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Resonant Riemann-Liouville Fractional Differential Equations with Periodic Boundary Conditions

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Abstract. In this paper, by using the coincidence degree theory due to J. Mawhin, we consider the solvability of a class of nonlinear fractional two-point boundary value problems at resonance. An example of application illustrates the existence result.

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1 Introduction

Fractional differential equations describe many phenomena in various fields of science and engineering such as physics, chemistry, biology, visco-elasticity, electromagnetics, economy, etc. Several methods have been used to deal with the question of solvability of boundary value problems (BVPs for short) for fractional differential equations; we quote the Laplace transform method, iteration methods, the upper and lower solution method, as well as topological methods (fixed point theory and Leray-Schauder degree theory) (see, e. g., [1,10], and references therein).

In [1] B. Ahmad and J. Nieto studied the following Riemann-Liouville fractional differential equation with fractional boundary conditions:

$$D_{0^+}^{\alpha} u(t) = f(t, u(t)), \quad t \in [0,T], \ 1 < \alpha \leq 2, \tag{1.1}$$

$$D_{0^+}^{\alpha-2} u\left(0^+\right) = b_0 D_{0^+}^{\alpha-2} u\left(T^-\right), \tag{1.2}$$

$$D_{0^+}^{\alpha-1} u\left(0^+\right) = b_1 D_{0^+}^{\alpha-1} u\left(T^-\right), \tag{1.3}$$

where $D_{0^+}^{\alpha}$ denotes the Riemann-Liouville fractional derivative of order α, $b_0 \neq 1$, $b_1 \neq 1$, and the function $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous. Clearly this is a nonresonant problem, i.e. the associated homogeneous problem admits only the following solution:

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2},$$

where the constants c_1, c_2 satisfy

$$\begin{aligned} c_1\Gamma(\alpha).0 + c_2\Gamma(\alpha-1) &= b_0(c_1\Gamma(\alpha).T + c_2\Gamma(\alpha-1)) \\ c_1\Gamma(\alpha) &= b_1 c_1\Gamma(\alpha), \end{aligned}$$

that is $c_1 = c_2 = 0$ for $b_0 \neq 1$ and $b_1 \neq 1$. Then a corresponding Green's function can be computed. A fixed point theorem was used to show that the operator $P : C_{2-\alpha} \longrightarrow C_{2-\alpha}$ defined by

$$\begin{aligned} (Pu)(t) &= \tfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s))\, ds + \tfrac{b_1 t^{\alpha-1}}{(1-b_1)\Gamma(\alpha)} \int_0^T f(s, u(s))\, ds \\ &\quad + \tfrac{b_0 t^{\alpha-2}}{(1-b_1)(1-b_0)\Gamma(\alpha-1)} \int_0^T (T - (1-b_1)s) f(s, u(s))\, ds \end{aligned}$$

has at least one fixed point.

By a similar method, G. Wang, W. Liu, and C. Ren investigated in [10], the existence and uniqueness of solutions for the fractional boundary-value problem:

$$\begin{cases} D_{0^+}^{\alpha} u(t) = f(t, u(t)), \quad t \in [0, T], \quad 1 < \alpha \leq 2, \\ I_{0^+}^{2-\alpha} u(t)|_{t=0} = 0, \quad D_{0^+}^{\alpha-2} u(T) = \sum_{i=1}^{m} a_i I_{0^+}^{\alpha-1}(\xi_i), \end{cases}$$

where $0 < \xi_i < T$, $a_i \in \mathbb{R}$, $m \geq 2$, and $I_{0^+}^{\alpha}$ stands for the Riemann-Liouville fractional integral. Standard fixed point principles have been employed.

In [11], the authors investigated higher-order fractional derivatives, i. e. for $2 < \alpha \leq 3$.

When the nonlinearity of f also depends on the first derivative, Z. Bai [2] discussed the solvability of m-point fractional BVPs at resonance; the coincidence degree theory as developed by Mawhin in [8] was employed. Concerning papers dealing with fractional-order BVPs at resonance, we refer, for example, to [4–6, 11, 12]. See also [9] for a resonant second-order boundary value problem.

In the present work, Mawhin's coincidence degree theory is used to deal with BVP (1.1), (1.2), (1.3) at the resonance case, i. e. for $b_0 = b_1 = 1$. An existence result illustrated by means of two examples of application is provided in Section 2.

We first present some definitions and auxiliary lemmas about fractional calculus theory.

Definition 1 (see [3, 7])**.** The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $h : (0, +\infty) \to \mathbb{R}$ is given by

$$I_{0^+}^{\alpha} h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\, ds,$$

where $\Gamma(.)$ refers to the function gamma, provided the right side is pointwise defined on $(0, +\infty)$.

Definition 2 (see [7, 11])**.** The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $h : (0, +\infty) \to \mathbb{R}$ is given by

$$D_{0^+}^{\alpha} h(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{h(s)}{(t-s)^{\alpha-n+1}} ds = \frac{d^n}{dt^n} I_{0^+}^{n-\alpha} h(t),$$

where $n = [\alpha] + 1$, provided the right side is pointwise defined on $(0, +\infty)$. Here $[\alpha]$ denotes the integer part of the real number α.

For $\alpha < 0$, we set by convention $D_{0^+}^{\alpha} h(t) = I_{0^+}^{-\alpha} h(t)$, and if $0 \leq \beta \leq \alpha$, we get $D_{0^+}^{\beta} I_{0^+}^{\alpha} h(t) = I_{0^+}^{\alpha-\beta} h(t)$.

