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General form transversals in groups

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Abstract. The classical notion of transversal in group to its subgroup is generalised. It is made with the help of reducing any conditions on the choice of representatives of the left (right) cosets in group to its subgroup. Obtained general form transversals are investigated and some its properties are studied.

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1 Introduction

In the theory of quasigroups and loops the following notion of left (right) transversal in group to its subgroup is well-known [1–4].

Definition 1. Let G be a group and H be its subgroup. Let $\{H_i\}_{i\in E}$ be the set of all left (right) cosets in G to H (E is a set of indexes with distinguished element 1), and we assume $H_1 = H$. A set $T = \{t_i\}_{i\in E}$ of representativities of the left (right) cosets (by one from each coset H_i and $t_1 = e \in H$) is called a **left (right) transversal** in G to H.

As is easy to see, in this definition the choice of representatives of left (right) cosets in G to H is not free – there exist two conditions: $H_1 = H$ and $t_1 = e \in H$. Let us reduce these two conditions and investigate obtained below general form transversals in group to its subgroup.

2 General form transversals in group to its subgroup

2.1 Definitions and elementary properties

Let G be a group and H be its subgroup. Below we shall use the following notations:

E is an index set (E contains a distinguished element 1);

left (right) cosets in the group G to its subgroup H are numbered by the indexes from E;

$\{H_i\}_{i\in E}$ is the set of all left (right) cosets in G to H;

e is the unit of group G;

Below all definitions and propositions will be formulated for the left cosets in G to H; for the right cosets in G to H it may be done analogously.

Definition 2. Let G be a group and H be its subgroup. Let $\{H_i\}_{i\in E}$ be the set of all left cosets in G to H. A set $T = \{t_i\}_{i\in E}$ of representativities of the left (right) cosets (by one from each coset H_i, i.e. $t_i \in H_i$) is called a **left general form transversal** in G to H (see also [6, 7]).

Remark 1. Generally speaking the numbering of left cosets $\{H_i\}_{i\in E}$ in G to H may be such that the subgroup H obtain an index $a \in E$ which is different from 1, i.e. $H = H_a \neq H_1$.

Remark 2. Generally speaking the unit e of the group G (and subgroup H) may not belong to the left general form transversal T in G to H, i.e. $e \notin T$.

Definition 3. If for left general form transversal $T = \{t_i\}_{i\in E}$ in G to H the following condition holds: $t_{i_0} = e$ for some $i_0 \in E$, then such transversal T is called a **left reduced transversal** in G to H. In opposite case T is called a **left non-reduced transversal** in G to H.

Definition 4. If for left general form transversal $T = \{t_i\}_{i\in E}$ in G to H the following condition holds: $H = H_1$ (i.e. the index of the subgroup H in the set of left cosets in G to H is equal to 1), then such transversal T is called a **left ordered transversal** in G to H. In opposite case T is called a **left non-ordered transversal** in G to H.

Definition 5. A left general form transversal $T = \{t_i\}_{i\in E}$ in G to H which is a left reduced and ordered transversal in G to H is usually called a **left transversal** in G to H.

Example 1. Let us have:

$$\begin{aligned} G &= S_3 = \{id, (12), (13), (23), (123), (132)\}, \\ H &= St_1(S_3) = \{id, (23)\}. \end{aligned}$$

Left cosets in G to H:

$$\begin{aligned} H_{i_1} &= H = \{id, (23)\}, \\ H_{i_2} &= \{(12), (123)\}, \\ H_{i_3} &= \{(13), (132)\}, \\ E &= \{i_1, i_2, i_3\} \equiv \{1, 2, 3\}. \end{aligned}$$

1. $i_1 \neq 1$ and $T = \{(23), (12), (132)\}$. Then T is a left non-reduced non-ordered general form transversal in G to H.
2. $i_1 = 1$ and $T = \{(23), (12), (132)\}$. Then T is a left non-reduced ordered general form transversal in G to H.
3. $i_1 \neq 1$ and $T = \{id, (123), (132)\}$. Then T is a left reduced non-ordered general form transversal in G to H.

4. $i_1 = 1$ and $T = \{id, (12), (13)\}$. Then T is a left (reduced and ordered) transversal in G to H.

