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Lattice of all topologies of countable module over countable rings

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Abstract. For any countable ring R with discrete topology τ_0 and any countable R-module M the lattice of all (R, τ_0)-module topologies contains:
– A sublattice which is isomorphic to the lattice of all real numbers with the usual order;
– Two to the power of continuum (R, τ_0)-module topologies each of which is a coatom.

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1 Introduction

For any ring R with discrete topology τ_0 and any R-module M the question of the existence of non-discrete Hausdorff (R, τ_0)-module topologies was considered in [1] and [2]. In particular, it was proved that any infinite module over any discrete ring R admits non-discrete Hausdorff module topology and an example of a topological ring (R, τ_0) and an R-module M was constructed for which the lattice of all (R, τ_0)-module topologies does not contain Hausdorff topologies.

In fact (see below Remark 3.1) for this topological ring (R, τ_0), the lattice of all (R, τ_0)-module topologies on this R-module M contains only anti-discrete topology.

The present paper is a continuation of these works and is devoted to the study of properties of the lattice of all topologies on countable modules over discrete ring.

The main result of this article is Theorem 3.2, in which it is proved that for any countable ring R with discrete topology τ_0 and any countable R-module M, the lattice of all (R, τ_0)-module topologies contains a sublattice which is isomorphic to the lattice of real numbers with the usual order and contains two to the power of continuum coatoms.

Similar results for countable groups and countable rings were obtained in [3, 4] and [5], respectively.

Furthermore, it was shown that the condition that the ring should be countable is essential in Theorem 3.2, namely, we constructed an example of an infinite discrete ring (R, τ_0) and a countable R-module M such that every (R, τ_0)-module topology on M which has a countable or finite basis of the filter of neighbourhoods of zero is anti-discrete.

2 Preliminary results

To present the main results we recall the following two well known theorems (see, for example, [1]).

Theorem 2.1. *A set $\Omega = \{V_\gamma | \gamma \in \Gamma\}$ of subsets of a ring R is a basis of the filter of neighbourhoods of zero for some ring topology τ on the ring R if and only if the following conditions are satisfied:*

1. $0 \in \bigcap\limits_{\gamma \in \Gamma} V_\gamma$;

2. For any subsets V_1 and $V_2 \in \Omega$ there exists a subset $V_3 \in \Omega$ such that $V_3 \subseteq V_1 \cap V_2$;

3. For any subset $V_1 \in \Omega$ there exists a subset $V_2 \in \Omega$ such that $V_2 + V_2 \subseteq V_1$;

4. For any subset $V_1 \in \Omega$ there exists a subset $V_2 \in \Omega$ such that $-V_2 \subseteq V_1$;

5. For any subset $V_1 \in \Omega$ and any element $r \in R$ there exists a subset $V_2 \in \Omega$ such that $r \cdot V_2 \subseteq V_1$ and $V_2 \cdot r \subseteq V_1$;

6. For any subset $V_1 \in \Omega$ there exists a subset $V_2 \in \Omega$ such that $V_2 \cdot V_2 \subseteq V_1$.

Theorem 2.2. *If (R, τ) is a topological ring and M is an R-module, then a set $\Lambda = \{U_\delta | \delta \in \Delta\}$ of subsets of the module M is a basis of the filter of neighborhoods of zero for some (R, τ)-module topology τ_1 of the module M if and only if the following conditions are satisfied:*

1. $0 \in \bigcap\limits_{\delta \in \Delta} U_\delta$*;*

2. For any subsets U_1 and $U_2 \in \Lambda$ there exists a subset $U_3 \in \Lambda$ such that $U_3 \subseteq U_1 \cap U_2$;

3. For any subset $U_1 \in \Lambda$ there exists a subset $U_2 \in \Lambda$ such that $U_2 + U_2 \subseteq U_1$;

4. For any subset $U_1 \in \Lambda$ there exists a subset $U_2 \in \Lambda$ such that $-U_2 \subseteq U_1$;

5. For any subset $U_1 \in \Lambda$ and any element $r \in R$ there exists a subset $U_2 \in \Lambda$ such that $r \cdot U_2 \subseteq U_1$;