Given these definitions, it can be checked that the Riemann-Liouvelle fractional integration and fractional differentiation operators of the power functions t^{λ} yield power functions of the same form. Indeed, for $\lambda > -1$ and $\alpha \geq 0$, we have

$$I_{0^+}^{\alpha} t^{\lambda} = \tfrac{\Gamma(\lambda+1)}{\Gamma(\lambda+\alpha+1)} t^{\lambda+\alpha} \text{ and } D_{0^+}^{\alpha} t^{\lambda} = \tfrac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)} t^{\lambda-\alpha}.$$

Also note that $D_{0^+}^{\alpha} t^{\lambda} = 0$, for all $\lambda = \alpha - i$ with $i = 1, 2, 3, \ldots, n$ (n is the smallest integer greater than or equal to α). Also we have

Lemma 1 (see [4])**.** *Suppose that $h \in L^1(0, +\infty)$ and α, β are positive real numbers. Then*

$$I_{0^+}^{\alpha} I_{0^+}^{\beta} h(t) = I_{0^+}^{\alpha+\beta} h(t) \quad \text{and} \quad D_{0^+}^{\alpha} I_{0^+}^{\alpha} h(t) = h(t).$$

If, in addition $D_{0^+}^{\alpha} h(t) \in L^1(0, +\infty)$, then

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} h(t) = h(t) + \sum_{i=1}^{i=n} c_i t^{\alpha-i},$$

for some constants $c_i \in \mathbb{R}\,(1 \leq i \leq n)$.

Finally, notice that the boundary value problem

$$\begin{cases} D_{0^+}^{\alpha} u(t) &= f(t, u(t)), \quad t \in [0, T], \ 1 < \alpha \leq 2 \\ D_{0^+}^{\alpha-2} u(0^+) &= D_{0^+}^{\alpha-2} u(T^-), \\ D_{0^+}^{\alpha-1} u(0^+) &= D_{0^+}^{\alpha-1} u(T^-) \end{cases}$$

is at resonance, i. e., the corresponding homogeneous boundary value problem:

$$\begin{cases} D_{0^+}^{\alpha} u(t) &= 0, \quad t \in [0, T], \ 1 < \alpha \leq 2 \\ D_{0^+}^{\alpha-2} u(0^+) &= D_{0^+}^{\alpha-2} u(T^-), \\ D_{0^+}^{\alpha-1} u(0^+) &= D_{0^+}^{\alpha-1} u(T^-) \end{cases}$$

has $u(t) = ct^{\alpha-2}$ as nontrivial solutions ($c \in \mathbb{R}$).

2 Main result

2.1 Functional framework

Since our main existence result is based on Mawhin's coincidence degree, we first recall some basic facts about this theory; more details can be found in [8].

Let X, Y be two real Banach spaces and $L : \operatorname{dom}(L) \subset X \to Y$ a Fredholm operator of index zero. Then there exist two continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L$, $X = \operatorname{Ker} L \oplus \operatorname{Ker} P$, and $Y = \operatorname{Im} L \oplus \operatorname{Im} Q$. It follows that the operator

$$L_P = L\left|_{\operatorname{dom}(L)\cap \operatorname{Ker} P} : \operatorname{dom}(L) \cap \operatorname{Ker} P \to \operatorname{Im} L\right.$$

is invertible; we denote its inverse by K_P (i.e. $L_P^{-1} = K_P$). Let Ω be an open bounded subset of X such that $\operatorname{dom}(L)\cap\overline{\Omega} \neq \emptyset$. The map $N : X \to Y$ is said to be L-compact on $\overline{\Omega}$ if $QN\left(\overline{\Omega}\right)$ is bounded and the operator $K_{P,Q} = K_P(I - Q)N : \overline{\Omega} \to X$ is compact. Since $\operatorname{Im} Q$ and $\operatorname{Ker} L$ have the same dimension, then there exists a linear isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$. Mawhin [8] established the following existence result for the abstract nonlinear equation $Lu = Nu$:

Theorem 1. *Let $L : X \to Y$ be a Fredholm operator of index zero and $N : X \to Y$ be L-compact operator on $\overline{\Omega}$. Then the equation $Lu = Nu$ has at least one solution in $\operatorname{dom}(L)\cap\overline{\Omega}$ if the following conditions are satisfied:*

1. *$Lu \neq Nu$ for each $(u, \lambda) \in [(\operatorname{dom}(L) \setminus \operatorname{Ker} L) \cap \partial\Omega] \times [0, 1]$;*
2. *$Nu \notin \operatorname{Im} L$, for each $u \in \operatorname{Ker} L \cap \partial\Omega$;*
3. *$\deg\left(QN\left|_{\operatorname{Ker} L}\right., \operatorname{Ker} L \cap \Omega, 0\right) \neq 0$.*

As usual, $C[0,T]$ will denote the Banach space of continuous real valued functions defined on $[0,T]$ with the norm $\|u\| = \sup_{t\in[0,T]} |u(t)|$. For all $t \in [0,T]$, we define the function u_r by $u_r(t) = t^r u(t)$, $r \geq 0$. Let $C_r[0,T]$ be the space of all functions u such that $u_r \in C[0,T]$. Then

Lemma 2. *$C_r[0,T]$ endowed with the norm $\|u\|_r = \sup_{t\in[0,T]} t^r |u(t)|$ is a real Banach space.*

