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On invariants and canonical form of matrices of second order with respect to semiscalar equivalence

B. Z. Shavarovskii

Abstract. We indicate a complete system of invariants and suggest a canonical form for one class of polynomial matrices of second order with respect to semiscalar equivalence.

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The notion of semiscalar equivalence of polynomial matrices is introduced and considered first in [1] (see also [2]). Related results are obtained in [3],[4]. These researches take on further development in [5] – [8]. The most important components of the problem of semiscalar equivalence are the search of invariants and the construction of normal forms for matrices with respect to such equivalence. Large difficulties in this problem arise already for matrices of second order. In this paper, some classes of order two polynomial matrices are singled out for which complete system of invariants is obtained and canonical form with respect to semiscalar equivalence is indicated. This form enables one to solve the classification problem for some polynomial matrices up to semiscalar equivalence.

We consider a ring $M(2,\ C[x])$ of order two polynomial matrices over the field of complex numbers C. According to [1] the matrices $A(x),\ B(x) \in M(2,\ C[x])$ are called semiscalarly equivalent if $CA(x)Q(x) = B(x)$ for some invertible matrices $C \in GL(2,\ C)$, $Q(x) \in GL(2,\ C[x])$. The determinant $|A(x)|$ is called the characteristic polynomial of $A(x)$ and its roots are called the characteristic roots of matrices $A(x)$. By Theorem 1 [1] (see also Theorem 1 §1, Section IV [2]) every matrix of full rank is semiscalarly equivalent to lower triangular form with invariant polynomials on the main diagonal. Without loss of generality, we can assume that first invariant polynomial of considered matrix is identity.

In this paper we use the standard notations. In particular, $c^{(t)}(\alpha)$ is the value at $x = \alpha$ of the t-th derivative of the polynomial $c(x)$.

Proposition 1. *Let be given a matrix*

$$A(x) = \left\| \begin{matrix} 1 & 0 \\ a(x) & \Delta(x) \end{matrix} \right\|, \quad \deg a(x) < \deg \Delta(x), \tag{1}$$

and a partition

$$M = M_1 \cup \ldots \cup M_w, \quad M_u \cap M_v = \emptyset, \quad u \neq v, \tag{2}$$

of the set M of characteristic roots of matrix $A(x)$ into subsets M_u such that $\alpha,\ \beta \in M_u$ if $a(\alpha) = a(\beta)$. Subsets M_u are uniquely defined by a class of semiscalarly equivalent matrices $\{CA(x)Q(x)\}$.

Proof. Let a matrix $A(x)$ be semiscalarly equivalent to a matrix

$$B(x) = \left\| \begin{array}{cc} 1 & 0 \\ b(x) & \Delta(x) \end{array} \right\|, \quad \deg b(x) < \deg \Delta(x). \tag{3}$$

Then there exists

$$\left\| \begin{array}{cc} s_{11} & s_{12} \\ s_{21} & s_{22} \end{array} \right\| \in GL(2,\ C), \left\| \begin{array}{cc} r_{11}(x) & r_{12}(x) \\ r_{21}(x) & r_{22}(x) \end{array} \right\| \in GL(2,\ C[x]),$$

such that

$$\left\| \begin{array}{cc} s_{11} & s_{12} \\ s_{21} & s_{22} \end{array} \right\| \left\| \begin{array}{cc} 1 & 0 \\ a(x) & \Delta(x) \end{array} \right\| = \left\| \begin{array}{cc} 1 & 0 \\ b(x) & \Delta(x) \end{array} \right\| \left\| \begin{array}{cc} r_{11}(x) & r_{12}(x) \\ r_{21}(x) & r_{22}(x) \end{array} \right\|. \tag{4}$$

On the basis of (4) we can write the relation

$$s_{21} + s_{22}a(x) = b(x)r_{11}(x) + \Delta(x)r_{21}(x). \tag{5}$$

Setting $x = \alpha$ and $x = \beta$ in (5), we obtain the relations

$$s_{21} + s_{22}a(\alpha) = b(\alpha)r_{11}(\alpha), \tag{6}$$

$$s_{21} + s_{22}a(\beta) = b(\beta)r_{11}(\beta). \tag{7}$$

From (4) it follows that $r_{11}(x) = s_{11} + s_{12}a(x)$. Since $a(\alpha) = a(\beta)$, then $r_{11}(\alpha) = r_{11}(\beta)$ and from (6) and (7) we have $r_{11}(\alpha)(b(\alpha) - b(\beta)) = 0$. Equality (4) implies that $r_{12}(x) = s_{12}\Delta(x)$. Therefore $r_{11}(\alpha) \neq 0$ and $b(\alpha) = b(\beta)$. The notion of semiscalar equivalence is a symmetrical relation. Then from $b(\alpha) = b(\beta)$ a similar argument yields $a(\alpha) = a(\beta)$. This completes the proof. □

Consider now the case in which in (2) $w = 1$, i.e., $a(\alpha) = a(\beta)$ for arbitrary roots $\alpha,\ \beta \in M$. We may assume (without loss of generality) that $a(\alpha) = 0$.

Let $M = \{\alpha_i,\ i = 1,\ \ldots,\ p\}$, n_i and m_i be the multiplicities of root α_i in the polynomials $\Delta(x)$ and $a(x)$, respectively. Since $\deg a(x) < \deg \Delta(x) = s$, for some root $\alpha_j \in M$ multiplicities n_j and m_j satisfy the condition $m_j < n_j$. Let it be the roots α_j, $j = 1,\ \ldots,\ q$, $1 \leq q \leq p$ and $m_{q+l} \geq n_{q+l}$, $l = 1,\ \ldots,\ p-q$ (the case in which $a(x) \equiv 0$ is trivial).