6. For any subset $U_1 \in \Lambda$ and any element $m \in M$ there exists a neighborhood V_2 of zero of the topological ring (R, τ) such that $V_2 \cdot m \subseteq U_1$;

7. For any subset $U_1 \in \Lambda$ there exists a neighborhood V_2 of zero of the topological ring (R, τ) and a subset $U_2 \in \Lambda$ such that $V_2 \cdot U_2 \subseteq U_1$.

Theorem 2.3. *(see the proof in [5], Theorem 3.1) If R is a countable ring and τ_0 is a non-discrete, Hausdorff ring topology such that the topological ring (R, τ_0) has a countable basis of the filter of neighborhoods of zero, then the following statements are true:*

1. For any infinite set A of natural numbers there exists a ring topology $\tau(A)$ such that the topological ring $(R, \tau(A))$ has a countable basis of the filter of neighborhoods of zero and such that $\tau_0 \leq \tau(A)$;

2. $\sup\{\tau(A), \tau(B)\}$ *is the discrete topology for any infinite sets A and B of natural numbers such that $A \cap B$ is a finite set;*

3. There exist the continuum of Hausdorff ring topologies each having a countable basis of the filter of neighbourhoods of zero and stronger than τ_0 and such that any two of them are comparable;

4. There are two to the power of continuum topologies such that $\sup\{\tau_1, \tau_2\}$ *is the discrete topology for any two different topologies;*

5. There are two to the power of continuum coatoms in the lattice of all ring topologies.

Remark 2.4. From the proof of Theorem 3.1 in [5] it is easy to see that all topologies which are indicated in this theorem are stronger than the topology τ_0.

Remark 2.5. As in the proof of the Statement 3.1.3 of Theorem 3.1 in [5] ring topology τ_r is defined for every real number r and $\tau_t \leq \tau_s$ if and only if $s \leq t$, then the lattice of all ring topologies contains a sublattice which is anti-isomorphic to the lattice of all real numbers with the usual order for any countable ring.

In addition, since the mapping σ such that $\sigma(r) = -r$ is an anti-isomorphism of the lattice of all real numbers on itself, then the lattice of all ring topologies contains a sublattice which is isomorphic to the lattice of all real numbers with the usual order for any countable ring.

3 Basic results

Remark 3.1. We will show that for the topological ring (R, τ_0) and for the R-module M, which are constructed in [3] and [4], any (R, τ_0)-module topology of the module M is anti-discrete.

Thus, let:

– R be the ring of polynomials of an argument x over the field of rational numbers Q;

– $M = \{r \cdot z | r \in Q\}$ be a one-dimensional vector space over the field of rational numbers Q;

– $\left(\sum\limits_{i=0}^{n} r_i \cdot x^i\right) \cdot (r \cdot z) = \left(\sum\limits_{i=0}^{n} r_i \cdot r\right) \cdot z$ for any element $\sum\limits_{i=0}^{n} r_i \cdot x^i \in R$ and any element $r \cdot z \in M$;

– The set $\Omega = \{R \cdot x^n | n = 1, 2, \ldots\}$ is a basis of the filter of neighbourhoods of zero in the topological ring (R, τ_0).

Now let τ be an (R, τ_0)-module topology of the module M and let U be an arbitrary neighbourhood of zero in the topological module (M, τ).

If $r \cdot z \in M$, then according to the condition 6 of Theorem 2.2, there exists a neighbourhood V of zero in the topological ring (R, τ_0) such that $V \cdot (r \cdot z) \subseteq U$, and hence $(R \cdot x^n) \cdot (r \cdot z) \subseteq V \cdot (r \cdot z) \subseteq U$ for some natural number n.

