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On a solution to equation with discrete multiplicative-additive derivative

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Abstract. As it is well known, a discrete differential equation (basically, with additive derivative) is called a difference equation [1–3]. The Cauchy problem for such kind of equations is considered in [4]. Several initial and boundary value problems for additive derivatives are also considered in [5].

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The subject of this paper is to study the solution to non-linear differential problems. The domain of solution determination is divided into a grid with step h for discretization of the problem for ordinary differential equations. Here, we accept $h = 1$. Therefore, we do not need the result as a continuous process.

As well as the equation, additional conditions can also be non-linear. It is based on discrete additive derivative, discrete multiplicative derivative and discrete integrals.

Let consider the equation with differentiation as follows:

$$y_{i+3} = y_{i+2} + \frac{f_i \cdot (y_{i+2} - y_{i+1})^2}{y_{i+1} - y_i}, \quad i \geq 0. \tag{1}$$

In order to solve this equation, we firstly find the equality describing how this sequence is obtained by giving values to i. If $i = 0$, then we obtain from (1)

$$y_3 = y_2 + f_0 \cdot \frac{(y_2 - y_1)^2}{y_1 - y_0}, \tag{2}$$

if $i = 1$, then

$$y_4 = y_3 + f_1 \cdot \frac{(y_3 - y_2)^2}{y_2 - y_1} = y_2 + f_0 \frac{(y_2 - y_1)^2}{y_1 - y_2} + f_1 \frac{(y_3 - y_2)^2}{y_2 - y_1} =$$

$$= y_2 + f_0 \frac{(y_2 - y_1)^2}{y_1 - y_0} + f_1 f_0^2 \frac{(y_2 - y_1)^2}{y_1 - y_0}, \tag{3}$$

if $i = 2$, then

$$y_5 = y_4 + f_2\frac{(y_4-y_3)^2}{y_3-y_2} = y_2 + f_0\frac{(y_2-y_1)^2}{y_2-y_0} + f_1 f_0^2\frac{(y_2-y_1)^2}{(y_1-y_0)^2} + f_2\frac{f_1^2(y_3-y_2)^3}{(y_2-y_1)^2} =$$

$$= y_2 + f_0\frac{(y_2-y_1)^2}{y_1-y_0} + f_1 f_0^2\frac{(y_2-y_1)^3}{(y_1-y_0)^2} + f_2 f_1^2 f_0^3\frac{(y_2-y_1)^4}{(y_1-y_0)^3} =$$

$$= y_2 + \sum_{k=0}^{2}\left(\prod_{p=1}^{k+1} f_{k+1-p}^p\right)\frac{(y_2-y_1)^{k+2}}{(y_2-y_0)^{k+1}}. \qquad (4)$$

So we obtain

$$y_{i+3} = y_2 + \sum_{k=0}^{i}\frac{(y_2-y_1)^{k+2}}{(y_1-y_0)^{k+1}}\cdot\prod_{p=1}^{k+1} f_{k+1-p}^p, \quad i\geq 0. \qquad (5)$$

Now let prove the relation obtained in (5) by mathematical induction.

We just proved that the statement (5) holds for $i=2$. Let show that if (5) holds for $i\leq q-1$, then also it holds for $i=q$:

$$y_{q+3} = y_{q+2} + \frac{(y_{q+2}-y_{q+1})^2}{y_{q+1}-y_q}\cdot f_q = y_2 + \sum_{k=0}^{q-1}\frac{(y_2-y_1)^{k+2}}{(y_1-y_0)^{k+1}}\cdot\prod_{p=1}^{k+1} f_{k+1-p}^p +$$

$$+f_q\frac{\left[y_2 + \sum_{k=0}^{q-1}\frac{(y_2-y_1)^{k+2}}{(y_1-y_0)^{k+1}}\cdot\prod_{p=1}^{k+1} f_{k+1-p}^p - y_2 - \sum_{k=0}^{q-2}\frac{(y_2-y_1)^{k+2}}{(y_2-y_0)^{k+1}}\cdot\prod_{p=1}^{k+1} f_{k+1-p}^p\right]^2}{y_2 + \sum_{k=0}^{q-2}\frac{(y_2-y_1)^{k+2}}{(y_1-y_0)^{k+1}}\cdot\prod_{p=1}^{k+1} f_{k+1-p}^p - y_2 - \sum_{k=0}^{q-3}\frac{(y_2-y_1)^{k+2}}{(y_2-y_0)^{k+1}}\cdot\prod_{p=1}^{k+1} f_{k+1-p}^p} =$$