Theorem 1. *Let every characteristic root $\alpha_i \in M$ of matrix $A(x)$ of the form (1) satisfy the condition $a(\alpha_i) = 0$. Let also multiplicities m_j and n_j of root $\alpha_j \in M$ in the polynomials $a(x)$ and $\Delta(x)$, respectively, satisfy the inequality $m_j < n_j$. Then multiplicities m_j are uniquely defined by a class of semiscalarly equivalent matrices $\{CA(x)Q(x)\}$ and rows $\| \, a_{j0} \quad a_{j1} \quad \dots \quad a_{j,\, l_j-m_j-1} \, \|$, $l_j = \min(2m_j,\ n_j)$, of coefficients from decompositions*

$$a(x) = \sum_{t=0}^{s-m_j-1} a_{jt}(x-\alpha_j)^{m_j+t} \tag{8}$$

are determined up to constant factor independent of $j = 1, \dots, q$.

Proof. Let matrices (1) and (3) be semiscalarly equivalent. If $a(\alpha_i) = b(\alpha_i) = 0$ then from relation (5) it follows that $s_{21} = 0$. Then

$$s_{22}a(x) - s_{11}b(x) - s_{12}a(x)b(x) = \Delta(x)r_{21}(x), \tag{9}$$

where $s_{11} \neq 0$, $s_{22} \neq 0$. Let for multiplicities m_j, m'_j, n_j of root $x = \alpha_j$ in the polynomials $a(x)$, $b(x)$, $\Delta(x)$, respectively, inequalities $m'_j < m_j < n_j$ be valid. Differentiating both members of equality (9) m'_j times at $x = \alpha_j$, we obtain $s_{11}b^{(m'_j)}(\alpha_j) = 0$. It is impossible, since $s_{11} \neq 0$ and $b^{(m'_j)}(\alpha_j) \neq 0$. Then $m'_j \geq m_j$. Considering that semiscalar equivalence is a symmetric relation, we have $m'_j \leq m_j$. Therefore $m'_j = m_j$. The first part of the theorem is proved.

By analogy to (8), write decomposition for the entry $b(x)$ of matrix (3):

$$b(x) = \sum_{t=0}^{s-m_j-1} b_{jt}(x-\alpha_j)^{m_j+t}. \tag{10}$$

Comparing the coefficients of equal degrees of binomial $x - \alpha_j$ on both sides of equality (9), we obtain

$$\begin{cases} s_{22}a_{j0} - s_{11}b_{j0} = 0, \\ s_{22}a_{j1} - s_{11}b_{j1} = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ s_{22}a_{j,\, l_j-m_j-1} - s_{11}b_{j,\, l_j-m_j-1} = 0, \end{cases} \tag{11}$$

where $l_j = \min(2m_j,\ n_j)$, $j = 1, \dots, q$, $s_{11} \neq 0$, $s_{22} \neq 0$. From equalities (11) it follows that $a_{j0} = kb_{j0}$, $a_{j1} = kb_{j1}, \dots, a_{j,\, l_j-m_j-1} = kb_{j,\, l_j-m_j-1}$, where $k = s_{11}s_{22}^{-1}$. This completes the proof of the theorem. □

Corollary 1. *Matrix (1) in the class $\{CA(x)Q(x)\}$ of semiscalarly equivalent matrices is determined up to a constant factor if multiplicities n_j and m_j in polynomials $\Delta(x)$ and $a(x)$ of every its characteristic root α_i, $i = 1, \dots, p$, satisfy the inequality $2m_i \geq n_i$.*

Proof. Let matrices (1) and (3) be semiscalarly equivalent. By Theorem 1 we have $a(\alpha_i) = b(\alpha_i) = 0$, $a^{(s_i)}(\alpha_i) = b^{(s_i)}(\alpha_i) = 0$, $s_i = 1, \ldots, m_i - 1$, $i = 1, \ldots, p$. From theorem we have also $a^{(h_i)}(\alpha_i) = kb^{(h_i)}(\alpha_i)$, $h_i = m_i, \ldots, n_i - 1$. Then the values of polynomial $a(x)$ and values of its derivative at α_i, $i = 1, \ldots, p$, of order $1, \ldots, n_i - 1$ are proportional to corresponding values of polynomial $b(x)$ and to corresponding values of the derivative of this polynomial. Since $\deg a(x)$, $\deg b(x) < \sum n_i = s$, then polynomials $a(x)$ and $b(x)$ differ from each other by a constant factor. Corollary is proved. □

Consider now the case when the conditions of the corollary are not satisfied, i.e., for some root α_i the inequality $2m_i < n_i$ is fulfilled.

