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On (σ-δ)-rings over Noetherian rings

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Abstract. For a ring R, an endomorphism σ of R and a σ-derivation δ of R, we introduce (σ-δ)-ring and (σ-δ)-rigid ring which are the generalizations of $\sigma(*)$-rings and δ-rings, and investigate their properties. Moreover, we prove that a (σ-δ)-ring is 2-primal and its prime radical is completely semiprime.

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1 Introduction and preliminaries

A ring R always means an associative ring with identity $1 \neq 0$, unless otherwise stated. The prime radical and the set of nilpotent elements of R are denoted by $P(R)$ and $N(R)$ respectively. The ring of integers is denoted by $\mathbb{Z}$, the field of rational numbers by $\mathbb{Q}$, the field of real numbers by $\mathbb{R}$, and the field of complex numbers by $\mathbb{C}$, unless otherwise stated.

Let R be a ring. This article concerns endomorphisms and derivations of a ring and we also discuss certain types of rings involving endomorphisms and derivations. We begin with the following:

Definition 1 (see Krempa [10])**.** An endomorphism σ of a ring R is said to be rigid if $a\sigma(a) = 0$ implies that $a = 0$, for all $a \in R$. A ring R is said to be σ-rigid if there exists a rigid endomorphism σ of R.

Example 1. Let $R = \mathbb{C}$ and $\sigma : \mathbb{C} \to \mathbb{C}$ be defined by $\sigma(a + ib) = a - ib$, for all $a, b \in R$. Then σ is a rigid endomorphism of R.

We recall a ring R is σ-rigid if there exists a rigid endomorphism σ of R and σ-rigid rings are reduced rings by Hong et. al. [6]. Properties of σ-rigid rings have been studied in Krempa [10], Hong et al. [6] and Hirano [5].

Definition 2 (see Kwak [12])**.** Let R be a ring and σ an endomorphism of R. Then R is said to be a $\sigma(*)$-ring if $a\sigma(a) \in P(R)$ implies that $a \in P(R)$, for $a \in R$.

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Example 2 (see Example 1 of Kwak [12])**.** Let $\mathbb{F}$ be a field, and $R = \begin{pmatrix} \mathbb{F} & \mathbb{F} \\ 0 & \mathbb{F} \end{pmatrix}$. Then $P(R) = \begin{pmatrix} 0 & \mathbb{F} \\ 0 & 0 \end{pmatrix}$. Let $\sigma : R \to R$ be defined by

$$\sigma\Big(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\Big) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}.$$

Then it can be seen that σ is an endomorphism of R and R is a $\sigma(*)$-ring.

We note that the above ring is not σ-rigid. Let $0 \neq a \in \mathbb{F}$. Then

$$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \sigma\Big(\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}\Big) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \text{ but } \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Example 3. Let $\mathbb{F}$ be a field, and $R = \mathbb{F}[x]$. Let $\sigma : R \to R$ be an endomorphism defined by $\sigma(f(x)) = f(0)$. Then R is not a $\sigma(*)$-ring.

Definition 3 (see [13])**.** An ideal I of a ring R is said to be completely semi-prime if $a^2 \in I$ implies that $a \in I$, for $a \in R$.

Definition 4. A ring R is said to be 2-primal if and only if $P(R) = N(R)$.

Example 4 (see Bhat [4])**.**

1. Let $R = \mathbb{F}[x]$ be the polynomial ring over a field $\mathbb{F}$. Then R is 2-primal with $P(R) = \{0\}$.
2. Let $M_2(\mathbb{Q})$ be the set of 2×2 matrices over $\mathbb{Q}$. Then $R[x]$ is a prime ring with non-zero nilpotent elements and so it cannot be 2-primal.

2-primal rings have been studied in recent years and are being treated by authors for different structures. We know that a ring R is 2-primal if the prime radical is completely semi-prime. Note that a reduced ring is 2-primal and a commutative ring is also 2-primal. For further detail on 2-primal rings refer to [2, 3, 7, 8, 9, 13, 15]. Furthermore, the concept of completely semi-prime ideals is also studied in this area. Kwak in [12] establishes a relation between a 2-primal ring and a $\sigma(*)$-ring. It is also known that if R is a Noetherian ring and σ an endomorphism of R, then R a $\sigma(*)$-ring implies that R is 2-primal (Proposition (2.4) of [4]), but the converse need not be true. For example, we have:

Example 2.5 of [4]: Let $R = F[x]$ be the polynomial ring over a field F. Then R is 2-primal with $P(R) = \{0\}$. Let $\sigma : R \to R$ be an endomorphism defined by

$$\sigma(f(x)) = f(0).$$

Then R is not a $\sigma(*)$-ring. For this consider $f(x) = xa, a \neq 0$.