Then $(r \cdot z) = x^n \cdot (r \cdot z) \in (R \cdot x^n) \cdot (r \cdot z) \subseteq V \cdot (r \cdot z) \subseteq U$. From the arbitrariness of the element $r \cdot z$ it follows that $U = M$, and hence the topology τ is anti-discrete.

Theorem 3.2. *If (R, τ_0) is a countable ring with the discrete topology τ_0 and M is a countable R-module then the following statements are true:*

1. For any infinite set A of natural numbers there exists an (R, τ_0)-module topology $\tau(A)$ which has a countable basis of the filter of neighborhoods of zero and such that $\sup\{\tau(A), \tau(B)\}$ is the discrete topology for any infinite sets A and B of natural numbers such that $A \cap B$ is a finite set;

2. There exist continuum of (R, τ_0)-module topologies which have a countable basis of the filter of neighbourhoods of zero and such that any two of them are comparable;

3. There exist two to the power of continuum coatoms in the lattice of all (R, τ_0)-module topologies on the module M;

4. The lattice of all (R, τ_0)-module topologies on the module M contains a sublattice which is anti-isomorphic to the lattice of all real numbers with the usual order, and contains a sublattice which is isomorphic to the lattice of all real numbers with the usual order.

Proof. We define the operation of multiplication on the group $\hat{R}(+) = \{(r, m) | r \in R, m \in M\}$, which is the direct sum of the groups $R(+)$ and $M(+)$, as follows: $(r_1, m_1) \cdot (r_2, m_2) = (r_1 \cdot r_2, r_1 \cdot m_2)$ for any elements $r_1, r_2 \in R$ and any elements $m_1, m_2 \in M$.

It is easy to see that $\hat{R}(+, \cdot)$ is a ring, and the set $\hat{I} = \{(0, m) | m \in M\}$ is an ideal of the ring $\hat{R}$.

If $\psi(0, m) = m$, then $\psi : \hat{I} \to M$ is a bijective mapping. Then putting $\hat{\psi}(\hat{U}) = \{\psi(0, m) | (0, m) \in \hat{U}\}$ for each subset $\hat{U} \subseteq \hat{I}$, we define a bijective mapping $\hat{\psi}$ of the set of all subsets of the set $\hat{I}$ on the set of all subsets of the set M.

Let $\hat{\Delta}$ be the lattice of all ring topologies on the ring $\hat{R}$ such that the ideal $\hat{I}$ is open, and let Δ be the lattice of all (R, τ_0)-module topologies on the module M. We show that the lattices $\hat{\Delta}$ and Δ are isomorphic.

Let $\hat{\tau} \in \hat{\Delta}$. As $\hat{I}$ is an open ideal in the topological ring $(\hat{R}, \hat{\tau})$ then the topological ring $(\hat{R}, \hat{\tau})$ has a basis $\hat{\Omega}$ of the filter of neighborhoods of zero such that $\hat{V} \subseteq \hat{I}$ for any $\hat{V} \in \hat{\Omega}$.

Since τ_0 is the discrete topology, then from Theorems 2.1 and 2.2 it follows that the set $\{\hat{\psi}(\hat{V}) | \hat{V} \in \hat{\Omega}\}$ is a basis of the filter of neighborhoods of zero for some $(R, \tau_0)-$module topology on the module M, and any (R, τ_0)-module topology on the module M can be obtained in this way.

Since any module topology is given in a unique way by any basis of the filter of neighborhoods of zero, we have identified mapping $\widetilde{\psi} : \hat{\Omega} \to \Omega$. It is easy to see that this map is bijective, and $\hat{\tau}_1 \leq \hat{\tau}_2$ if and only if $\widetilde{\psi}(\hat{\tau}_1) \leq \widetilde{\psi}(\hat{\tau}_2)$, i.e. $\widetilde{\psi} : (\hat{\Omega}, \leq) \to (\Omega, \leq)$ is a lattice isomorphism.