Theorem 1. *For an arbitrary left general form transversal $T = \{t_i\}_{i \in E}$ in G to H the following statements are true:*

1. *For every $h \in H$ the set $T_h = Th = \{t_i h\}_{i \in E}$ is a left general form transversal in G to H too.*
2. *There exists an element $h_0 \in H$ such that the set $T_{h_0} = Th_0$ is a left reduced (maybe non-ordered) general form transversal in G to H.*
3. *For every $\pi \in G$ the set ${}_\pi T = \pi T = \{\pi t_i\}_{i \in E}$ is a left general form transversal in G to H too.*
4. *There exists an element $\pi_0 \in G$ such that the set ${}_{\pi_0} T = \pi_0 T = \{\pi_0 t_i\}_{i \in E}$ is a left (reduced and ordered) transversal in G to H.*

Proof. **1**. For every $i \in E$ and $h \in H$ we have

$$t_i \in H_i \quad \Longrightarrow \quad t_i h \in H_i,$$

and so

$$(Th) \cap H_i = \{t_i h\},$$

i.e. Th is a left general form transversal in G to H.

2. Let

$$T \cap H = h^*,$$

i.e. h^* is a representative of general form transversal T in the subgroup H. Then we put

$$h_0 = (h^*)^{-1}.$$

We obtain

$$h^* \in T \quad \Longrightarrow \quad e = h^* \cdot (h^*)^{-1} \in (Th_0),$$

i.e. due to item **1** general form transversal $T_1 = Th_0$ is a left reduced (maybe non-ordered) general form transversal in G to H

3. Let us take an arbitrary element $\pi \in G$ and consider the set

$${}_\pi T = \pi T = \{\pi t_i\}_{i \in E}.$$

Because T is a left general form transversal in G to H then

$$G = \bigcup_{i \in E} (t_i H).$$

So we obtain

$$G = \pi G = \pi \cdot \left(\bigcup_{i \in E} (t_i H) \right) = \bigcup_{i \in E} ((\pi t_i) H),$$

i.e. every element $g \in G$ may be presented in the form $g = t^* h$, where $h \in H$ and $t^* \in \pi T$.

Now let us show that for every $i, j \in E$, $i \neq j$, the following equality is true

$$((\pi t_i)H) \cap ((\pi t_j)H) = \varnothing.$$

Let us assume that it is not true, and so

$$\pi t_i h_1 = \pi t_j h_2 = g_0$$

for some $h_1, h_2 \in H$. Then we obtain

$$t_i h_1 = t_j h_2 \quad \Longrightarrow \quad t_i = t_j h_2 h_1^{-1} \in t_j H \quad \Longrightarrow \quad (t_i H) \cap (t_j H) \neq \varnothing,$$

that is in contradiction to the fact that T is a left general form transversal in G to H.

4. Let us consider the left coset H_1 and take the element

$$\pi^* = t_1 = H_1 \cap T.$$

Then we may take $\pi_0 = (\pi^*)^{-1}$. Really we have

$$e = (\pi^*)^{-1} \cdot \pi^* = \pi_0 t_1 \in \pi_0 T,$$

i.e. with the help of item **3** the left general form transversal $\pi_0 T$ is a left (reduced and ordered) transversal in G to H. □

2.2 A transversal operation

Definition 6. Let $T = \{t_i\}_{i \in E}$ be a left general form transversal in G to H. Define the following operation on the set E:

$$x \overset{(T)}{\cdot} y = z \quad \Leftrightarrow \quad t_x t_y = t_z h, \quad h \in H.$$

Theorem 2. *For an arbitrary left general form transversal $T = \{t_i\}_{i \in E}$ in G to H the following statements are true:*

1. *There exists an element $a_0 \in E$ such that the system $\left\langle E, \overset{(T)}{\cdot}, a_0 \right\rangle$ is a left quasigroup with right unit a_0.*
2. *If a left general form transversal $T = \{t_i\}_{i \in E}$ is a reduced (but non-ordered) transversal in G to H, then there exists an element $a_0 \in E$ such that the system $\left\langle E, \overset{(T)}{\cdot}, a_0 \right\rangle$ is a left loop with unit a_0.*
3. *If a left general form transversal $T = \{t_i\}_{i \in E}$ is an ordered (but non-reduced) transversal in G to H, then the system $\left\langle E, \overset{(T)}{\cdot}, 1 \right\rangle$ is a left quasigroup with right unit 1.*

4. If a left general form transversal $T = \{t_i\}_{i\in E}$ is an ordered and reduced transversal in G to H, then the system $\left\langle E, \overset{(T)}{\cdot}, 1 \right\rangle$ is a left loop with unit 1.

