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Almost periodicity of functions on universal algebras

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Abstract. The Bohr compactification is well known for groups and semigroups [1,4,7,11,13]. In the present paper the analogue of Bohr compactification is considered for universal algebras. Some questions posed by J. E. Hart and K. Kunen [9] are answered.

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1 Introduction

The aim of the present article is to study the compactifications of topological universal algebras generated by special functions. Any space is considered to be Tychonoff and non-empty. We use the terminology from [8].

The discrete sum $\Omega = \oplus\{\Omega_n : n \in N = \{0, 1, 2, ...\}\}$ of the pairwise disjoint discrete spaces $\{\Omega_n : n \in N\}$ is called a signature. A topological Ω-algebra or a topological universal algebra of the signature Ω is a family $\{G, e_{nG} : n \in N\}$, where G is a non-empty topological space and $e_{nG} : \Omega_n \times G^n \to G$ is a continuous mapping for each $n \in N$.

Subalgebras, homomorphisms, isomorphisms and Cartesian products of topological Ω-algebras are defined in traditional way [4, 5, 7, 9].

Let G be a topological space and $n \in N$. A continuous mapping $\lambda : G^n \to G$ is called an n-ary operation on G.

If G is a topological Ω-algebra and $\omega \in \Omega_n$, then $\omega : G^n \to G$, where $\omega(x) = e_{nG}(\omega, x)$ for every $x \in G^n$, is an n-ary operation on G.

A pair (Y, φ) is a generalized compactification or a g-compactification of a topological space X if Y is a compact space, $\varphi : X \to Y$ is a continuous mapping and the set $\varphi(X)$ is dense in Y. If (Z, φ) and (Y, ψ) are g-compactifications of X, then $(Z, \varphi) \le (Y, \psi)$ if and only if there exists a continuous mapping $g : Y \to Z$ such that $\varphi = g \circ \psi$. If $\varphi : X \to Y$ is an embedding, then a pair (Y, φ) is called a compactification and we consider that $X \subseteq Y$ and $\varphi(x) = x$ for each $x \in X$.

If (Y, φ) and (Z, ψ) are g-compactifications of X, $(Y, \varphi) \le (Z, \psi)$ and $(Z, \psi) \le (Y, \varphi)$, then the g-compactifications (Y, φ), (Z, ψ) are called equivalent and there exists a unique homeomorphism $g : Y \to Z$ such that $\psi = g \circ \varphi$. We identify the equivalent g-compactifications. In this case the class of all g-compactifications of the space X is a set.

A pair (E,φ) is an algebraical g-compactification or an ag-compactification of a topological Ω-algebra G if E is a compact topological Ω-algebra, $\varphi : G \to E$ is a continuous homomorphism and the set $\varphi(G)$ is dense in E. If (Z,φ) and (Y,ψ) are ag-compactifications of G and $(Y,\varphi) \leq (Z,\psi)$, then the unique continuous mapping $g : Y \to Z$, for which $\psi = g \circ \varphi$, is a continuous homomorphism of Y onto Z. If $(Y,\varphi) \leq (Z,\psi)$ and $(Z,\psi) \leq (Y,\varphi)$, then the ag-compactifications (Y,φ), (Z,ψ) are called equivalent and there exists a unique topological isomorphism $g : Y \to Z$ such that $\psi = g \circ \varphi$.

If a pair (E,φ) is an ag-compactification and a compactification of a topological Ω-algebra G, then (E,φ) is called an a-compactification of G. If $\Omega = \Omega_0$, then any g-compactification of a topological Ω-algebra G is an ag-compactification.

If G is a topological Ω-algebra, then $Com_\Omega(G)$ is the set of all ag-compactifications of the topological Ω-algebra G.

The following properties are obvious.

Property 1. *The set $Com_\Omega(G)$ is a complete lattice for every topological Ω-algebra G and for every non-empty subset $L \subseteq Com_\Omega(X)$ there exist the maximal element $\vee L$ and the minimal element $\wedge L$.*

Property 2. *In the lattice of all ag-compactifications of a topological Ω-algebra G there exists the maximal a-compactification $(\beta_\Omega G, \beta_G)$, which is called the Bohr compactification of the topological Ω-algebra G.*

Property 3. *In the lattice of all ag-compactifications of a topological Ω-algebra G there exists the minimal ag-compactification $(\mu_a G, \mu_G)$, which is the singleton Ω-algebra.*

As a rule, the Bohr compactification of a topological Ω-algebra G is an ag-compactification of G.