Let $Y = L^1[0,T]$ be the Lebesgue space of measurable functions y such that $s \longmapsto |y(s)|$ is Lebesgue integrable equipped with the norm $\|y\|_1 = \int_0^T |y(s)|\, ds$ and $X = C_{2-\alpha}[0,T]$ endowed with the norm $\|u\|_{2-\alpha} = \sup_{t\in[0,T]} t^{2-\alpha} |u(t)|$. Define the linear operator $L : \operatorname{dom}(L) \cap X \longrightarrow Y$ by

$$Lu = D_{0^+}^{\alpha} u, \tag{2.1}$$

where

$$\operatorname{dom}(L) = \left\{u \in X : D_{0^+}^{\alpha} u \in Y, u \text{ satisfies conditions } (1.2), (1.3) \text{ with } b_0 = b_1 = 1\right\}.$$

Finally, define the Nemytskii operator $N : X \longrightarrow Y$ by

$$(Nu)(t) = f(t, u(t)), \quad t \in [0, T]. \tag{2.2}$$

Thus, BVP (1.1), (1.2), (1.3) with $b_0 = b_1 = 1$ can be written as

$$Lu = Nu, \ u \in \operatorname{dom}(L).$$

In a series of lemmas, we next investigate the properties of operators L and N.

2.2 Auxiliary lemmas

Lemma 3. *Let L be the operator defined by (2.1); then*

$$\operatorname{Ker} L = \left\{ct^{\alpha-2} : c \in \mathbb{R}\right\} \ \textit{and} \ \operatorname{Im} L = \left\{y \in L[0,T] : \int_0^T y(s)ds = 0\right\}.$$

Proof. The equation $D^{\alpha}_{0^+} u(t) = 0$ admits $u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2}$ as solutions, where c_1, c_2 are arbitrary constants. Then

$$D^{\alpha-2}_{0^+} u(t) = I^{2-\alpha}_{0^+} u(t) = c_1 \Gamma(\alpha) t + c_2 \Gamma(\alpha - 1) \ \text{ and } \ D^{\alpha-1}_{0^+} u(t) = c_1 \Gamma(\alpha).$$

Combining this with (1.2) and (1.3), we find that

$$c_2 \Gamma(\alpha - 1) = c_1 \Gamma(\alpha) T + c_2 \Gamma(\alpha - 1)$$

and hence $c_1 = 0$ while c_2 is any constant.

If $y \in \operatorname{Im}(L)$, then there exists $u \in \operatorname{dom}(L)$ such that $D^{\alpha}_{0^+}(t) = y(t)$. Hence

$$u(t) = I^{\alpha}_{0^+} y(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}$$

and

$$\begin{aligned} D^{\alpha-2}_{0^+} u(t) &= I^{2}_{0^+} y(t) + c_1 \Gamma(\alpha) t + c_2 \Gamma(\alpha - 1), \\ D^{\alpha-1}_{0^+} u(t) &= I^{1}_{0^+} y(t) + c_1 \Gamma(\alpha). \end{aligned}$$

By the boundary conditions (1.2), (1.3), we infer that

$$c_1 = -\frac{1}{\Gamma(\alpha) T} \int_0^T (T - s) y(s) ds \ \text{ and } \ \int_0^T y(s)ds = 0.$$

Let $y \in Y$ satisfy $\int_0^T y(s)ds = 0$. If $u(t) = I^{\alpha}_{0^+} y(t) - \frac{t^{\alpha-1}}{\Gamma(\alpha)T} \int_0^T (T - s) y(s)ds$, then $u \in \operatorname{dom}(L)$ and $D^{\alpha}_{0^+} D^{\alpha}_{0^+} u(t) = y(t)$. As a consequence $y \in \operatorname{Im}(L)$. □

Lemma 4.

(a) $L : \operatorname{dom}(L) \cap X \longrightarrow Y$ is a Fredholm operator of index 0.

(b) The linear continuous projectors $Q : Y \to Y$ and $P : X \to X$ are such that

$$Qy = \frac{1}{T} \int_0^T y(s)ds \ \textit{ and } \ (Pu)(t) = \frac{1}{\Gamma(\alpha - 1)} I^{2-\alpha}_{0^+} u(t) \mid_{t=T} t^{\alpha-2}.$$

Proof. It is easy to see that $Q^2y = Qy$ and $P^2u = Pu$, for $y \in Y$, $u \in X$. For all $y \in Y$, $y_1 = y - Qy \in \mathrm{Im}\,(L)$ because $\int_0^T y_1(s)ds = 0$. Hence $Y = \mathrm{Im}\,(L) + \mathrm{Im}\,(Q)$, $(\mathrm{Im}\,(Q) = \mathbb{R})$. For $m \in \mathrm{Im}\,(L) \cap \mathbb{R}$, we have $\int_0^T mds = Tm = 0$; therefore $m = 0$ and $Y = \mathrm{Im}\,(L) \oplus \mathrm{Im}\,(Q)$. Thus $\dim(\mathrm{Ker}\, L) = co\dim(\mathrm{Im}\, L) = \dim(\mathrm{Im}\, Q) = \dim(\mathbb{R}) = 1$. So L is a Fredholm operator of index 0. ∎

Lemma 5. *Let* $L_P = L\mid_{\mathrm{dom}(L)\cap \mathrm{Ker}\,P}: \mathrm{dom}(L) \cap \mathrm{Ker}\,P \to \mathrm{Im}\,(L)$. *The inverse* K_P *of* L_P *is given by*

$$(K_P y)(t) = I_{0^+}^{\alpha} y(t) - \frac{t^{\alpha-1}}{T\Gamma(\alpha)} I_{0^+}^{2} y(T).$$

Moreover

$$\|K_P y\|_{2-\alpha} \le \frac{2T}{\Gamma(\alpha)} \|y\|_1,$$

for all $y \in \mathrm{Im}\,(L)$.