Theorem 2. *Let n_j be the multiplicity of the root α_j in the characteristic polynomial $\Delta(x)$, $\deg \Delta(x) = s$, of the matrices (1) and (3). Besides, let $w = 1$ in the partition (2) of set M of theirs characteristic roots and*

$$a(x) = \sum_{t=0}^{s-m_j-1} a_{jt}(x-\alpha_j)^{m_j+t}, \quad b(x) = \sum_{t=0}^{s-m_j-1} b_{jt}(x-\alpha_j)^{m_j+t}$$

be binomial decompositions of the entries $a(x)$, $b(x)$ of these matrices. Matrices (1) and (3) are semiscalarly equivalent if and only if for every characteristic root α_j such that $m_j < n_j$ and for every pair of characteristic roots α_i, α_l such that $2m_i < n_i$, $2m_l < n_l$, there exists the same number $k \neq 0$, the following conditions hold:

1) $\left\| \begin{matrix} a_{j0} & a_{j1} & \ldots & a_{j,\, l_j-m_j-1} \end{matrix} \right\| = k \left\| \begin{matrix} b_{j0} & b_{j1} & \ldots & b_{j,\, l_j-m_j-1} \end{matrix} \right\|$, $l_j = \min(2m_j,\, n_j)$;

2)

$$\begin{vmatrix} a_{j1} & a_{j2} & \ldots & a_{j,\, s_j-1} & a_{js_j} \\ a_{j0} & a_{j1} & \ddots & a_{j,\, s_j-2} & a_{j,\, s_j-1} \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & a_{j1} & a_{j2} \\ 0 & & & a_{j0} & a_{j1} \end{vmatrix} = k^{s_j} \begin{vmatrix} b_{j1} & b_{j2} & \ldots & b_{j,\, s_j-1} & b_{js_j} \\ b_{j0} & b_{j1} & \ddots & b_{j,\, s_j-2} & b_{j,\, s_j-1} \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & b_{j1} & b_{j2} \\ 0 & & & b_{j0} & b_{j1} \end{vmatrix}, \quad (12)$$

$s_j = 1, \ldots, m_j - 1, m_j + 1, \ldots, n_j - m_j - 1$;

3)

$$a_{im_i} a_{i0}^{-2} - a_{lm_l} a_{l0}^{-2} = k^{-1}(b_{im_i} b_{i0}^{-2} - b_{lm_l} b_{l0}^{-2}). \quad (13)$$

Proof. Necessity. Let matrices (1) and (3) be semiscalarly equivalent. The condition 1) follows from Theorem 1. If for characteristic root α_j such that $m_j < n_j$ satisfies the inequality $2m_j \geq n_j$, then the condition 2) follows from the condition 1). In the opposite case such that $2m_j < n_j$ from the equality (9) we obtain the systems

$$\begin{cases} s_{22}a_{j0} - s_{11}b_{j0} = 0, \\ s_{22}a_{j1} - s_{11}b_{j1} = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ s_{22}a_{j,\, m_j-1} - s_{11}b_{j,\, m_j-1} = 0, \end{cases} \quad (14)$$

$$\left\{\begin{array}{c} s_{22}a_{jm_j} - s_{11}b_{jm_j} - s_{12}a_{j0}b_{j0} = 0, \\ s_{22}a_{j,\,m_j+1} - s_{11}b_{j,\,m_j+1} - s_{12}(a_{j0}b_{j1} + a_{j1}b_{j0}) = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ s_{22}a_{j,\,n_j-m_j-1} - s_{11}b_{j,\,n_j-m_j-1} - s_{12}\sum_{u=0}^{n_j-2m_j-1} a_{ju}b_{j,\,n_j-2m_j-u-1} = 0. \end{array}\right. \tag{15}$$

Since $s_{11},\ s_{22} \neq 0$, from (14) we can write

$$a_{j0} = kb_{j0}, a_{j1} = kb_{j1}, \dots, a_{j,\,m_j-1} = kb_{j,\,m_j-1}, k = s_{11}s_{22}^{-1}. \tag{16}$$

From this is follows that equality (12) is satisfied for $s_j = 1,\ \dots,\ m_j - 1$. As appears from (15), if $a_{jm_j} = kb_{jm_j}$, that $s_{12} = 0$ and $a_{js_j} = kb_{js_j}$ for $s_i = m_i + 1,\ \dots,\ n_i - m_i - 1$. From this it follows that equality (12) is valid for the same $s_i = m_i + 1,\ \dots,\ n_i - m_i - 1$. For this reason we think in what follows $a_{jm_j} \neq kb_{jm_j}$, $k = s_{11}s_{22}^{-1}$. From the first and second equations (15) by excluding s_{12} we obtain

$$a_{j0}a_{j,\,m_j+1} - a_{jm_j}(kb_{j1} + a_{j1}) = k^2 b_{j0}b_{j,\,m_j+1} - kb_{jm_j}(kb_{j1} + a_{j1}). \tag{17}$$

If $m_j = 1$, then $a_{j0}a_{j2} - a_{j1}^2 = k^2(b_{j0}b_{j2} - b_{j1}^2)$. This means that conditions (12) are fulfilled for $s_j = m_j + 1$. If $m_j > 1$, then $a_{j1} = kb_{j1}$ and from (17) by multiplication $a_{j0}^{m_j-1} = k^{m_j-1}b_{j0}^{m_j-1}$ can be obtained

$$a_{j0}^{m_j}a_{j,\,m_j+1} - 2a_{j0}^{m_j-1}a_{j1}a_{jm_j} = k^{m_j+1}(b_{j0}^{m_j}b_{j,\,m_j+1} - 2b_{j0}^{m_j-1}b_{j1}b_{jm_j}). \tag{18}$$