Proof. **1**. For any arbitrary $a, b \in E$ consider the following equivalent equations on the set E:

$$\begin{aligned} a \overset{(T)}{\cdot} x &= b, \\ t_a t_x &= t_b h, \quad h \in H, \\ t_x &= t_a^{-1} t_b h = t_c h^*, \quad h^* \in H, \\ x &= c, \end{aligned}$$

for some $c \in E$; moreover, the element $c = c(a, b)$ is uniquely determined by the elements $a, b \in E$. So the system $\left\langle E, \overset{(T)}{\cdot} \right\rangle$ is a left quasigroup. If a_0 is the index of subgroup H as a left coset in G to H, i.e. $H \equiv H_{a_0}$, then $t_{a_0} = h_0 \in H$ for some element h_0. For every $x \in E$ we have the following equivalent equations on the set E:

$$\begin{aligned} x \overset{(T)}{\cdot} a_0 &= u, \\ t_x t_{a_0} &= t_u h, \quad h \in H, \\ t_x h_0 &= t_u h, \quad h \in H, \\ t_x &= t_u h h_0^{-1} = t_u h^*, \quad h^* \in H, \\ u &= x, \end{aligned}$$

i.e. for every $x \in E$: $x \overset{(T)}{\cdot} a_0 = x$. It means that the system $\left\langle E, \overset{(T)}{\cdot}, a_0 \right\rangle$ is a left quasigroup with right unit a_0.

2. If a left general form transversal $T = \{t_i\}_{i\in E}$ is a reduced (but non-ordered) transversal in G to H, then $t_{a_0} = e \in H$. For every $x \in E$ we have the following equivalent equations on the set E:

$$\begin{aligned} a_0 \overset{(T)}{\cdot} x &= u, \\ t_{a_0} t_x &= t_u h, \quad h \in H, \\ e t_x &= t_u h, \quad h \in H, \\ t_x &= t_u h, \quad h \in H, \\ u &= x, \end{aligned}$$

i.e. for every $x \in E$: $a_0 \overset{(T)}{\cdot} x = x$. It means that the system $\left\langle E, \overset{(T)}{\cdot}, a_0 \right\rangle$ is a left loop with two-sided unit a_0.

3. If a left general form transversal $T = \{t_i\}_{i\in E}$ is an ordered (but non-reduced) transversal in G to H, then the proof is analogous to the proof of the item **1**, but we have $a_0 = 1$ too (because $H_1 \equiv H_{a_0} \equiv H$). So we obtain that the system $\left\langle E, \overset{(T)}{\cdot}, 1 \right\rangle$ is a left loop with unit 1.

4. It is an evident corollary of the items **2** and **3** □

2.3 Permutation representation

Definition 7. Let G be a group and H be its subgroup. A **permutation representation** $\hat{G}$ of the group G by left cosets to its subgroup H is the following map φ:

$$\varphi : G \to S_E,$$

$$\varphi : g \to \hat{g},$$

$$\hat{g}(x) = y \quad \overset{def}{\Leftrightarrow} \quad g \cdot (H_x) = H_y, \qquad x, y \in E.$$

If some left general form transversal $T = \{t_i\}_{i \in E}$ in G to H is chosen, then the last formula may be rewritten in the following form:

$$\hat{g}(x) = y \quad \overset{def}{\Leftrightarrow} \quad g \cdot (t_x \cdot H) = t_y \cdot H.$$

The map φ is a homomorphism from the group G to the symmetric group S_E. The kernel of this homomorphism is called a *core* of G to H:

$$Core_G H = \bigcap_{\pi \in G} (\pi H \pi^{-1}).$$

If $Core_G H = \{e\}$, then the above-mentioned representation is a strict representation and φ is an isomorphism.