Proof. For all $y \in \mathrm{Im}\,(L)$, we have

$$(LK_P y)(t) = D_{0^+}^{\alpha}\left(I_{0^+}^{\alpha} y(t) - \frac{t^{\alpha-1}}{T\Gamma(\alpha)} I_{0^+}^{2} y(T)\right) = y(t).$$

Recall that

$$\mathrm{Ker}\,P = \left\{u \in \mathrm{dom}\,(L):\ I_{0^+}^{2-\alpha} u(t)\mid_{t=T} = 0\right\}.$$

Thus, for $u \in \mathrm{dom}\,(L) \cap \mathrm{Ker}\,P$, we have

$$\begin{aligned}(K_P L)\,u(t) &= I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) - \tfrac{t^{\alpha-1}}{T.\Gamma(\alpha)} I_{0^+}^{2} D_{0^+}^{\alpha} u(T)\\ &= u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} - \tfrac{I_{0^+}^{2-\alpha} u(T)}{T.\Gamma(\alpha)} t^{\alpha-1}.\end{aligned}$$

Since $u \in \mathrm{dom}(L) \cap \mathrm{Ker}\,P$, then

$$(K_P L)\,u \in \mathrm{dom}(L) \cap \mathrm{Ker}\,P$$

and so

$$I_{0^+}^{2-\alpha} u(T) = 0 \ \text{ and } \ c_1 t^{\alpha-1} + c_2 t^{\alpha-2} \in \mathrm{dom}(L) \cap \mathrm{Ker}\,P.$$

Moreover

$$I_{0^+}^{2-\alpha}\left(c_1 t^{\alpha-1} + c_2 t^{\alpha-2}\right) = c_1 \Gamma(\alpha)\, t + c_2 \Gamma(\alpha-1);$$

hence

$$c_2 \Gamma(\alpha-1) = c_1 \Gamma(\alpha)\, T + c_2 \Gamma(\alpha-1) = 0.$$

Finally $c_2 = c_1 = 0$ and

$$(K_P L)\,u(t) = u(t),$$

which shows that $K_P = (L_P)^{-1}$.

Keeping in mind that

$$t^{2-\alpha}(K_P y)(t) = \frac{t^{2-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)ds - \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)\, y(s)ds,$$

we deduce that

$$t^{2-\alpha}\left|\left(K_{P}y\right)(t)\right| \leq \frac{T^{2-\alpha}}{\Gamma(\alpha)}T^{\alpha-1}\int_{0}^{T}|y(s)|\,ds+\frac{T}{T\Gamma(\alpha)}T\int_{0}^{T}|y(s)|\,ds=\frac{2T}{\Gamma(\alpha)}\|y\|_{1}.$$

Finally

$$\|K_{P}y\|_{2-\alpha}=\sup_{t\in[0,T]}t^{2-\alpha}\left|\left(K_{P}y\right)(t)\right| \leq \frac{2T}{\Gamma(\alpha)}\|y\|_{1}.$$

□

Lemma 6. *For all* $u\in X$, $t\in[0,T]$, *we have*

$$t^{2-\alpha}K_{P}(I-Q)\,Nu(t)=\frac{1}{\Gamma(\alpha)}\int_{0}^{T}G(t,s)\,f(s,u(s))\,ds,$$

where

$$G(t,s)=\begin{cases} t^{2-\alpha}(t-s)^{\alpha-1}+\frac{ts}{T}-\frac{t}{2}-\frac{t^{2}}{\alpha T}, & 0\leq s<t\leq T,\\ \frac{ts}{T}-\frac{t}{2}-\frac{t^{2}}{\alpha T}, & 0\leq t<s\leq T.\end{cases}$$

Lemma 7. *Let* $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ *be a continuous function. Assume that* Ω *is an open bounded subset from* X *such that* $\operatorname{dom}(L)\cap\overline{\Omega}\neq\emptyset$; *then* N *is* L*-compact on* $\overline{\Omega}$.

Proof. In order to prove that N is L-compact on $\overline{\Omega}$, we only need to show that $QN\left(\overline{\Omega}\right)$ is bounded and $K_{P}(I-Q)\,N:\overline{\Omega}\to Y$ is compact.

Since $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ is continuous, $\overline{\Omega}$ is bounded; therefore there exists a constant $M>0$ such that $|f(t,u(t))|\leq M$, $\forall\,u\in\overline{\Omega}$, $\forall\,t\in[0,T]$. Consequently, for all $u\in\overline{\Omega}$, we have

$$\begin{aligned}\|QN(u)\|_{1} &= \int_{0}^{T}\left[\frac{1}{T}\left|\int_{0}^{T}f(s,u(s))\,ds\right|\right]ds=\left|\int_{0}^{T}f(s,u(s))\,ds\right|\\ &\leq \int_{0}^{T}|f(s,u(s))|\,ds\leq MT.\end{aligned}$$

Since $(I-Q)$ and K_{P} are continuous linear operators, then $(I-Q)\,N(u)$ and $K_{P}(I-Q)\,N(u)$ are bounded. Hence

$$\begin{aligned}\|(I-Q)\,N(u)\|_{1} &\leq \|N(u)\|_{1}+\|QN(u)\|_{1}\leq 2TM,\\ \|K_{P}(I-Q)\,N(u)\|_{2-\alpha} &\leq \frac{2T}{\Gamma(\alpha)}\|(I-Q)\,N(u)\|_{1}\leq\frac{4T^{2}M}{\Gamma(\alpha)}.\end{aligned}$$