Denote by A_{juv}, B_{juv} submatrices obtained, respectively, from matrices

$$\left\| \begin{array}{ccccc} a_{j1} & a_{j2} & \dots & a_{jm_j} & a_{j,\,m_j+1} \\ a_{j0} & a_{j1} & \ddots & a_{j,\,m_j-1} & a_{jm_j} \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & a_{j1} & a_{j2} \\ 0 & & & a_{j0} & a_{j1} \end{array} \right\|, \quad \left\| \begin{array}{ccccc} b_{j1} & b_{j2} & \dots & b_{jm_j} & b_{j,\,m_j+1} \\ b_{j0} & b_{j1} & \ddots & b_{j,\,m_j-1} & b_{jm_j} \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & b_{j1} & b_{j2} \\ 0 & & & b_{j0} & b_{j1} \end{array} \right\|, \tag{19}$$

by obliterating of two last columns and u-th and v-th rows. Denote also by $\delta_{j,\,m_j+1}(A)$, $\delta_{j,\,m_j+1}(B)$ the determinants of matrices (19) respectively. Decompose them for minors of order two that are contained in the last two columns. Because $|A_{juv}| = |B_{juv}| = 0$ for $u \neq m_j + 1$, we have

$$\delta_{j,\,m_j+1}(A) = (-1)^{m_j+1}\left(\begin{vmatrix} a_{jm_j} & a_{j,\,m_j+1} \\ a_{j0} & a_{j1} \end{vmatrix} \left|A_{j,\,1,\,m_j+1}\right| - \right.$$

$$\left. - \begin{vmatrix} a_{j,\,m_j-1} & a_{jm_j} \\ a_{j0} & a_{j1} \end{vmatrix} \left|A_{j,\,2,\,m_j+1}\right| + \ \dots + \begin{vmatrix} a_{j1} & a_{j2} \\ a_{j0} & a_{j1} \end{vmatrix} \left|A_{j,\,m_j,\,m_j+1}\right| \right),$$

$$\delta_{j,\,m_j+1}(B)=(-1)^{m_j+1}\left(\begin{vmatrix} b_{jm_j} & b_{j,\,m_j+1} \\ b_{j0} & b_{j1}\end{vmatrix}\left|B_{j,\,1,\,m_j+1}\right|-\right.$$

$$\left.-\begin{vmatrix} b_{j,\,m_j-1} & b_{jm_j} \\ b_{j0} & b_{j1}\end{vmatrix}\left|B_{j,\,2,\,m_j+1}\right|+\;\ldots+\begin{vmatrix} b_{j1} & b_{j2} \\ b_{j0} & b_{j1}\end{vmatrix}\left|B_{j,\,m_j,\,m_j+1}\right|\right).$$

Since the rows $\|\ a_{j0}\quad a_{j1}\quad \ldots\quad a_{j,\,m_j-1}\ \|$, $\|\ b_{j0}\quad b_{j1}\quad \ldots\quad b_{j,\,m_j-1}\ \|$ differ by a multiplier k (see (16)), each summand of expression in parenthesis for $\delta_{j,\,m_j+1}(A)$, except first two, differs from the corresponding summand for $\delta_{j,\,m_j+1}(B)$ by a multiplier k^{m_j+1}. From this fact and from the equality (18) follows equality (12) for $s_j=m_j+1$.

Denote by $\delta_{js_j}(A)$ and $\delta_{js_j}(B)$ the determinants in left and right parts of equality (12), respectively. Suppose by induction $\delta_{jr}(A)=k^r\delta_{jr}(B)$ for all r such that $m_j<r<n_j-m_j-1$. Accept for the sake of determinacy $r>2m_j$. In the case where $r\leq 2m_j$ the proof radically is not different. From first r-th equality (15) exclude s_{12} and by sufficiently evident transformations we obtain

$$\left\{\begin{array}{c}
(a_{j,\,m_j+1}-(a_{j0}b_{j0})^{-1}a_{jm_j}\sum_{u=0}^{1}a_{ju}b_{j,\,1-u})(-a_{j0})^{m_j}\delta_{j,\,r-m_j}(A)= \\
=k^{r+1}(b_{j,\,m_j+1}-(a_{j0}b_{j0})^{-1}b_{jm_j}\sum_{u=0}^{1}a_{ju}b_{j,\,1-u})(-b_{j0})^{m_j}\delta_{j,\,r-m_j}(B), \\
(a_{j,\,m_j+2}-(a_{j0}b_{j0})^{-1}a_{jm_j}\sum_{u=0}^{2}a_{ju}b_{j,\,2-u})(-a_{j0})^{m_j+1}\delta_{j,\,r-m_j-1}(A)= \\
k^{r+1}(b_{j,\,m_j+2}-(a_{j0}b_{j0})^{-1}b_{jm_j}\sum_{u=0}^{2}a_{ju}b_{j,\,2-u})(-b_{j0})^{m_j+1}\delta_{j,\,r-m_j-1}(B), \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
(a_{j,\,r-m_j+1}-(a_{j0}b_{j0})^{-1}a_{jm_j}\sum_{u=0}^{r-2m_j+1}a_{ju}b_{j,\,r-2m_j-u+1}- \\
-a_{j,\,r-m_j+1}+(a_{j0}b_{j0})^{-1}a_{jm_j}\sum_{u=0}^{r-2m_j+1}a_{ju}b_{j,\,r-2m_j-u+1})(-a_{j0})^{r-m_j}\delta_{jm_j}(A)= \\
k^{r+1}(b_{j,\,r-m_j+1}-(a_{j0}b_{j0})^{-1}b_{jm_j}\sum_{u=0}^{r-2m_j+1}a_{ju}b_{j,\,r-2m_j-u+1}- \\
-b_{j,\,r-m_j+1}+(a_{j0}b_{j0})^{-1}b_{jm_j}\sum_{u=0}^{r-2m_j+1}a_{ju}b_{j,\,r-2m_j-u+1})(-b_{j0})^{r-m_j}\delta_{jm_j}(B), \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
(a_{jr}-(a_{j0}b_{j0})^{-1}a_{jm_j}\sum_{u=0}^{r-m_j}a_{ju}b_{j,\,r-m_j-u})(-a_{j0})^{r-1}\delta_1(A)= \\
k^{r+1}(b_{jr}-(a_{j0}b_{j0})^{-1}b_{jm_j}\sum_{u=0}^{r-m_j}a_{ju}b_{j,\,r-m_j-u})(-b_{j0})^{r-1}\delta_1(B), \\
(a_{j,\,r+1}-(a_{j0}b_{j0})^{-1}a_{jm_j}\sum_{u=0}^{r-m_j+1}a_{ju}b_{j,\,r-m_j-u+1})(-a_{j0})^{r}= \\
k^{r+1}(b_{j,\,r+1}-(a_{j0}b_{j0})^{-1}b_{jm_j}\sum_{u=0}^{r-m_j+1}a_{ju}b_{j,\,r-m_j-u+1})(-b_{j0})^{r}.
\end{array}\right. \tag{20}$$