It is easy to show that with the help of factorisation on the core it is always possible to take into consideration the strict permutation representation $\hat{G}$ of the group G by left cosets to its subgroup H. So below we assume that the above-mentioned representation is a strict representation.

Theorem 3. *For an arbitrary left general form transversal $T = \{t_i\}_{i \in E}$ in G to H the following statements are true:*

1. *There exists an element $a_0 \in E$ such that for every $h \in H$: $\hat{h}(a_0) = a_0$.*
2. *The following identities are fulfilled:*
   - (a) *For all $x, y \in E$: $\widehat{t_x}(y) = x \overset{(T)}{\cdot} y$;*
   - (b) *For all $x, y \in E$: $\widehat{t_x^{-1}}(y) = x \overset{(T)}{\diagdown} y$, where "$\overset{(T)}{\diagdown}$" is a left division for the operation $\left\langle E, \overset{(T)}{\cdot}, a_0 \right\rangle$ (i.e. $x \overset{(T)}{\diagdown} y = z \Longleftrightarrow x \overset{(T)}{\cdot} z = y$);*
   - (c) *For every $x \in E$: $\widehat{t_x}(a_0) = x$.*

3. If a left general form transversal $T = \{t_i\}_{i\in E}$ is a reduced (but non-ordered) transversal in G to H, then the following identities are fulfilled:

(a) For all $x, y \in E$: $\widehat{t_x}(y) = x \overset{(T)}{\cdot} y$;

(b) For all $x, y \in E$: $\widehat{t_x}^{-1}(y) = x \overset{(T)}{\diagdown} y$;

(c) For every $x \in E$: $\widehat{t_x}(a_0) = \widehat{t_{a_0}}(x) = x$.

4. If a left general form transversal $T = \{t_i\}_{i\in E}$ is an ordered (but non-reduced) transversal in G to H, then the following identities are fulfilled:

(a) For all $x, y \in E$: $\widehat{t_x}(y) = x \overset{(T)}{\cdot} y$;

(b) For all $x, y \in E$: $\widehat{t_x}^{-1}(y) = x \overset{(T)}{\diagdown} y$;

(c) For every $x \in E$: $\widehat{t_x}(1) = x$.

5. If a left general form transversal $T = \{t_i\}_{i\in E}$ is an ordered and reduced transversal in G to H, then the following identities are fulfilled:

(a) For all $x, y \in E$: $\widehat{t_x}(y) = x \overset{(T)}{\cdot} y$;

(b) For all $x, y \in E$: $\widehat{t_x}^{-1}(y) = x \overset{(T)}{\diagdown} y$;

(c) For every $x \in E$: $\widehat{t_x}(1) = \widehat{t_1}(x) = x$.

Proof. **1**. According to item **1** of Theorem 2 there exists an element $a_0 \in E$ such that $H \equiv H_{a_0}$ (i.e. $t_{a_0} = h_0 \in H$). Then for every $h \in H$ we have the following equivalent equalities:

$$\begin{aligned} \hat{h}\,(a_0) &= a_1, \\ ht_{a_0} &= t_{a_1}h^*, \quad h^* \in H, \\ hh_0 &= t_{a_1}h^*, \quad h^* \in H, \\ t_{a_1} &= hh_0(h^*)^{-1} \in H, \\ t_{a_1} &= t_{a_0}, \\ a_1 &= a_0. \end{aligned}$$

So we obtain that $\hat{h}\,(a_0) = a_0$.

2. **a**. For all $x, y \in E$ we have the following equivalent equalities:

$$\begin{aligned} x \overset{(T)}{\cdot} y &= u, \\ t_x t_y &= t_u h, \quad h \in H, \\ t_x t_y H &= t_u H, \\ \widehat{t_x}(y) &= u. \end{aligned}$$

So we obtain that $\widehat{t}_x(y) = x \overset{(T)}{\cdot} y$.