For all $t_{1}\in[0,T]$, $t_{2}\in[0,T]$, $(t_{1}<t_{2})$, and $u\in\overline{\Omega}$, we have

$$\begin{aligned}&\left|t_{2}^{2-\alpha}K_{P}(I-Q)\,Nu(t_{2})-t_{1}^{2-\alpha}K_{P}(I-Q)\,Nu(t_{1})\right|\\ &= \frac{1}{\Gamma(\alpha)}\left|\int_{0}^{T}G(t_{2},s)\,f(s,u(s))\,ds-\int_{0}^{T}G(t_{1},s)\,f(s,u(s))\,ds\right|\\ &= \frac{1}{\Gamma(\alpha)}\left|\int_{0}^{T}(G(t_{2},s)-G(t_{1},s))f(s,u(s))\,ds\right|\\ &\leq \frac{M}{\Gamma(\alpha)}\int_{0}^{T}|G(t_{2},s)-G(t_{1},s)|\,ds.\end{aligned}$$

Next, we distinguish between three different cases:

1. Case $t_1 < t_2 < s$. We have
$$\begin{aligned} |G(t_2,s) - G(t_1,s)| &= |t_2 - t_1| \left|\tfrac{s}{T} - \left(\tfrac{1}{2} + \tfrac{t_2+t_1}{\alpha T}\right)\right| \\ &\leq |t_2 - t_1| \left(\tfrac{s}{T} + \left(\tfrac{1}{2} + \tfrac{t_2+t_1}{\alpha T}\right)\right); \end{aligned}$$
then
$$\begin{aligned} \textstyle\int_0^T |G(t_2,s) - G(t_1,s)|\, ds &\leq |t_2 - t_1| \textstyle\int_0^T \left(\tfrac{s}{T} + \left(\tfrac{1}{2} + \tfrac{t_2+t_1}{\alpha T}\right)\right) ds \\ &= (T + \tfrac{t_2+t_1}{\alpha}) |t_2 - t_1|. \end{aligned}$$

2. Case $s < t_1 < t_2$. We have
$$\begin{aligned} |G(t_2,s) - G(t_1,s)| &= \Big| t_2^{2-\alpha}(t_2 - s)^{\alpha-1} - t_1^{2-\alpha}(t_1 - s)^{\alpha-1} \\ &\quad + (t_2 - t_1)\left(\tfrac{s}{T} - \left(\tfrac{1}{2} + \tfrac{t_2+t_1}{\alpha T}\right)\right)\Big| \\ &\leq \left| t_2^{2-\alpha}(t_2 - s)^{\alpha-1} - t_1^{2-\alpha}(t_1 - s)^{\alpha-1}\right| \\ &\quad + \left|(t_2 - t_1)\left(\tfrac{s}{T} - \left(\tfrac{1}{2} + \tfrac{t_2+t_1}{\alpha T}\right)\right)\right|. \end{aligned}$$
Note that the function Ψ_s defined by
$$\Psi_s(t) = t^{2-\alpha}(t-s)^{\alpha-1},$$
where $t \in [0,T]$ and $0 \leq s < t$, is increasing on $[0,T]$ because its derivative
$$\Psi_s'(t) = (2-\alpha)\left(\frac{t-s}{t}\right)^{\alpha-1} + (\alpha-1)\left(\frac{t}{t-s}\right)^{2-\alpha}$$
is positive. Then
$$t_2^{2-\alpha}(t_2 - s)^{\alpha-1} - t_1^{2-\alpha}(t_1 - s)^{\alpha-1} > 0$$
and
$$\begin{aligned} &\textstyle\int_0^T (t_2^{2-\alpha}(t_2 - s)^{\alpha-1} - t_1^{2-\alpha}(t_1 - s)^{\alpha-1}) ds \\ = \;& t_2^{2-\alpha} \textstyle\int_0^{t_2} (t_2 - s)^{\alpha-1} ds - t_1^{2-\alpha} \int_0^{t_1} (t_1 - s)^{\alpha-1} ds \\ = \;& \tfrac{t_2 - t_1}{\alpha}. \end{aligned}$$
Finally
$$\begin{aligned} \textstyle\int_0^T |G(t_2,s) - G(t_1,s)|\, ds &\leq \tfrac{t_2-t_1}{\alpha} + (T + \tfrac{t_2+t_1}{\alpha}) |t_2 - t_1| \\ &= (T + \tfrac{t_2+t_1+1}{\alpha}) |t_2 - t_1|. \end{aligned}$$

3. Case $t_1 < s < t_2$. We have
$$\begin{aligned} |G(t_2,s) - G(t_1,s)| &= \left| t_2^{2-\alpha}(t_2 - s)^{\alpha-1} + (t_2 - t_1)\left(\tfrac{s}{T} - \left(\tfrac{1}{2} + \tfrac{t_2+t_1}{\alpha T}\right)\right)\right| \\ &\leq t_2^{2-\alpha}(t_2 - s)^{\alpha-1} + \left|(t_2 - t_1)\left(\tfrac{s}{T} - \left(\tfrac{1}{2} + \tfrac{t_2+t_1}{\alpha T}\right)\right)\right| \\ &\leq t_2^{2-\alpha}(t_2 - t_1)^{\alpha-1} + (T + \tfrac{t_2+t_1}{\alpha}) |t_2 - t_1|. \end{aligned}$$

This shows that $K_P(I-Q)N$ is equicontinuous, as claimed. □

2.3 Existence theorem

We are now in position to state and prove our main existence result.