If we add left parts of equality (20) and separately right parts we obtain

$$(-a_{j0})^{r}a_{j,\,r+1}+(-a_{j0})^{r-1}a_{jr}\delta_{j1}(A)+\;\ldots\;+(-a_{j0})^{r-m_j}a_{j,\,r-m_j+1}\delta_{jm_j}(A)+\;\ldots\;+$$
$$+(-a_{j0})^{m_j}a_{j,\,m_j+1}\delta_{j,\,r-m_j}(A)+(-a_{j0})^{m_j-1}a_{jm_j}\delta_{j,\,r-m_j+1}(A)-$$
$$-(a_{j0}b_{j0})^{-1}a_{jm_j}(b_{j1}\delta_{j,\,r-m_j}(A)(-a_{j0})^{m_j+1}+b_{j2}\delta_{j,\,r-m_j-1}(A)(-a_{j0})^{m_j+2}+\;\ldots\;+$$
$$+b_{j,\,r-m_j}\delta_{j1}(A)(-a_{j0})^{r})+(a_{j0}b_{j0})^{-1}a_{jm_j}(b_{j1}\delta_{j,\,r-m_j}(A)(-a_{j0})^{m_j+1}+$$
$$+b_{j2}\delta_{j,\,r-m_j-1}(A)(-a_{j0})^{m_j+2}+\;\ldots\;+b_{j,\,r-m_j}\delta_{j1}(A)(-a_{j0})^{r}+b_{j,\,r-m_j+1}(-a_{j0})^{r+1})-$$

$$-(a_{j,\,r-m_j+1}-(a_{j0}b_{j0})^{-1}a_{jm_j}\sum_{u=0}^{r-2m_j+1}a_{ju}b_{j,\,r-2m_j-u+1})(-a_{j0})^{r-m_j}\delta_{jm_j}(A)=$$
$$k^{r+1}((-b_{j0})^{r}b_{j,\,r+1}+(-b_{j0})^{r-1}b_{jr}\delta_{j1}(B)+\;\dots\;+(-b_{j0})^{r-m_j}b_{j,\,r-m_j+1}\delta_{jm_j}(B)+\;\dots\;+$$
$$+(-b_{j0})^{m_j}b_{j,\,m_j+1}\delta_{j,\,r-m_j}(B)+(-b_{j0})^{m_j-1}b_{jm_j}\delta_{j,\,r-m_j+1}(B)-$$
$$-(a_{j0}b_{j0})^{-1}b_{jm_j}(a_{j1}\delta_{j,\,r-m_j}(B)(-b_{j0})^{m_j+1}+a_{j2}\delta_{j,\,r-m_j-1}(B)(-b_{j0})^{m_j+2}+\;\dots\;+$$
$$+a_{j,\,r-m_j}\delta_{j1}(B)(-b_{j0})^{r})+(a_{j0}b_{j0})^{-1}b_{jm_j}(a_{j1}\delta_{j,\,r-m_j}(B)(-b_{j0})^{m_j+1}+$$
$$+a_{j2}\delta_{j,\,r-m_j-1}(B)(-b_{j0})^{m_j+2}+\;\dots\;+a_{j,\,r-m_j}\delta_{j1}(B)(-b_{j0})^{r}+a_{j,\,r-m_j+1}(-b_{j0})^{r+1})-$$
$$-(b_{j,\,r-m_j+1}-(a_{j0}b_{j0})^{-1}b_{jm_j}\sum_{u=0}^{r-2m_j+1}a_{ju}b_{j,\,r-2m_j-u+1})(-b_{j0})^{r-m_j}\delta_{jm_j}(B))\,. \quad (21)$$