b. For all $x, y \in E$ we have the following equivalent equalities:

$$\begin{aligned} \widehat{t}_x^{-1}(y) &= u, \\ \widehat{t}_x(u) &= y, \\ x \overset{(T)}{\cdot} u &= y, \\ u &= x \overset{(T)}{\diagdown} y, \end{aligned}$$

where "$\overset{(T)}{\diagdown}$" is a left division for the operation $\left\langle E, \overset{(T)}{\cdot}, a_0 \right\rangle$ (i.e. $x \overset{(T)}{\diagdown} y = z \Longleftrightarrow x \overset{(T)}{\cdot} z = y$). So we obtain that $\widehat{t}_x^{-1}(y) = x \overset{(T)}{\diagdown} y$.

c. According to item **1** of Theorem 2 there exists an element $a_0 \in E$ such that for every $x \in E$ $x \overset{(T)}{\cdot} a_0 = x$. Then due to item **2a** we have for every $x \in E$

$$\widehat{t}_x(a_0) = x \overset{(T)}{\cdot} a_0 = x.$$

3. Let the left general form transversal $T = \{t_i\}_{i \in E}$ be a reduced (but non-ordered) transversal in G to H. Then $t_{a_0} = e$. So all identities from the item **2** of present Theorem are true; moreover, we have for every $x \in E$

$$\widehat{t}_{a_0}(x) = \hat{e}(x) = id(x) = x.$$

4. Let the left general form transversal $T = \{t_i\}_{i \in E}$ be an ordered (but non-reduced) transversal in G to H. Then $a_0 = 1$. So all identities from the item **2** of present Theorem are true; moreover, we have for every $x \in E$

$$\widehat{t}_x(1) = x.$$

5. It is an evident corollary of the items **3** and **4**. $\square$

Theorem 4. *For an arbitrary left general form transversal $T = \{t_i\}_{i \in E}$ in G to H the following statements are true:*

1. *If $P = \{p_i\}_{i \in E}$ is a left general form transversal in G to H such that for every $x \in E$:*

$$\begin{aligned} P &= Th_0, \\ p_{x'} &= t_x h_0, \end{aligned}$$

*where $h_0 \in H$ is an arbitrary fixed element (see item **1** from Theorem 1), then the transversal operation $\left\langle E, \overset{(P)}{\cdot} \right\rangle$ is isotopic to the transversal operation $\left\langle E, \overset{(T)}{\cdot} \right\rangle$, and this isotopy has the form $(id, \hat{h}_0, id)$.*

2. *If $S=\{s_i\}_{i\in E}$ is a left general form transversal in G to H such that for every $x\in E$:*

$$\begin{aligned} S &= \pi T, \\ s_{x'} &= \pi t_x, \end{aligned}$$

*where $\pi\in G$ is an arbitrary fixed element (see item **3** from Theorem 1), then the transversal operation $\left\langle E, \overset{(S)}{\cdot}\right\rangle$ is isotopic to the transversal operation $\left\langle E, \overset{(T)}{\cdot}\right\rangle$, and this isotopy has the form (π^{-1}, id, π).*

Proof. **1**. Let $P=\{p_i\}_{i\in E}$ be a left general form transversal in G to H such that for every $x\in E$:

$$\begin{aligned} P &= Th_0, \\ p_{x'} &= t_x h_0, \end{aligned}$$

where $h_0\in H$ is an arbitrary fixed element. According to items **1** and **2** from Theorem 3 there exists an element $a_0\in E$ such that for every $h\in H$

$$\begin{aligned} \hat{h}\,(a_0) &= a_0, \\ \widehat{t}_x(a_0) &= x, \\ \hat{p}_{x'}(a_0) &= x', \end{aligned}$$

for all $x,x'\in E$. Then we have for all $x\in E$

$$x' = \hat{p}_{x'}(a_0) = \hat{t}_x\hat{h}_0(a_0) = \widehat{t}_x(a_0) = x,$$

i.e. for all $x\in E$

$$p_x = t_x h_0.$$

According to item **2** from Theorem 3 we obtain for all $x,y\in E$:

$$x \overset{(P)}{\cdot} y = \hat{p}_x(y) = \hat{t}_x\hat{h}_0(y) = x \overset{(T)}{\cdot} \hat{h}_0(y),$$

i.e. the transversal operation $\left\langle E, \overset{(P)}{\cdot}\right\rangle$ is isotopic to the transversal operation $\left\langle E, \overset{(T)}{\cdot}\right\rangle$, and this isotopy has the form $(id, \hat{h}_0, id)$.