Theorem 2. *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be continuous. Assume that*

- *(H_1) there exist two functions a, $r \in L^1[0,T]$ such that for all $t \in [0,T]$ and $x \in \mathbb{R}$, we have $|f(t,x)| \le t^{2-\alpha} a(t)|x| + r(t)$,*
- *(H_2) there exists a constant $M > 0$ such that for all $u \in \operatorname{dom}(L)$, if $|u(t)| > M$ for all $t \in [0,T]$, then $\int_0^T f(s, u(s))\, ds \neq 0$,*
- *(H_3) there exists a constant $M^* > 0$ such that for all $c \in \mathbb{R}$, if $|c| > M^*$, then either*
$$c \int_0^T f\left(s, cs^{\alpha-2}\right) ds < 0 \ \text{ or } \ c \int_0^T f\left(s, cs^{\alpha-2}\right) ds > 0.$$

Then the boundary value problem (1.1), (1.2), (1.3) with $b_0 = b_1 = 1$ has at least one solution $u \in C_{2-\alpha}[0,T]$ provided that $\|a\|_1 < \frac{\Gamma(\alpha)}{2T}$.

Proof. Let
$$\Omega_1 = \{u \in \operatorname{dom}(L) \setminus \operatorname{Ker} L : \ Lu = \lambda N u, \ \lambda \in (0,1)\}.$$

For $u \in \Omega_1$, we have $u \in \operatorname{dom}(L) \cap \operatorname{Ker} P$ and $Lu = \lambda N u$ with $\lambda \neq 0$ because $u \notin \operatorname{Ker} L$; then
$$\begin{aligned} \|u\|_{2-\alpha} &= \|K_P L u\|_{2-\alpha} \\ &\le \tfrac{2T}{\Gamma(\alpha)} \|Lu\|_1 = \tfrac{2T\lambda}{\Gamma(\alpha)} \|Nu\|_1 \\ &\le \tfrac{2T}{\Gamma(\alpha)} \int_0^T |f(s,u(s))|\, ds. \end{aligned}$$

From condition (H_1), we have
$$|f(s,u(s))| \le s^{2-\alpha} a(s)|u(s)| + r(s) \le a(s) \sup_{s\in[0,T]} s^{2-\alpha}|u(s)| + r(s).$$

Hence
$$\int_0^T |f(s,u(s))|\, ds \le \|a\|_1 \|u\|_{2-\alpha} + \|r\|_1.$$

Then
$$\|u\|_{2-\alpha} \le \frac{2T}{\Gamma(\alpha)} \left(\|a\|_1 \|u\|_{2-\alpha} + \|r\|_1\right).$$

Finally
$$\|u\|_{2-\alpha} \le \frac{2T\|r\|_1}{\Gamma(\alpha) - 2T\|a\|_1} = M_1.$$

Consider the set
$$\Omega_2 = \{u \in \operatorname{Ker} L : \ Nu \in \operatorname{Im} L\}.$$

For $u \in \Omega_2$, we have $u(t) = ct^{\alpha-2}$ and $\int_0^T f\left(s, cs^{\alpha-2}\right) ds = 0$. Then, from the condition (H_2), there exists $t_0 \in [0,T]$ such that $\left|ct_0^{\alpha-2}\right| \leq M$, with $t_0 \neq 0$. Therefore

$$\|u\|_{2-\alpha} = \sup_{t\in[0,T]} t^{2-\alpha} \left|ct^{\alpha-2}\right| = |c| \leq Mt_0^{2-\alpha} = M_2.$$

Let

$$\Omega_3 = \{u \in \operatorname{Ker} L : -\lambda Ju + (1-\lambda) QNu = 0,\ \lambda \in [0,1]\},$$

where $J : \operatorname{Ker} L \to \operatorname{Im} Q$ is the linear isomorphism defined by $J(u) = c$.

In case (H_3) is satisfied, assume that $c\int_0^T f\left(s, cs^{\alpha-2}\right) ds < 0$ holds. For all $u \in \Omega_3$, we can write $u = ct^{\alpha-2}$ and

$$\lambda c^2 = \frac{(1-\lambda)}{T} c \int_0^T f\left(s, cs^{\alpha-2}\right) ds.$$

If $\lambda = 1$, then $c = 0$. Otherwise, if Hypothesis $|c| > M^*$, then by (H_3), one has

$$\frac{(1-\lambda)}{T} c \int_0^T f\left(s, cs^{\alpha-2}\right) ds < 0,$$

which contradicts $\lambda c^2 \geq 0$. Thus

$$\|u\|_{2-\alpha} = |c| \leq\ M^*.$$

If $c\int_0^T f\left(s, cs^{\alpha-2}\right) ds > 0$ holds, then Ω_3 can be defined as follows:

$$\Omega_3 = \{u \in \operatorname{Ker} L : \lambda Ju + (1-\lambda) QNu = 0,\ \lambda \in [0,1]\}.$$

Next, we shall prove that all conditions of Theorem 1 are fulfilled.