Group similar terms in both parts of obtained equality to have

$$(-a_{j0})^{r}a_{j,\,r+1}+(-a_{j0})^{r-1}a_{jr}\delta_{j1}(A)+\;\dots\;+(-a_{j0})^{r-m_j}a_{j,\,r-m_j+1}\delta_{jm_j}(A)+$$
$$+\;\dots\;+(-a_{j0})^{m_j}a_{j,\,m_j+1}\delta_{j,\,r-m_j}(A)+(-a_{j0})^{m_j-1}a_{j,\,m_j}\delta_{j,\,r-m_j+1}(A)+$$
$$+(a_{j0}b_{j0})^{-1}a_{jm_j}b_{j,\,r-m_j+1}(-a_{j0})^{r+1}-(a_{j,\,r-m_j+1}-$$
$$-(a_{j0}b_{j0})^{-1}a_{jm_j}\sum_{u=1}^{r-2m_j+1}a_{ju}b_{j,\,r-2m_j-u+1})(-a_{j0})^{r-m_j}\delta_{jm_j}(A)=$$
$$k^{r+1}((-b_{j0})^{r}b_{j,\,r+1}+(-b_{j0})^{r-1}b_{jr}\delta_{j1}(B)+\;\dots\;+(-b_{j0})^{r-m_j}b_{j,\,r-m_j+1}\delta_{jm_j}(B)+$$
$$+\;\dots\;+(-b_{j0})^{m_j}b_{j,\,m_j+1}\delta_{j,\,r-m_j}(B)+(-b_{j0})^{m_j-1}b_{j,\,m_j}\delta_{j,\,r-m_j+1}(B)+$$
$$+(a_{j0}b_{j0})^{-1}b_{jm_j}a_{j,\,r-m_j+1}(-b_{j0})^{r+1}-(b_{j,\,r-m_j+1}-$$
$$-(a_{j0}b_{j0})^{-1}b_{jm_j}\sum_{u=1}^{r-2m_j+1}a_{ju}b_{j,\,r-2m_j-u+1})(-b_{j0})^{r-m_j}\delta_{jm_j}(B). \quad (22)$$

It follows from (15) that

$$a_{j,\,r-m_j+1}+(a_{j0}b_{j0})^{-1}a_{jm_j}\sum_{u=0}^{r-2m_j+1}a_{ju}b_{j,\,r-2m_j-u+1}=$$
$$=k(b_{j,\,r-m_j+1}+(a_{j0}b_{j0})^{-1}b_{jm_j}\sum_{u=0}^{r-2m_j+1}a_{ju}b_{j,\,r-2m_j-u+1}).$$

From this relation it is easy to be sure that the following equality is true

$$(a_{j0}b_{j0})^{-1}a_{jm_j}b_{j,\,r-m_j+1}(-a_{j0})^{r+1}-(a_{j,\,r-m_j+1}-$$
$$-(a_{j0}b_{j0})^{-1}a_{jm_j}\sum_{u=0}^{r-2m_j+1}a_{ju}b_{j,\,r-2m_j-u+1})(-a_{j0})^{r-m_j}\delta_{jm_j}(A)=$$
$$=k^{r+1}((a_{j0}b_{j0})^{-1}b_{jm_j}a_{j,\,r-m_j+1}(-b_{j0})^{r+1}-(b_{j,\,r-m_j+1}-$$

$$-(a_{j0}b_{j0})^{-1}b_{jm_j}\sum_{u=0}^{r-2m_j+1}a_{ju}b_{j,\,r-2m_j-u+1})(-b_{j0})^{r-m_j}\delta_{jm_j}(B). \tag{23}$$

From (16) and induction hypothesis we can write

$$(-a_{j0})^{m_j-2}a_{j,\,m_j-1}\delta_{j,\,r-m_j+2}(A)+\ \dots\ +(-a_{j0})a_{j2}\delta_{j,\,r-1}(A)+a_{j1}\delta_{jr}(A)=$$
$$=k^{r+1}((-b_{j0})^{m_j-2}b_{j,\,m_j-1}\delta_{j,\,r-m_j+2}(B)+\ \dots\ +(-b_{j0})b_{j2}\delta_{j,\,r-1}(B)+b_{j1}\delta_{jr}(B). \tag{24}$$

Comparing (22), (23) and (24), we obtain equality

$$(-a_{j0})^{r}a_{j,\,r+1}+(-a_{j0})^{r-1}a_{jr}\delta_{j1}(A)+\ \dots\ +(-a_{j0})a_{j2}\delta_{j,\,r-1}(A)+a_{j1}\delta_{jr}(A)=$$
$$=k^{r+1}((-b_{j0})^{r}b_{j,\,r+1}+(-b_{j0})^{r-1}b_{jr}\delta_{j1}(B)+\ \dots\ +(-b_{j0})b_{j2}\delta_{j,\,r-1}(B)+b_{j1}\delta_{jr}(B)), \tag{25}$$

i.e., $\delta_{j,\,r+1}(A)=k^{r+1}\delta_{j,\,r+1}(B)$, $k=s_{11}s_{22}^{-1}$. The necessity of conditions 2) of the theorem is proved.

Let

$$a(x)=\sum_{t=0}^{s-m_i-1}a_{it}(x-\alpha_i)^{m_i+t},\quad a(x)=\sum_{t=0}^{s-m_l-1}a_{lt}(x-\alpha_l)^{m_l+t},$$
$$b(x)=\sum_{t=0}^{s-m_i-1}b_{it}(x-\alpha_i)^{m_i+t},\quad b(x)=\sum_{t=0}^{s-m_l-1}b_{lt}(x-\alpha_l)^{m_l+t}$$

$$s_{22}a_{im_i}-s_{11}b_{im_i}-s_{12}a_{i0}b_{i0}=0,$$

be decompositions for entries $a(x)$, $b(x)$ of matrices (1), (3) into degrees of binomials $x-\alpha_i$, $x-\alpha_l$. From (9) it may be written

$$s_{22}a_{im_i}-s_{11}b_{im_i}-s_{12}a_{i0}b_{i0}=0,$$
$$s_{22}a_{lm_l}-s_{11}b_{lm_l}-s_{12}a_{l0}b_{l0}=0.$$

From these equalities exclude s_{12}. Considering that $a_{i0}=kb_{i0}$, $a_{l0}=kb_{l0}$, we have (13). The necessity of the conditions 1) – 3) of theorem is proved.