2. Let $S=\{s_i\}_{i\in E}$ be a left general form transversal in G to H such that for every $x\in E$:

$$\begin{aligned} S &= \pi T, \\ s_{x'} &= \pi t_x, \end{aligned}$$

where $\pi \in G$ is an arbitrary fixed element. Analogously to the item **1** of this Theorem we have
$$x' = \hat{s}_{x'}(a_0) = \hat{\pi}\,\hat{t}_x(a_0) = \hat{\pi}(x),$$
i.e. for every $x \in E$
$$\begin{aligned} s_{\pi(x)} &= \pi t_x, \\ s_x &= \pi t_{\pi^{-1}(x)}. \end{aligned}$$
Then according to item **2** from Theorem 3 we obtain for all $x, y \in E$:
$$x \overset{(S)}{\cdot} y = \hat{s}_x(y) = \hat{\pi}\,\hat{t}_{\hat{\pi}^{-1}(x)}(y) = \hat{\pi}(\hat{\pi}^{-1}(x) \overset{(T)}{\cdot} y),$$
i.e. the transversal operation $\left\langle E, \overset{(S)}{\cdot} \right\rangle$ is isotopic to the transversal operation $\left\langle E, \overset{(T)}{\cdot} \right\rangle$, and this isotopy has the form (π^{-1}, id, π). □

Remark 3. The last statement allows us to see a new sense of Theorem 2. Now it is evident that the transition from a general form transversal to the reduced (or ordered) transversal is just a transition from a left quasigroup transversal operation to a left loop transversal operation (which is its isotope).

3 Quasigroup and loop general form transversals

Definition 8. Let $T = \{t_i\}_{i \in E}$ be a left general form transversal in G to H. If its transversal operation $\left\langle E, \overset{(T)}{\cdot} \right\rangle$ is a quasigroup, then the transversal T is called a left **quasigroup** general form transversal in G to H (in [5] such transversal is called a **stable** transversal in G to H).

Remark 4. According to item **1** from Theorem 2 there exists an element $a_0 \in E$ such that a_0 is a right unit in the operation $\left\langle E, \overset{(T)}{\cdot} \right\rangle$; so if T is a left quasigroup general form transversal in G to H, then the system $\left\langle E, \overset{(T)}{\cdot}, a_0 \right\rangle$ is a quasigroup with the right unit a_0.

Theorem 5. *If $T = \{t_i\}_{i \in E}$ is a left quasigroup general form transversal in G to H, then there exists an element $a_0 \in E$ such that the system $\left\langle E, \overset{(T)}{\cdot}, a_0 \right\rangle$ is a loop.*

Proof. It is an evident corollary from the item **2** of Theorem 2. □

Definition 9. A left reduced quasigroup general form transversal in G to H is usually called a left **loop** general form transversal in G to H.

Theorem 6. *The following statements are equivalent:*

1. *A set $T = \{t_x\}_{x\in E}$ is a left quasigroup general form transversal in G to H;*
2. *For every $\pi \in G$ the set $T\pi = \{t_x\pi\}_{x\in E}$ is a left general form transversal in G to H;*
3. *For all $\pi_1, \pi_2 \in G$ the set $\pi_1 T\pi_2 = \{\pi_1 t_x \pi_2\}_{x\in E}$ is a left general form transversal in G to H;*
4. *For every $\pi \in G$ the set $T = \{t_x\}_{x\in E}$ is a left general form transversal in G to $H^\pi = \pi H\pi^{-1}$.*

Proof. **1⇒2**. Let a set $T = \{t_x\}_{x\in E}$ be a left quasigroup general form transversal in G to H. Then the system $\left\langle E, \overset{(T)}{\cdot} \right\rangle$ is a quasigroup. Let an element $\pi \in G$ be an arbitrary fixed element from G. We shall consider the set $T\pi = \{t_x\pi\}_{x\in E}$ and prove that this set is a left general form transversal in G to H.