Let Ω be bounded open such that $\overline{\Omega}_1 \cup \overline{\Omega}_2 \cup \overline{\Omega}_3 \subset \Omega$. We have already proved that L is a Fredholm operator of index 0 and that N is L-compact on $\overline{\Omega}$. Also, we have

1. $Lu \neq Nu$, for each $(u,\lambda) \in [(\operatorname{dom}(L) \diagdown \operatorname{Ker} L) \cap \partial\Omega] \times [0,1]$ for $\overline{\Omega}_1 \subset \Omega$.
2. $Nu \notin \operatorname{Im} L$ for each $u \in \operatorname{Ker} L \cap \partial\Omega$ for $\overline{\Omega}_2 \subset \Omega$.
3. In order to take into account the subset Ω_3 in the above two cases, we consider the homotopy $H(u,\lambda) = \pm\lambda Ju + (1-\lambda) QNu$. Then $H(u,\lambda) \neq 0$, for each $u \in \operatorname{Ker} L \cap \partial\Omega$. As $\overline{\Omega}_3 \subset \Omega$. By the homotopy property of the degree, we finally deduce that

$$\begin{aligned} \deg\left(QN|_{\operatorname{Ker} L}, \operatorname{Ker} L \cap \Omega, 0\right) &= \deg\left(H(u,0), \operatorname{Ker} L \cap \Omega, 0\right) \\ &= \deg\left(H(u,1), \operatorname{Ker} L \cap \Omega, 0\right) \\ &= \deg\left(\pm J, \operatorname{Ker} L \cap \Omega, 0\right) \neq 0, \end{aligned}$$

which completes the proof of Theorem 2. □

2.4 Example 1

Consider the boundary value problem:

$$\begin{cases} D_{0^+}^{\frac{3}{2}} u(t) & = & \frac{3\sqrt{t}}{5\pi\sqrt{\pi}} u(t)(2\sin u(t) - 3) + \pi\sqrt{\pi}\cos t, \quad 0 < t < \frac{\pi}{4}, \\ D_{0^+}^{\frac{1}{2}} u(0^+) & = & D_{0^+}^{\frac{1}{2}} u\left(\frac{\pi}{4}^-\right), \\ I_{0^+}^{\frac{1}{2}} u(0^+) & = & I_{0^+}^{\frac{1}{2}} u\left(\frac{\pi}{4}^-\right). \end{cases} \tag{1}$$

In this example,

$$\alpha = \frac{3}{2},\ T = \frac{\pi}{4}, \text{ and } f(t,x) = \frac{3\sqrt{t}}{5\pi\sqrt{\pi}} x(2\sin x - 3) + \pi\sqrt{\pi}\cos t.$$

In addition, we have

1.

$$|f(t,x)| \le \frac{3\sqrt{t}}{5\pi\sqrt{\pi}} |x| (2|\sin x| + 3) + \pi\sqrt{\pi}\cos t \le \frac{3}{\pi\sqrt{\pi}}\sqrt{t}\,|x| + \pi\sqrt{\pi}\cos t.$$

Then

$$a(t) = \frac{3}{\pi\sqrt{\pi}},\ \|a\|_1 = \frac{3}{4\sqrt{\pi}} < \frac{\Gamma\left(\frac{3}{2}\right)}{2\frac{\pi}{4}} = \frac{1}{\sqrt{\pi}}, \text{ and } r(t) = \pi\sqrt{\pi}\cos t.$$

2. Let $M = 80$. For each $u \in \mathrm{dom}(L)$, suppose that $|u(t)| > M$, for all $t \in [0, \frac{\pi}{4}]$. If $u(t) > M$, for all $t \in [0, \frac{\pi}{4}]$, then $2\sin u(t) - 3 \le -1$ and thus

$$f(t, u(t)) \le -\frac{3\sqrt{t}}{5\pi\sqrt{\pi}} u(t) + \pi\sqrt{\pi}\cos t \le -\frac{3\sqrt{t}}{5\pi\sqrt{\pi}} M + \pi\sqrt{\pi}\cos t.$$

Notice that since $-u(t) < -M$, then

$$\int_0^{\frac{\pi}{4}} f(t, u(t))\, dt \le \int_0^{\frac{\pi}{4}} \left(-\frac{3\sqrt{t}}{5\pi\sqrt{\pi}} M + \pi\sqrt{\pi}\cos t\right) dt = -0.06 < 0.$$

If $u(t) < -M$, for all $t \in [0, \frac{\pi}{4}]$, then $0 < M < -u(t)$ and

$$\frac{3\sqrt{t}}{5\pi\sqrt{\pi}} M < -\frac{3\sqrt{t}}{5\pi\sqrt{\pi}} u(t) \le \frac{3\sqrt{t}}{5\pi\sqrt{\pi}} u(t)(2\sin u(t) - 3).$$

Hence $f(t, u(t)) \ge \frac{3\sqrt{t}}{5\pi\sqrt{\pi}} M + \pi\sqrt{\pi}\cos t$, for all $t \in [0, \frac{\pi}{4}]$. Consequently

$$\int_0^{\frac{\pi}{4}} f(t, u(t))\, dt \ge \int_0^{\frac{\pi}{4}} \left(\frac{3\sqrt{t}}{5\pi\sqrt{\pi}} M + \pi\sqrt{\pi}\cos t\right) ds = 7.93 > 0.$$

Finally $\int_0^{\frac{\pi}{4}} f(t, u(t))\, dt \neq 0$.