Sufficiency. For each characteristic root $x=\alpha_j$ of matrix (1) such that $m_j<n_j$ and $2m_j\geq n_j$, from condition 1) of theorem it follows that

$$s_{22}a(x)-s_{11}b(x)-s_{12}a(x)b(x)\equiv 0\ (mod\ (x-\alpha_j)^{n_j}), \tag{26}$$

where $s_{22}=1$, $s_{11}=k=a_{j0}b_{j0}^{-1}$, $s_{12}\in C$.

Let now $x=\alpha_j$ be an arbitrary characteristic root of matrices (1), (3) such that $2m_j<n_j$. Consider equalities (14) and (15) as one system of equations with coefficients a_{ju}, b_{ju}, $u=0,\ 1,\ \dots,\ n_j-m_j-1$, $a_{j0}\neq 0$, $b_{j0}\neq 0$, in three unknowns

$s_{22},\ s_{11},\ s_{12}$. We shall show that conditions of theorem imply that there is nonzero solution of this system such that $s_{22}=1,\ s_{11}=k=a_{j0}b_{j0}^{-1}$ the same for every characteristic root α_j of matrices (1), (3) such that $2m_j<n_j$. We shall prove this fact by induction. The condition 1) implies that system (14) has nonzero solution such that it does not dependent on the choice of the characteristic root α_j. After annihilation of equal summands on the both sides of equality (12) for $s_j=m_j+1$ and after division by $a_{j0}^{m_j}=k^{m_j}b_{j0}^{m_j}$ with the help of simple transformations we can obtain the following relation

$$a_{j,\,m_j+1}-kb_{j,\,m_j+1}-(a_{j0}b_{j0})^{-1}(a_{jm_j}-kb_{jm_j})(a_{j0}b_{j1}+a_{j1}b_{j0})=0.$$

This means that

$$s_{22}=1,\quad s_{11}=k,\quad s_{12}=(a_{j0}b_{j0})^{-1}(a_{jm_j}-kb_{jm_j}). \tag{27}$$

is a common solution of first two equations of system (15). From (13) it follows that $(a_{i0}b_{i0})^{-1}(a_{im_i}-kb_{im_i})=(a_{l0}b_{l0})^{-1}(a_{lm_l}-kb_{lm_l})$. This result suggests that this solution (27) of first two equations of system (15) does not depend on the choice of the root α_j such that $2m_j<n_j$.

Assume by induction that (27) satisfies first $r-m_j+1$ equations of system (15), i.e.,

$$\begin{cases} a_{jm_j}-kb_{jm_j}-(a_{j0}b_{j0})^{-1}(a_{jm_j}-kb_{jm_j})a_{j0}b_{j0}=0,\\ a_{j,\,m_j+1}-kb_{j,\,m_j+1}-(a_{j0}b_{j0})^{-1}(a_{jm_j}-kb_{jm_j})\sum_{u=0}^{1}a_{ju}b_{j,\,1-u}=0,\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ a_{jr}-kb_{jr}-(a_{j0}b_{j0})^{-1}(a_{jm_j}-kb_{jm_j})\sum_{u=0}^{r-m_j}a_{ju}b_{j,\,r-m_j-u}=0. \end{cases} \tag{28}$$

In so doing, we may think for the sake of determinacy $r>2m_j$. In opposite case proof is completely analogous. Taking into account the conditions 1), 2) and inductive assumption we can write equalities (23), (24) and (25). From these equalities we obtain equality (22). This relation implies the equality (21). It is evident that from the second and all following equalities of (28) we find that first $r-m_j$ equalities of (20) are valid. The first $r-m_j$ equalities of (20) along with relation (21) yield the last equality of (20). This equality after shortening in $(-a_{j0})^r=k^r(-b_{j0})^r$ and after some simplifications can be written in the form

$$a_{j,\,r+1}-kb_{j,\,r+1}-(a_{j0}b_{j0})^{-1}(a_{jm_j}-kb_{jm_j})\sum_{u=0}^{r-m_j+1}a_{ju}b_{j,\,r-m_j-u+1}=0.$$

This means that (27) is the solution of $(r-m_j+1)$-th equation of system (15). This solution does not dependent on the choice of the root α_j.

Thus, congruence (26) holds true for each characteristic root α_j of matrices (1), (3) and for the same set of numbers (27), where $s_{22}\neq 0,\ s_{11}\neq 0$. It enables us to write the congruence

$$s_{22}a(x) - s_{11}b(x) - s_{12}a(x)b(x) \equiv 0 \ (mod\ \Delta(x)). \tag{29}$$

We introduce the following notation:

$$r_{11}(x) = s_{11} - s_{12}b(x), r_{12}(x) = s_{12}\Delta(x),$$

$$r_{22}(x) = s_{22} - s_{12}b(x), r_{21}(x) = \frac{s_{22}a(x) - s_{11}b(x) - s_{12}a(x)b(x)}{\Delta(x)}.$$

It is clear that $r_{21}(x) \in C$. With this notations check that equality (4) is true. From this it follows that matrices (1) and (3) are semiscalarly equivalent. The theorem is proved. □

Theorem 3. *In the partition (2) for matrix $A(x)$ of the form (1) let us have $w = 1$; n_i and m_i be the multiplicities of some root $\alpha_i \in M$ in the characteristic polynomial $\Delta(x)$ and in polynomial $a(x)$ of matrix, $A(x)$ respectively, moreover $2m_i < n_i$. Then in the class of semiscalarly equivalent matrices $\{CA(x)Q(x)\}$ there exists a matrix $B(x)$ of the form (3), where entry $b(x)$ satisfies the following conditions: $b(\alpha_i) = 0$, $b^{(m_i)}(\alpha_i) = m_i!$, $b^{(2m_i)}(\alpha_i) = 0$. For a fixed root α_i the matrix $B(x)$ is defined uniquely.*