Also if R is a Noetherian ring and σ an endomorphism of R, then R a $\sigma(*)$-ring implies that $P(R)$ is completely semi-prime (Proposition (1) of [11]), but the converse need not be true. For example, we have

Example [12]: Let $\mathbb{F}$ be a field, $R = \mathbb{F} \times \mathbb{F}$. Let $\sigma : R \to R$ be an automorphism defined as

$$\sigma((a, b)) = (b, a), a, b \in \mathbb{F}.$$

Here $P(R) = \{0\}$ is a completely semi-prime ring, as R is a reduced ring. But R is not a $\sigma(*)$-ring. Since $(1, 0)\sigma((1, 0)) = (0, 0)$, but $(1, 0)$ does not belong to $P(R)$.

Definition 5 (see [14])**.** Let R be a ring, σ an endomorphism of R and $\delta : R \to R$ an additive map such that

$$\delta(ab) = \delta(a)\sigma(b) + a\delta(b), \text{ for all } a, b \in R.$$

Then δ is a σ-derivation of R.

Example 5. Let $R = \mathbb{Z}[\sqrt{2}]$. Then $\sigma : R \to R$ defined as

$$\sigma(a + b\sqrt{2}) = (a - b\sqrt{2}), \text{ for } a + b\sqrt{2} \in R.$$

is an endomorphism of R. For any $s \in R$, define $\delta_s : R \to R$ by

$$\delta_s(a + b\sqrt{2}) = (a + b\sqrt{2})s - s\sigma(a + b\sqrt{2}), \text{ for } a + b\sqrt{2} \in R.$$

Then δ_s is a σ-derivation of R.

Definition 6 (see Bhat [1])**.** Let R be a ring. Let σ be an automorphism of R and δ a σ-derivation of R. Then R is a δ-ring if $a\delta(a) \in P(R)$ implies that $a \in P(R)$.

Note that a δ-ring is without identity, as $1\delta(1) = 0$, but $1 \neq 0$.

Example 6. Let S be a ring without identity and $R = S \times S$ with $P(R) = \{0\}$ (for example we take $S = 2\mathbb{Z}$).

Then $\sigma : R \to R$ is an endomorphism defined by

$$\sigma((a, b)) = (b, a).$$

For any $s \in R$, define $\delta_s : R \to R$ by

$$\delta_s(a, b) = (a, b)s - s\sigma(a, b), \text{ for } (a, b) \in R.$$

Let $(a, b)\delta_s(a, b) \in P(R)$, then $(a, b)\{(a, b)s - s\sigma(a, b)\} \in P(R)$ or $(a, b)\{(a, b)s - s(b, a)\} \in P(R)$, i.e. $(a, b)(as - bs, bs - sa) \in P(R)$. Therefore, $(a(as - bs), b(bs - sa)) \in P(R) = \{0\}$ which implies that $a = 0$, $b = 0$, i.e. $(a, b) = (0, 0) \in P(R)$. Thus R is a δ-ring.

It is known that if R is a δ-ring, σ an endomorphism of R, δ a σ-derivation of R such that $\delta(P(R)) \in P(R)$, then R is 2-primal (Theorem 2.2 of [1]).

In this note we generalize the $\sigma(*)$-rings and δ-rings as follows:

Definition 7. Let R be a ring. Let σ be an endomorphism of R and δ a σ-derivation of R. Then R is said to be a $(\sigma$-$\delta)$-ring if $a(\sigma(a) + \delta(a)) \in P(R)$ implies that $a \in P(R)$, for $a \in R$.

Example 7. Let $\mathbb{F}$ be a field, and $R = \begin{pmatrix} \mathbb{F} & \mathbb{F} \\ 0 & \mathbb{F} \end{pmatrix}$. Then $P(R) = \begin{pmatrix} 0 & \mathbb{F} \\ 0 & 0 \end{pmatrix}$. Let $\sigma : R \to R$ be defined by

$$\sigma\Big(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\Big) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}.$$

Then it can be seen that σ is an endomorphism of R. For any $s \in R$, define $\delta_s : R \to R$ by

$$\delta_s(a) = as - s\sigma(a), \text{ for } a \in R.$$

Let $s = \begin{pmatrix} p & q \\ 0 & r \end{pmatrix}, \quad x = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \quad y = \begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}.$

Now $\delta_s(xy) = (xy)s - s\sigma(xy) = \begin{pmatrix} 0 & aa_1q + ab_1r + bc_1r - cc_1q \\ 0 & 0 \end{pmatrix}.$

Also $\delta_s(x)\sigma(y) + x\delta_s(y) = \begin{pmatrix} 0 & aa_1q + ab_1r + bc_1r - cc_1q \\ 0 & 0 \end{pmatrix}.$

Hence $\delta_s(xy) = \delta_s(x)\sigma(y) + x\delta_s(y)$. Thus δ_s is a σ-derivation on R.