As noted above (see Introduction), there exists a non-discrete Hausdorff (R, τ_0)-module topology $\bar{\tau}_0$ on the module M. If $\hat{\tau}_0 = \hat{\Psi}^{-1}(\bar{\tau}_0)$, then $\hat{I}$ is an open ideal in the topological ring $(\hat{R}, \hat{\tau}_0)$. Then the topological ring $(\hat{R}, \hat{\tau}_0)$ has a basis $\hat{B}$ of the filter of neighborhoods of zero such that $\hat{U} \subseteq \hat{I}$ for every $\hat{U} \in \hat{B}$ and $\bigcap_{\hat{U} \in \hat{B}} \hat{U} = \{0\}$.

From countability of the ring $\hat{R}$, it follows that there exists a countable subset $\hat{B}_0 \subseteq \hat{B}$ such that $\bigcap_{\hat{U} \in \hat{B}_0} \hat{U} = \{0\}$ and the conditions of Theorem 2.1 are satisfied. Hence, there is a Hausdorff topology $\tilde{\tau}_0 \in \hat{\Omega}$ such that topological ring $(\hat{R}, \tilde{\tau}_0)$ has a countable basis of the filter of neighborhoods of zero and $\hat{I}$ is an open ideal.

Then Statements 1 – 5 of Theorem 2.3 are true for the topological ring $(\hat{R}, \tilde{\tau}_0)$, and from Remark 2.4 it follows that $\hat{I}$ is an open ideal for any topology, which is obtained according of Statements 1 – 5 of Theorem 2.3, i.e. all these topologies belong to $\hat{\Omega}$. As the lattice $\hat{\Omega}$ is isomorphic to the lattice Ω, then Statements 1 – 3 of Theorem 3.2 are true.

In addition, the Statement 4 of Theorem 3.2 follows from Remark 2.5.

The theorem is proved. □

Remark 3.3. We will construct an example of a ring (R, τ_0) with discrete ring topology τ_0 and countable R-module M such that every non-discrete (R, τ_0)-module topology which has a finite or countable basis of the filter of neighborhoods of zero, is anti-discrete.

This example shows that the requirement that the ring R should be countable is essential in Statements 1 and 2 of Theorem 3.2.

As for any ring R with the discrete topology τ_0 any infinite module allows a non-discrete Hausdorff (R, τ_0)-module topology, then the lattice of all (R, τ_0)-module topologies contains coatoms.

However, the following questions remain unresolved:

– How many coatoms are in the lattice of all module topologies on any infinite module over any ring with discrete topology?

– Do there exist a ring with discrete topology and an infinite module for which the lattice of all module topologies has only one coatom?

– Do there exist a ring with discrete topology and an infinite module for which the lattice of all module topologies is a chain?

Example 3.4. Let X be a set with the cardinality of continuum and let $Y = \{y_1, y_2, \ldots\}$ be a countable set. We consider the free associative algebra R over the two-element field Z_2 which is generated by the set X and the linear space M over Z_2 for which the set Y is a basis.

We consider the set $\tilde{N}$ of all countable strictly increasing sequences of natural numbers.

If ω_0 is the smallest countable transfinite number and ω_c is the smallest transfinite number with the cardinality of continuum, then:

$\tilde{N} = \{\tilde{m}_\alpha | \omega_0 \leq \alpha < \omega_c\}$ and $X = \{x_\alpha | 1 \leq \alpha < \omega_c\}$.