Proof. **Existence.** We may take, that already the entry $a(x)$ of the matrix $A(x)$ satisfies the condition $a^{(m_i)}(\alpha_i) = m_i!$. In the opposite case, for this purpose we divide the first column of matrix $A(x)$ and multiply its first row by $\frac{a^{(m_i)}(\alpha_i)}{m_i!}$. Let α_j denote an arbitrary characteristic root of matrix $A(x)$ of multiplicity n_j such that in the decomposition

$$a(x) = \sum_{t=0}^{s-m_j-1} a_{jt}(x-\alpha_j)^{m_j+t}, \tag{30}$$

where $s = \deg \Delta(x)$, the index m_j is less than n_j. We set

$$\left\| \, b_{j0} \quad b_{j1} \quad \ldots \quad b_{j,\, l_j-m_j-1} \, \right\| = \left\| \, a_{j0} \quad a_{j1} \quad \ldots \quad a_{j,\, l_j-m_j-1} \, \right\|,$$

where $l_j = \min(2m_j,\ n_j)$. Let $\alpha_l \in M$, $\alpha_l \neq \alpha_i$, be an arbitrary characteristic root such that $2m_l < n_l$. We write the formal equality $b_{im_i}b_{i0}^{-2} - b_{lm_l}b_{l0}^{-2} = a_{im_i}a_{i0}^{-2} - a_{lm_l}a_{l0}^{-2}$, where $a_{l0},\ a_{i0},\ a_{im_i},\ a_{lm_l}$ are coefficients of the decomposition (30) for $j = i$ and $j = l$. Setting $b_{i0} = a_{i0}$, $b_{l0} = a_{l0}$ and $b_{im_i} = 0$ in this relation, we calculate b_{lm_l}. Using this value b_{lm_l} and determined above $b_{l0} = a_{l0}$, $b_{l1} = a_{l1}$, …, $b_{l,\, m_l-1} = a_{l,\, m_l-1}$, from formal equalities

$$\begin{vmatrix} b_{l1} & b_{l2} & \dots & b_{l,\,s_l-1} & b_{ls_l} \\ b_{l0} & b_{l1} & \ddots & b_{l,\,s_l-2} & b_{l,\,s_l-1} \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & b_{l1} & b_{l2} \\ 0 & & & b_{l0} & b_{l1} \end{vmatrix} = \begin{vmatrix} a_{l1} & a_{l2} & \dots & a_{l,\,s_l-1} & a_{ls_l} \\ a_{l0} & a_{l1} & \ddots & a_{l,\,s_l-2} & a_{l,\,s_l-1} \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & a_{l1} & a_{l2} \\ 0 & & & a_{l0} & a_{l1} \end{vmatrix}, \qquad (31)$$

$s_l = m_l + 1, \dots, n_l - m_l - 1$, we find recurrently $b_{l,\,m_l+1}, \dots, b_{l,\,n_l-m_l-1}$. Setting $l = i$, $b_{im_i} = 0$ and using determined above $b_{i0} = a_{i0}$, $b_{i1} = a_{i1}, \dots, b_{i,\,m_i-1} = a_{i,\,m_i-1}$, similarly from (31) we find recurrently $b_{i,\,m_i+1}, \dots, b_{i,\,n_i-m_i-1}$. Thus, for every root $\alpha_j \in M$ such that in the decomposition (30) $m_j < n_j$, some numbers $b_{i0}, b_{i1}, \dots, b_{j,\,n_j-m_j-1} \in C$ are defined. We construct the matrix $B(x)$ of the form (3) whose entry $b(x)$, where $\deg b(x) < s$, satisfies such conditions: $b(\alpha_j) = 0$, $b^{(1)}(\alpha_j) = 0, \dots, b^{(m_j-1)}(\alpha_j) = 0$, $b^{(m_j)}(\alpha_j) = m_j!b_{j0}, \dots, b^{(n_j-1)}(\alpha_j) = (n_j - 1)!b_{j,\,n_j-m_j-1}$, and $b(\alpha) = 0$, $b^{(1)}(\alpha) = 0, \dots, b^{(n-1)}(\alpha_j) = 0$ for each root $\alpha \in M$ of multiplicity n which is different from α_j. Since matrix (1) and constructed matrix of the form (3) satisfy the conditions of Theorem 2, they are semiscalarly equivalent. The first part of theorem is proved.

The **uniqueness** of the matrix $B(x)$ of the form (3) whose entry $b(x)$ satisfies the conditions described in theorem follows from the uniqueness of construction of the polynomial $b(x)$, $\deg b(x) < s = \deg \Delta(x)$, by known its values and values of its derivatives of respective orders at roots of the polynomial $\Delta(x)$. The theorem is completely proved.

□

Definition 1. The matrix $B(x)$ of the form (3) whose existence and uniqueness in the class $\{CA(x)Q(x)\}$ are established in theorem 3 is called α_i-*canonical*. The matrix $A(x)$ of the form (1) is called also α_i-*canonical* if for each root $\alpha_j \in M$ of multiplicity n_j in the decomposition (30) of its entry $a(x)$ index m_j satisfies the condition $2m_j \geq n_j$ and for some root $\alpha_i \in M$ we have $m_i < n_i$, $a^{(m_i)}(\alpha_i) = (m_i)!$.

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B. Z. Shavarovskii
Institute for Applied Problems of Mechanics
and Mathematics, NAS Ukraine, L'vov
E-mail: *bshavarovskii@gmail.com*

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