Because $T = \{t_x\}_{x\in E}$ is a left quasigroup general form transversal in G to H, then

$$\pi = t_{c_0} h_0$$

for some $t_{c_0} \in T$ and $h_0 \in H$. Because the operation $\left\langle E, \overset{(T)}{\cdot} \right\rangle$ is a quasigroup, then for every $x \in E$ we have

$$t_x\pi = t_x t_{c_0} h_0 = t_{x \overset{(T)}{\cdot} c_0} h_1 = t_{R_{c_0}(x)} h_1$$

for some $h_1 \in H$. Then every element $g \in G$ may be represented in the following form:

$$g = t_{c_1} h^* = t_{R_{c_0}(c_1/c_0)} h_1 h_1^{-1} h^* = t_{c_1/c_0}\pi h_1^{-1} h^* = (t_{c_1/c_0}\pi) h^{**}, \qquad h^{**} \in H.$$

Let us assume that this representation is not unique, i.e. there exist $a, b \in E$, $a \neq b$ and $h_1, h_2 \in H$ such that

$$t_a \pi h_1 = g = t_b \pi h_2.$$

According to item **1** of Theorem 3 there exists an element $a_0 \in E$ such that we have the following equivalent equalities

$$\begin{array}{rcl}
\hat{t}_a\hat{\pi}\,\hat{h}_1(a_0) & = & \hat{t}_b\hat{\pi}\,\hat{h}_2(a_0) \\
\hat{t}_a\hat{\pi}\,(a_0) & = & \hat{t}_b\hat{\pi}\,(a_0) \\
\hat{t}_a\hat{t}_{c_0}\hat{h}_0(a_0) & = & \hat{t}_b\hat{t}_{c_0}\hat{h}_0(a_0) \\
\hat{t}_a\hat{t}_{c_0}(a_0) & = & \hat{t}_b\hat{t}_{c_0}(a_0) \\
\hat{t}_a(c_0) & = & \hat{t}_b(c_0)
\end{array}$$

$$\begin{array}{rcl} a \overset{(T)}{\cdot} c_0 & = & b \overset{(T)}{\cdot} c_0 \\ a & = & b \end{array}$$

because the operation $\left\langle E, \overset{(T)}{\cdot} \right\rangle$ is a quasigroup. We obtain a contradiction and so the above mentioned representation is unique. Then the set $T\pi = \{t_x\pi\}_{x\in E}$ is a left general form transversal in G to H.

2$\Rightarrow$**3**. It is evident due to item **3** of Theorem 1.

3$\Rightarrow$**4**. If the condition of item **3** holds then a fortiori is true that for every $\pi \in G$ the set $\pi T\pi^{-1} = \{\pi t_x \pi^{-1}\}_{x\in E}$ is a left general form transversal in G to H. So for all $a, b \in E$, $a \neq b$ we have the following equivalent statements:

$$\begin{cases} G = \bigcup\limits_{x\in E} (\pi t_x \pi^{-1})H, \\ \varnothing = (\pi t_a \pi^{-1} H) \cap (\pi t_b \pi^{-1} H), \end{cases}$$

$$\begin{cases} G = \pi^{-1} G \pi = \pi^{-1} \left(\bigcup\limits_{x\in E} (\pi t_x \pi^{-1})H \right) \pi = \bigcup\limits_{x\in E} t_x(\pi^{-1} H \pi), \\ \varnothing = \pi^{-1} \cdot \varnothing \cdot \pi = \pi^{-1}((\pi t_a \pi^{-1} H) \cap (\pi t_b \pi^{-1} H))\pi = (t_a(\pi^{-1} H\pi)) \cap (t_b(\pi^{-1} H \pi)). \end{cases}$$

Because the element $\pi \in G$ is an arbitrary element from G then the element π^{-1} will be an arbitrary element from G too. So the set $T = \{t_x\}_{x\in E}$ is a left general form transversal in G to $H^{\pi'} = \pi' H \pi'^{-1}$ for every $\pi' \in G$ (where $\pi' = \pi^{-1}$).

4$\Rightarrow$**1**. Let for every $\pi \in G$ a set T be a left general form transversal in G to $H^{\pi} = \pi H \pi^{-1}$. In order to prove that the set T is a left quasigroup general form transversal in G to H, it is sufficient to prove that for all arbitrary fixed elements $a, b \in E$ the equation

$$x \overset{(T)}{\cdot} a = b$$

has unique solution in the set E.