We define the multiplication of elements of the set $Y \cup \{0\}$ by elements of the set X as follows:

– If $\alpha < \omega_c$, then we let $x_\alpha \cdot 0 = 0$;

– If $\alpha < \omega_0$, i.e. α is a natural number, then we let $x_\alpha \cdot y_k = y_{\alpha+k-1}$ for any natural number k;

– If $\omega_0 \leq \alpha < \omega_c$, then $\tilde{m}_\alpha$ is an increasing sequence of natural numbers, i.e. $\tilde{m}_\alpha = (m_1, m_2, \ldots)$, and then we let $x_\alpha \cdot y_k = y_1$ if $k \in \{m_1, m_2, \ldots\}$ and $x_\alpha \cdot y_k = 0$ if $k \notin \{m_1, m_2, \ldots\}$.

Then, using the associative and distributive laws, we can extend the operation of the multiplication of elements of the ring R on the elements of the group M so that the group M will be a R-module.

We show now that every non-discrete module topology on the R-module M which has a finite or countable basis of the filter of neighbourhoods of zero is anti-discrete.

Assume the contrary, i.e. that on the R-module M there exists a non-discrete module topology τ which has a finite or countable basis Ω of the filter of neighbourhoods of zero and which is not anti-discrete.

If $\{0\} \neq \bigcap\limits_{V \in \Omega} V$ and $0 \neq g \in \bigcap\limits_{V \in \Omega} V\}$, then there exists a natural number n such that $g = k_1 \cdot y_1 + k_2 \cdot y_2 + \ldots + k_n \cdot y_n$ and $k_n \neq 0$, i.e. $k_n = 1$.

Since the sequence $(n, n+1, n+2, \ldots) \in \tilde{N}$, then $(n, n+1, n+2, \ldots) = \tilde{m}_\alpha$ for some transfinite number $\omega_0 \leq \alpha < \omega_c$.

Now if $V \in \Omega$, then (see Theorem 2.2, the property 5) for the element $x_\alpha \in R$, there exists a neighbourhood $V_1 \in \Omega$ such that $x_\alpha \cdot V_1 \subseteq V$. Then (see above, the definition of multiplication of elements from M by elements from R) $y_1 = x_\alpha \cdot y_n = x_\alpha \cdot g = x_\alpha \cdot V_1 \subseteq V$.

So, we have proved that $y_1 \in V$ for every neighbourhood $V \in \Omega$.

If $V \in \Omega$ and $h = y_{k_1} + y_{k_2} + \ldots + y_{k_s} \in M$, then there exists a neighbourhood $V'_1 \in \Omega$ such that $\underbrace{V'_1 + V'_1 + \ldots V'_1}_{s\ items} \subseteq V$ and there exist neighbourhoods of $V_{k_1}, V_{k_2}, \ldots, V_{k_s} \in \Omega$ such that $x_{k_i} \cdot V_{k_i} \subseteq V'_1$ for every natural number $1 \leq i \leq s$.

Then

$$h = y_{k_1} + y_{k_2} + \ldots + y_{k_s} = x_{k_1} \cdot y_1 + x_{k_2} \cdot y_1 + \ldots + x_{k_s} \cdot y_1 \subseteq \underbrace{V'_1 + V'_1 + \ldots + V'_1}_{s\ items} \subseteq V.$$

The arbitrariness of the element $h \in M$ implies that $V = M$, and from the arbitrariness of the neighbourhood V we have that the topology τ is anti-discrete for the case when $\{0\} \neq \bigcap\limits_{V \in \Omega} V$.

Now let $\{0\} = \bigcap\limits_{V \in \Omega} V$. The further proof will be realized in several steps.

Step I. *We show that for any natural number n and any neighborhood $V_0 \in \Omega$ there exists an element $h \in V_0$ such that*

$$h = k_{n+1} \cdot y_{n+1} + k_{n+2} \cdot y_{n+2} + \ldots + k_{n+t} \cdot y_{n+t}$$

and $k_i \in \{0, 1\}$ for $n+1 \leq i \leq n+t$.

Let V_1 be a neighbourhood of zero in (M, τ) such that $V_1 - V_1 \subseteq V_0$.

