td><td>-1</td><td>$-b$</td><td>1</td><td>$\frac{1}{b}$</td><td>$\frac{ab}{a-b}$</td><td>1</td><td>$-b$</td></tr>
<tr><td>$b<0$</td><td rowspan="2">ND^s</td><td>N^i</td><td rowspan="2">ND^i</td><td>N^s</td><td>N^i</td><td rowspan="2">S</td><td>S</td><td>19f)</td></tr>
<tr><td>$b>1$</td><td>S</td><td>S</td><td>N^s</td><td>N^s</td><td>19g)</td></tr>
</table>

System 24). This system has three real parameters under conditions $ab(a-1)(b-1)(c-1) \neq 0$, $a > b$, so the space of coefficients must be three-dimensional. We can simplify this by restraining the parameter c and obtaining three simpler cases. If $c \neq 0$ then the system has seven singular point in the finite part of the phase plane and if $c = 0$, then the system has six singular points (see Table 6).

Using the above conditions and the information provided by characteristic roots of singular points, we get six sectors $S_1, \ldots, S_6$ ilustrated in Fig. 27.

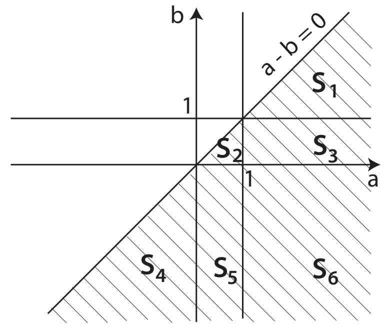


Fig. 27

Table 6. System 24)

<table>
<tr><td colspan="2">SP</td><td>$(0,0)$</td><td>$(0,1)$</td><td>$\left(0,\frac{1}{c}\right)$</td><td>$\left(-\frac{1}{a},0\right)$</td><td>$\left(-\frac{1}{b},0\right)$</td><td>$\left(\frac{1}{1-a},\frac{1}{1-a}\right)$</td><td>$\left(\frac{1}{1-b},\frac{1}{1-b}\right)$</td><td rowspan="2">Fig.</td></tr>
<tr><td colspan="2">λ_1
λ_2</td><td>1
1</td><td>$c-1$
$c-1$</td><td>$\frac{c-1}{c}$
$-\frac{c-1}{c}$</td><td>$\frac{b-a}{a}$
$\frac{1-c}{b}$</td><td>$\frac{a-b}{a}$
$\frac{1-c}{b}$</td><td>$\frac{b-a}{a-1}$
$\frac{c-1}{a-1}$</td><td>$\frac{a-b}{b-1}$
$\frac{c-1}{b-1}$</td></tr>
<tr><td rowspan="6">$c < 1$
$c \neq 0$</td><td>S_1</td><td rowspan="18">N^i</td><td rowspan="6">N^s</td><td rowspan="12">S</td><td rowspan="6">S</td><td rowspan="3">N^i</td><td>N^s</td><td rowspan="6">S</td><td>23a)</td></tr>
<tr><td>S_2</td><td>N^i</td><td>23b)</td></tr>
<tr><td>S_3</td><td>N^s</td><td rowspan="3">23a)</td></tr>
<tr><td>S_4</td><td rowspan="3">N^i</td><td rowspan="2">N^i</td></tr>
<tr><td>S_5</td></tr>
<tr><td>S_6</td><td>N^s</td><td rowspan="2">23c)</td></tr>
<tr><td rowspan="6">$c > 1$</td><td>S_1</td><td rowspan="6">N^i</td><td rowspan="3">N^s</td><td rowspan="6">S</td><td rowspan="6">S</td><td>N^i</td></tr>
<tr><td>S_2</td><td rowspan="5">N^s</td><td rowspan="2">23a)</td></tr>
<tr><td>S_3</td></tr>
<tr><td>S_4</td><td>N^i</td><td>23c)</td></tr>
<tr><td>S_5</td><td rowspan="2">N^s</td><td rowspan="2">23a)</td></tr>
<tr><td>S_6</td></tr>
<tr><td rowspan="6">$c = 0$</td><td>S_1</td><td rowspan="6">N^s</td><td rowspan="6">$I_\infty(0,1,0)$
S</td><td rowspan="6">S</td><td rowspan="3">N^i</td><td>N^s</td><td rowspan="6">S</td><td>23d)</td></tr>
<tr><td>S_2</td><td>N^i</td><td>23e)</td></tr>
<tr><td>S_3</td><td>N^s</td><td rowspan="3">23d)</td></tr>
<tr><td>S_4</td><td rowspan="3">N^i</td><td rowspan="2">N^i</td></tr>
<tr><td>S_5</td></tr>
<tr><td>S_6</td><td>N^s</td><td>23e)</td></tr>
</table>

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Received May 27, 2016

Alexandru Şubă
Institute of Mathematics
and Computer Science A.S.M.
Chişinău, Republic of Moldova
E-mail: *alexandrusuba@gmail.com*

Vadim Repeşco
State University of Tiraspol
Chişinău, Republic of Moldova
E-mail: *repescov@gmail.com*