Now let $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \quad s = \begin{pmatrix} p & q \\ 0 & r \end{pmatrix}.$

$A[\sigma(A) + \delta(A)] \in P(R)$ which implies that

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\Big\{\sigma\Big(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\Big) + As - s\sigma(A)\Big\} \in P(R),$$

i.e. $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\Big\{\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} + \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} p & q \\ 0 & r \end{pmatrix} - \begin{pmatrix} p & q \\ 0 & r \end{pmatrix}\sigma(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix})\Big\} \in P(R)$

or $\begin{pmatrix} a^2 & a^2q + abr + bc - acq \\ 0 & c^2 \end{pmatrix} \in P(R) = \begin{pmatrix} 0 & \mathbb{F} \\ 0 & 0 \end{pmatrix}$ which implies that

$a^2 = 0, \quad c^2 = 0$, i.e. $a = 0, \quad c = 0$.

Therefore, $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in P(R)$. Hence $P(R)$ is a (σ-δ)-ring.

Remark 1.

1. If $\delta(a) = 0$, then a (σ-δ)-ring is a $\sigma(*)$-ring.
2. If $\sigma(a) = 0$, then a (σ-δ)-ring is a δ-ring.
3. If $\sigma(a) = a$, $\delta(a) = 0$, then a (σ-δ)-ring is completely semi-prime.

Definition 8. Let R be a ring. Let σ be an endomorphism of R and δ a σ-derivation of R. Then R is said to be a (σ-δ)-rigid ring if

$$a(\sigma(a) + \delta(a)) = 0 \text{ implies that } a = 0, \text{ for } a \in R.$$

Example 8. Let $R = \mathbb{C}$ and $\sigma : \mathbb{C} \to \mathbb{C}$ be defined by

$$\sigma(a + ib) = a - ib, \text{ for all } a, b \in R.$$

Then σ is an endomorphism on R.
Define a σ-derivation δ on R as

$$\delta(A) = A - \sigma(A),$$

i.e. $\delta(a + ib) = a + ib - \sigma(a + ib) = a + ib - (a - ib) = 2ib$.
Now $A[\sigma(A) + \delta(A)] = 0$ which implies that $(a + ib)[\sigma(a + ib) + \delta(a + ib)] = 0$, i.e. $(a + ib)[(a - ib) - 2ib] = 0$ or $(a + ib)(a + ib) = 0$ which implies that $a = 0, b = 0$. Therefore, $A = a + ib = 0$. Hence R is a (σ-δ)-rigid ring.

With this we prove the following

Theorem A: Let R be a Noetherian integral domain which is also an algebra over $\mathbb{Q}$. σ an automorphism on R and δ a σ-derivation of R. If R is a (σ-δ)-ring, then R is 2-primal. (This has been proved in Theorem 2.2).

Theorem B: Let R be a Noetherian integral domain which is also an algebra over $\mathbb{Q}$, σ an automorphism on R and δ a σ-derivation of R. If R is a (σ-δ)-ring, then $P(R)$ is completely semi-prime. (This has been proved in Theorem 2.5).

Example of a ring satisfying the hypothesis of Theorem A and Theorem B is $R = \mathbb{Z}$. It is a Noetherian integral domain which is also an algebra over $\mathbb{Q}$. Let $\sigma : R \to R$ be defined by

$$\sigma(a) = 2a.$$

Then it can be seen that σ is an endomorphism of R.
For any $s \in R$, define $\delta_s : R \to R$ by

$$\delta_s(a) = as - s\sigma(a), \text{ for } a \in R.$$

Then δ_s is a σ-derivation on R. Also R is a (σ-δ)-ring.

2 Proof of the main results

For the proof of the main result, we need the following

Proposition 1. *Let R be a ring, σ an automorphism of R and δ a σ-derivation of R. Then for $u \neq 0$, $\sigma(u) + \delta(u) \neq 0$.*

Proof. Let $0 \neq u \in R$, we show that $\sigma(u) + \delta(u) \neq 0$. Let for $0 \neq u$, $\sigma(u) + \delta(u) = 0$ which implies that

$$\delta(u) = -\sigma(u). \tag{1}$$

We know that for $a, b \in R$, $\delta(ab) = \delta(a)\sigma(b) + a\delta(b)$. By using (2.1), this implies that $\delta(ab) = -\sigma(a)\sigma(b) + a(-\sigma(b))$ or $-\sigma(ab) = -[a + \sigma(a)]\sigma(b)$. Since σ is an endomorphism of R, this gives $-\sigma(a)\sigma(b) = -[a + \sigma(a)]\sigma(b)$, i.e. $\sigma(a) = a + \sigma(a)$. Therefore, $a = 0$, which is not possible. Hence the result is proved. □