$$= y_2 + \sum_{k=0}^{q-1}\frac{(y_2-y_1)^{k+2}}{(y_1-y_0)^{k+1}}\cdot\prod_{p=1}^{k+1} f_{k+1-p}^p + f_q\frac{\left[\frac{(y_2-y_1)^{q+1}}{(y_1-y_0)^q}\prod_{p=1}^{q} f_{q-p}^p\right]^2}{\frac{(y_2-y_1)^q}{(y_2-y_0)^{q-1}}\prod_{p=1}^{q-1} f_{q-1-p}^p} =$$

$$= y_2 + \sum_{k=0}^{q-1}\frac{(y_2-y_1)^{k+2}}{(y_1-y_0)^{k+1}}\cdot\prod_{p=1}^{k+1} f_{k+1-p}^p + f_q\frac{(y_2-y_1)^{q+2}}{(y_1-y_0)^{q+1}}\cdot\frac{f_{q-1}^2 f_{q-2}^1 \cdots f_0^2}{f_{q-2} f_{q-3}^2 \cdots f_0^{q-1}} =$$

$$= y_2 + \sum_{k=0}^{q-1}\frac{(y_2-y_1)^{k+2}}{(y_1-y_0)^{k+1}}\cdot\prod_{p=1}^{k+1} f_{k+1-p}^p + \frac{(y_2-y_1)^{q+2}}{(y_1-y_0)^{q+1}}\cdot f_q\cdot f_{q-1}^2\cdot f_{i-2}^3 \cdots\times$$

$$\times f_1^q \cdots f_0^{q+1} = y_2 + \sum_{k=0}^{q}\frac{(y_2-y_1)^{k+2}}{(y_1-y_0)^{k+1}}\cdot\prod_{p=1}^{k+1} f_{k+1-p}^p \qquad (6)$$

thereby showing that indeed (5) holds for $i = q$. Since both the basis and the inductive step have been performed, by mathematical induction, the statement (5) holds for all natural numbers i. So we proved the following theorem.

Theorem 1. *If there is an order-bounded sequence with true values f_i for the given non-linear third order difference equation (1), then the general solution to this equation has the form (5), where y_0, y_1 and y_2 are arbitrary constants.*

Cauchy problem. If the Cauchy problem is considered for the equation (1), then the following initial conditions shall be provided

$$y_k = \alpha_k, \quad \alpha = 0, 1, 2. \tag{7}$$

Given these conditions, the solution to the problem (1), (7) takes the form

$$y_{i+3} = \alpha_2 + \sum_{k=0}^{i} \frac{(\alpha_2 - \alpha_1)^{k+2}}{(\alpha_1 - \alpha_0)^{k+1}} \cdot \prod_{p=1}^{k+1} f_{k+1-p}^{p}, \quad i = \overline{0, N-3}.$$

Boundary value problem. If we consider the boundary value problem for equation (1) with boundary conditions

$$y_2 - y_1 = \alpha_1, \quad y_1 - y_0 = \alpha_0, \quad y_N = \alpha_N, \tag{8}$$

then in accordance with (5) we obtain the following solution

$$y_{i+3} = y_2 + \sum_{k=0}^{i} \frac{\alpha_1^{k+2}}{\alpha_0^{k+1}} \cdot \prod_{p=1}^{k+1} f_{k+1-p}^{p}, \quad i = \overline{0, N-3},$$

$$\alpha_N = y_2 + \sum_{k=0}^{N-3} \frac{\alpha_1^{k+2}}{\alpha_0^{k+1}} \cdot \prod_{p=1}^{k+1} f_{k+1-p}^{p},$$

Substituting the last expression into the equation, one can get the following

$$y_2 = \alpha_N - \sum_{k=0}^{N-3} \frac{\alpha_1^{k+2}}{\alpha_0^{k+1}} \cdot \prod_{p=1}^{k+1} f_{k+1-p}^{p} \quad i = \overline{0, N-3}.$$

Thus, a single-valued solution could be obtained.

Conclusion. We have studied the Cauchy problem and the boundary value problem for the third order difference equation with discrete derivative and the analytical expressions for their solutions were obtained. Once the form of the general solution to the equation was determined, it was proved by means of mathematical induction. Finally, the constants included in the general solution were studied and defined.

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