3. Let $M^* = 95$. For every $c \in \mathbb{R}$ with $|c| > M^*$, we have $\left(2\sin\frac{c}{\sqrt{t}} - 3\right) \leq -1$. Then

$$\frac{3}{5\pi\sqrt{\pi}}c^2\left(2\sin\frac{c}{\sqrt{t}} - 3\right) + \pi\sqrt{\pi}c\cos t \leq -\frac{3}{5\pi\sqrt{\pi}}c^2 + \pi\sqrt{\pi}c\cos t.$$

Finally

$$\begin{aligned} c\int_0^{\frac{\pi}{4}} f\left(t, \tfrac{c}{\sqrt{t}}\right)dt &\leq \int_0^{\frac{\pi}{4}}\left(-\tfrac{3}{5\pi\sqrt{\pi}}c^2 + \pi\sqrt{\pi}c\cos t\right)dt \\ &= -\tfrac{3}{20\sqrt{\pi}}c^2 + \tfrac{\pi\sqrt{\pi}}{\sqrt{2}}c < 0, \end{aligned}$$

for all $c \notin \left[0, \frac{20\pi^2}{3\sqrt{2}}\right]$. We conclude that all conditions of Theorem 2 hold, proving that problem 1 has at least one solution u in $C_{\frac{1}{2}}[0, \frac{\pi}{4}]$.

2.5 Example 2

Consider the following boundary value problem

$$\begin{cases} D_{0_+}^{\frac{3}{2}}u(t) &= f(t, u(t)), \quad 0 < t < 1, \\ D_{0_+}^{\frac{1}{2}}u(0^+) &= D_{0_+}^{\frac{1}{2}}u(1^-), \\ I_{0_+}^{\frac{1}{2}}u(0^+) &= I_{0_+}^{\frac{1}{2}}u(1^-), \end{cases} \tag{2}$$

where

$$f(t,x) = \begin{cases} -\frac{\sqrt{t}}{10}, & t \in [0,1],\ x \in (-\infty, 0) \\ \frac{\sqrt{t}}{10}\left(x - 1 + \frac{1}{3}\ln\left(|x|\sqrt{t} + 1\right)\right), & t \in [0,1],\ x \in [0, +\infty). \end{cases}$$

Next, we check all of assumptions of Theorem 2:

1. Since for all $s > 0, \ln s \leq s - 1 < s$, then

$$|f(t,x)| \leq \frac{\sqrt{t}}{10}\left(|x| + \frac{1}{3}\left(|x|\sqrt{t} + 1\right)\right) + \frac{\sqrt{t}}{10} = \sqrt{t}\left(\frac{1}{10} + \frac{\sqrt{t}}{30}\right)|x| + 4\frac{\sqrt{t}}{30}.$$

Then we take

$$a(t) = \left(\frac{1}{10} + \frac{\sqrt{t}}{30}\right) \text{ and } r(t) = 4\frac{\sqrt{t}}{30}$$

with $a, r \in L^1[0,1]$ and

$$\|a\|_1 = \int_0^1\left(\frac{1}{10} + \frac{\sqrt{t}}{30}\right)dt = \frac{1}{10} + \frac{2}{90} = \frac{11}{90} < \frac{\Gamma\left(\frac{3}{2}\right)}{2} \simeq 0.443.$$

2. For $M = 91$, assume that $u(t) > M$, for all $t \in [0,1]$. Then

$$f(s, u(s)) \geq \frac{\sqrt{s}}{10}\left(M - 1 + \frac{1}{3}\ln\left(M\sqrt{s} + 1\right)\right).$$

As a consequence, we derive the estimates:

$$\begin{aligned}\int_0^1 f(s,u(s))ds &\geq (M-1)\int_0^1 \frac{\sqrt{s}}{10}ds + \frac{1}{30}\int_0^1 \sqrt{s}\ln(M\sqrt{s}+1)ds\\ &= \frac{2}{30}(M-1) + \frac{2}{90}\left(\left(1+\frac{1}{M^3}\right)\ln(M+1) - \frac{(M+1)^3}{3M^3}\right.\\ &\quad \left. + \frac{3(M+1)^2}{2M^3} - \frac{3(M+1)}{M^3} + \frac{11}{6M^3}\right)\\ &\geq \frac{2}{30}(M-1) - \frac{2}{90}\frac{(M+1)^3+9(M+1)}{3M^3} \simeq 5.99.\end{aligned}$$

Now suppose that $u(t) < -M$, for all $t \in [0,1]$. Then

$$\int_0^1 f(s,u(s))ds = \int_0^1 -\frac{\sqrt{s}}{10}ds = -\frac{2}{30} < 0$$

which shows that

$$\int_0^1 f(s,u(s))ds \neq 0,$$

for all $u \in \mathrm{dom}(L)$ satisfying $|u(t)| > M$, for all $t \in [0,1]$.

3. Let $M^* = \frac{2}{3}$. For all $c > M^*$, we have

$$\begin{aligned}c\int_0^1 f\left(s,\frac{c}{\sqrt{s}}\right)ds &= \int_0^1 c\frac{\sqrt{s}}{10}\left(\frac{c}{\sqrt{s}} - 1 + \frac{1}{3}\ln\left(\frac{|c|}{\sqrt{s}}\sqrt{s}+1\right)\right)ds\\ &= \frac{c^2}{10} - \frac{2}{30}c + \frac{2}{90}c\ln(|c|+1)\\ &= \frac{c}{10}\left(c - \frac{2}{3} + \frac{2}{9}\ln(|c|+1)\right) > 0,\end{aligned}$$

while for $c < -M^*$, we have

$$c\int_0^1 f\left(s,\frac{c}{\sqrt{s}}\right)ds = c\int_0^1 -\frac{\sqrt{s}}{10}ds = -\frac{2}{30}c > 0.$$

Therefore we have showed that problem 2 has at least one solution u in $C_{\frac{1}{2}}[0,1]$.

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