We now state and prove the main results of this paper in the form of the following Theorems:

Theorem 1. *Let R be a Noetherian integral domain which is also an algebra over $\mathbb{Q}$, σ an automorphism of R and δ a σ-derivation of R. If R is a (σ-δ), then R is 2-primal.*

Proof. R is a (σ-δ)-ring. We know that a reduced ring is 2-primal. We use the principle of Mathematical Induction to prove that R is a reduced ring. Let for $x \in R$, $x^n = 0$. We use induction on n and show that $x = 0$. The result is trivially true for $n = 1$, as $x^n = x^1 = a(\sigma(a) + \delta(a)) = 0$. Now Proposition 1, implies that $a = 0$, hence $x = 0$. Therefore, the result is true for $n = 1$. Let us assume that the result is true for $n = k$, i.e. $x^k = 0$ implies that $x = 0$. Let $n = k + 1$. Then $x^{k+1} = 0$ which implies that

$$a^{k+1}(\sigma(a) + \delta(a))^{k+1} = 0.$$

Again by Proposition 1 we get $a = 0$. Hence $x = 0$. Therefore, the result is true for $n = k + 1$ too. Thus the result is true for all n by the principle of Mathematical Induction. Hence the theorem is proved. □

The converse of the above is not true.

Example 9. Let $R = F(x)$ be the field of rational polynomials in one variable x. Then R is 2-primal with $P(R) = \{0\}$.
Let $\sigma : R \to R$ be an endomorphism defined by

$$\sigma(f(x)) = f(0).$$

For $r \in R$, $\delta_r : R \to R$ be a σ-derivation defined as

$$\delta_r(a) = ar - r\sigma(a).$$

Then R is not a $(\sigma - \delta)$-ring.
Take $f(x) = xa + b, r = \frac{-b}{xa}$. Then

$$\begin{aligned} f(x)\Big\{\sigma(f(x)) + \delta_r(f(x))\Big\} &= f(x)\Big\{b + (xa+b)(\frac{-b}{xa}) - (\frac{-b}{xa})\sigma(f(x))\Big\} \\ &= f(x)\Big\{b - b - \frac{b^2}{xa} + \frac{b}{xa}b\Big\} \\ &= f(x)\Big\{b - b - \frac{b^2}{xa} + \frac{b^2}{xa}\Big\} = 0 \in P(R). \end{aligned}$$

But $f(x) \neq 0$. Therefore, $f(x)$ is not an element of $P(R)$. Hence R is not a (σ-δ)-ring.

For the proof of the next theorem, we require the following:
J. Krempa [10] has investigated the relation between minimal prime ideals and completely prime ideals of a ring R. With this he proved the following:

Theorem 2. *For a ring R the following conditions are equivalent:*
(1) R is reduced.
(2) R is semiprime and all minimal prime ideals of R are completely prime.
(3) R is a subdirect product of domains.

Theorem 3. *Let R be a Noetherian integral domain which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R and δ a σ-derivation of R. If R is a (σ-δ)-ring, then $P(R)$ is completely semi-prime.*

Proof. As proved in Theorem 1, R is a reduced ring and by using Theorem 2, the result follows. □

The converse of the above is not true.

Example 10. Let $\mathbb{F}$ be a field, $R = \mathbb{F} \times \mathbb{F}$. Let $\sigma : R \to R$ be an automorphism defined as

$$\sigma((a,b)) = (b,a), a, b \in \mathbb{F}.$$

Here $P(R)$ is a completely semi-prime ring, as R is a reduced ring.
For $r \in F$, define $\delta_r : R \to R$ by

$$\delta_r((a,b)) = (a,b)r - r\sigma((a,b)) \text{ for } a,b \in F.$$

Then δ_r is a σ-derivation on R. Take $A = (1,-1), r = \frac{1}{2}$.

Now $A\Big\{\sigma(A) + \delta_r(A)\Big\} = (1,-1)\Big\{\sigma((1,-1)) + (1,-1)\frac{1}{2} - \frac{1}{2}\sigma((1,-1))\Big\} = (1,-1)\Big\{(-1,1) + (\frac{1}{2}, \frac{-1}{2}) - \frac{1}{2}(-1,1)\Big\} = (0,0) \in P(R) = \{0\}$. But $(1,-1) \neq 0$. Hence it is not a (σ-δ)-ring.

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