As for the natural number n the set

$$M_n = \{l_1 \cdot y_1 + l_2 \cdot y_2 + \ldots + l_{n-1} \cdot y_{n-1} | l_i \in \{0, 1\}, 1 \leq i \leq n-1\}$$

is finite, then there exist elements $g = k_1 \cdot y_1 + k_2 \cdot y_2 + \ldots + k_m \cdot y_m \in V_1$ and $g' = k'_1 \cdot y_1 + k'_2 \cdot y_2 + \ldots + k'_m \cdot y_m \in V_1$ such that $k_i = k'_i$ for $1 \leq i \leq n$. Then

$$h = g - g' = (k_{n+1} - k'_{n+1}) \cdot y_{n+1} + (k_{n+2} - k'_{n+2}) \cdot y_{n+2} + \ldots + (k_m - k'_m) \cdot y_m \in V_1 - V_1 \subseteq V_0.$$

By this the statement indicated in Step I is proved.

Step II. *By induction we construct an increasing sequence $n_1, n_2, \ldots$ of natural numbers and a sequence $g_1, g_2, \ldots$ of elements of the module M.*

If $\Omega = \{V_1, V_2, \ldots\}$, then we take an element

$$g_1 = k_1 \cdot y_1 + k_2 \cdot y_2 + \ldots + k_{n_1} \cdot y_{n_1} \in V_1.$$

According to the statement indicated in Step I, for the natural number n_1 and the neighbourhood V_2 there exists an element

$$g_2 = k_{n_1+1} \cdot y_{n_1+1} + k_{n_1+2} \cdot y_{n_1+2} + \ldots + k_{n_2} \cdot y_{n_2} \in V_2.$$

Assume that for any number $2 \leq i \leq k$ we have constructed a natural number n_i and an element

$$g_i = k_{n_{i-1}+1} \cdot y_{n_{i-1}+1} + k_{n_{i-1}2} \cdot y_{n_{i-1}+2} + \ldots + k_{n_i} \cdot y_{n_i} \in V_i.$$

Then according to the statement indicated in Step I, for the natural number n_k and the neighbourhood V_{k+1} there exists an element

$$g_{k+1} = k_{n_k+1} \cdot y_{n_k+1} + k_{n_k+2} \cdot y_{n_k+2} + \ldots + k_{n_{k+1}} \cdot y_{n_{k+}} \in V_{k+1}.$$

So, we have identified an increasing sequence $n_1, n_2, \ldots$ of natural numbers and the sequence $g_1, g_2, \ldots$ of elements of the module M such that

$$g_i = k_{n_{i-1}1} \cdot y_{n_{i-1}+1} + k_{n_{i-1}+2} \cdot y_{n_{i-1}+2} + \ldots + k_{n_i} \cdot y_{n_i} \in V_i$$

for any natural number i.

Step III. *We verify that $y_1 \in \bigcap\limits_{i=1}^{\infty} V_i$.*

If $n_1, n_2, \ldots$ is the sequence of natural numbers which was built in the second Step, then it belongs to $\tilde{N}$, and hence, $(n_1, n_2, \ldots) = \tilde{m}_\alpha$ for some transfinite number $\omega_0 \leq \alpha < \omega_c$.

If i is any natural number, then for the element x_α and the neighbourhood of zero V_i there exists a natural number j such that $x_\alpha \cdot V_j \subseteq V_i$. Then, the definition of multiplication of elements from M by elements from R implies that $y_1 = x_\alpha \cdot g_j \in x_\alpha \cdot V_j \subseteq V_i$.

The arbitrariness of the natural number i implies that $y_1 \in \bigcap\limits_{i=1}^{\infty} V_i$. This contradicts the assumption that $\{0\} = \bigcap\limits_{i=1}^{\infty} V_i$, and hence the case $\{0\} = \bigcap\limits_{i=1}^{\infty} V_i$ is impossible.

Thus, any non-discrete module topology on R-module M which has a finite or countable basis of the filter of neighbourhoods of zero is anti-discrete.

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