We have the following equivalent equalities:

$$\begin{array}{rcl} x \overset{(T)}{\cdot} a & = & b \\ t_x t_a & = & t_b h, \quad h \in H \end{array}$$

$$t_x = t_b h t_a^{-1} = (t_b t_a^{-1}) \cdot (t_a h t_a^{-1}) \tag{1}$$

Because the set T is a left general form transversal in G to $H^{t_a} = t_a H t_a^{-1}$ (when $\pi = t_a$), then there exists the unique element $c = c(a, b) \in E$ such that

$$t_b t_a^{-1} \in t_c \cdot (t_a H t_a^{-1}).$$

Substituting this product in (1) we obtain:

$$t_x = t_c \cdot (t_a h' t_a^{-1}) \cdot (t_a h t_a^{-1}) = t_c \cdot (t_a h^* t_a^{-1}), \quad h^* \in H.$$

Because the set T is a left general form transversal in G to $H^{t_a} = t_a H t_a^{-1}$, then $x = c$. The proof is finished. $\square$

Corollary 1. *The following statements are equivalent:*

1. *A set $T = \{t_x\}_{x\in E}$ is a left quasigroup general form transversal in G to H;*
2. *For every $\pi \in G$ the set $T\pi = \{t_x\pi\}_{x\in E}$ is a left quasigroup general form transversal in G to H;*
3. *For all $\pi_1, \pi_2 \in G$ the set $\pi_1 T\pi_2 = \{\pi_1 t_x \pi_2\}_{x\in E}$ is a left quasigroup general form transversal in G to H.*

Theorem 7. *The following statements are equivalent:*

1. *A set $T = \{t_x\}_{x\in E}$ is a left loop general form transversal in G to H;*
2. *For every $\pi \in G$ the set $T\pi = \{t_x\pi\}_{x\in E}$ is a left general form transversal in G to H;*
3. *For all $\pi \in G$ the set $\pi T\pi^{-1} = \{\pi t_x \pi^{-1}\}_{x\in E}$ is a left reduced general form transversal in G to H;*
4. *For every $\pi \in G$ the set $T = \{t_x\}_{x\in E}$ is a left reduced general form transversal in G to $H^\pi = \pi H\pi^{-1}$.*

Proof. It is an evident corollary from Theorems 1 and 6. □

Corollary 2. *The following statements are equivalent:*

1. *A set $T = \{t_x\}_{x\in E}$ is a left loop general form transversal in G to H;*
2. *For every $\pi \in G$ the set $T\pi = \{t_x\pi\}_{x\in E}$ is a left quasigroup general form transversal in G to H;*
3. *For all $\pi \in G$ the set $\pi T\pi^{-1} = \{\pi t_x \pi^{-1}\}_{x\in E}$ is a left loop general form transversal in G to H.*

Theorem 8. *Let $T = \{t_x\}_{x\in E}$ be a left loop general form transversal in G to H. According to Definition 9 and Theorem 3 there exists an element $a_0 \in E$ such that $\widehat{t}_{a_0} = id$. Then for every $x \in E$, $x \neq a_0$, the permutation $\widehat{t}_x$ is a fixed-point-free permutation on the set E.*

Proof. Let the conditions of Theorem hold and assume that it is not true, i.e. there exist $c_0 \in E$ and $a_1 \in E$, $a_1 \neq a_0$, such that

$$\begin{cases} \widehat{t}_{a_1}(c_0) = c_0, \\ a_1 \neq a_0. \end{cases}$$

Then according to Theorem 2 we have the following equivalent equalities

$$\widehat{t}_{a_1}(c_0) \quad = \quad c_0,$$

$$\begin{aligned} a_1 \overset{(T)}{\cdot} c_0 &= c_0 = a_1 \overset{(T)}{\cdot} c_0, \\ a_1 \overset{(T)}{\cdot} c_0 &= a_1 \overset{(T)}{\cdot} c_0, \\ a_1 &= a_0, \end{aligned}$$

since the system $\left\langle E, \overset{(T)}{\cdot}, a_0 \right\rangle$ is a loop. But the last equality contradicts to the assumption that $a_1 \neq a_0$. The proof is finished. $\square$

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