Fix a topological space G. Let $C(G)$ be the space of real-valued continuous functions on the space G in the topology of uniform convergence. The topology on $C(G)$ is generated by the metric $d(f,g) = \sup\{|f(x) - g(x)| : x \in X\}$. Let $C^\circ(G)$ be the subspace of bounded functions. Then $C^\circ(G)$ is a Banach algebra (ring) with the norm $||f|| = \sup\{|f(x)| : x \in G\}$. For some $f, g \in C(G)$ it is possible that $d(f,g) = \infty$. We have $C(G) = C^\circ(G)$ if and only if the space G is pseudocompact. If $C(G) \neq C^\circ(G)$, then $C(G)$ is a linear space, but is not a topological linear space. The space $C(G)$ is a topological ring relative to the operations $f + g$ and $f \cdot g$. For any number $\lambda \in \mathbb{R}$ the correspondence $t_\lambda(f) = \lambda f$ is a continuous mapping of $C(G)$ into $C(G)$. For $\lambda \neq 0$, the correspondence $t_\lambda : C(G) \to C(G)$ is a homeomorphism.

Compactifications of the spaces can be produced in a variety of ways. One way is by use of subspaces of the space $C^\circ(G)$.

Let $F \subseteq C^\circ(G)$ be a non-empty subspace. Consider the mapping $e_F : G \to \mathbb{R}^F$, where $e_F(x) = (f(x) : f \in F)$. Denote by $b_F G$ the closure of the set $e_F(G)$ in $\mathbb{R}^F$. Then $(b_F G, e_F)$ is a g-compactification of G and $\beta G = \beta_{C^\circ(G)} G$ is the Stone-Čech maximal compactification of G [8]. Moreover, the family of functions $\bar{F} = \{g \in C(b_F G) : g \circ e_F \in F\}$ separates points of the space $b_F G$. Hence, if F is a ring which contains all constant functions and is closed in $C^\circ(G)$, then, by Stone-Weierstrass theorem ([8], Theorem 3.2.21), we have $C(b_F G) = \bar{F}$ and $F = \{g \circ e_F : g \in C(b_F G\}$.

Let (E, φ) be a g-compactification of a topological Ω-algebra G. If $C_E(G) = \{f \circ \varphi : f \in C(E)\}$, then $C_E(G)$ is the maximal subalgebra of the Banach algebra $C^\circ(G)$ such that $(E, \varphi) = (b_{C_E(G)}G, e_{C_E(G)})$. Denote $C^\Omega(G) = C_{\beta_\Omega G}(G)$.

Question A. Let G be a topological Ω-algebra and $F \subseteq C^\circ(G)$. Under which conditions $(b_F G, e_F)$ is an Ω-algebra ag-compactification of G and $e_F : G \to b_F G$ is a homomorphism?

Question B. Let G be a topological Ω-algebra and $f \in C(G)$. Under which conditions $f \in C^\Omega(G)$?

In [9] J. E. Hart and K. Kunen had formulated the next problems for the class E of all compact Ω-algebras:

Problem 1. To define the compactification $\beta_\Omega G$ as for groups directly with some notion of almost periodicity for functions ([9], Remark 2.4.1).

Problem 2. To give a method of construction of the Bohr compactification of an arbitrary algebra ([9], Remarks 2.4.1 and 3.1.6).

In this paper these problems are considered for arbitrary algebras.

We need the following elementary assertion.

Lemma 1. *Let (X, d) be a complete metric space. For a non-empty subset L of the space X the following assertions are equivalent:*

1. The closure $H = cl_X L$ of the set L in X is a compact subset of X.

2. For every $\epsilon > 0$ there exists a finite subset $S(\epsilon)$ of X such that $d(x, S(\epsilon)) = inf\{d((x, y) : y \in S(\epsilon)\} \leq \epsilon$ for each $x \in L$.

Proof. Follows immediately from Theorem 4.3.29 from [8], which affirms that a metrizable space Y is compact if and only if on Y there exists a metric ρ which is both totally bounded and complete. □

2 Almost periodicity on topological spaces

Fix a topological space G. Denote by $\Pi(G)$ the set of all continuous mappings $\varphi : G \to G$. Relative to the operation of composition $\varphi \circ \psi$, where $(\varphi \circ \psi)(x) = \varphi(\psi(x))$ for $\psi, \psi \in \Pi(G)$ and $x \in G$, the set $\Pi(G)$ is a semigroup with identity e_G, where $e_G(x) = x$ for each $x \in G$. A semigroup with identity is called a monoid. We say that $\Pi(G)$ is the monoid of all continuous translations of G. If $f \in C(G)$ and $\varphi \in \Pi(G)$, then $f_\varphi = f \circ \varphi$ ($f_\varphi(x) = f(\varphi(x))$ for any $x \in G$). Evidently, $f_\varphi \in C(G)$.

Fix a non-empty subset $P \subseteq \Pi(G)$. We say that P is a set of continuous translations of G. The set P is called a transitive set of translations of G if for any two points $x, y \in G$ there exists $\varphi \in P$ such that $\varphi(x) = y$. Obviously, the monoid $\Pi(G)$ is transitive.

For any function $f \in C(G)$ we put $P(f) = \{f_\varphi : \varphi \in P\}$. If $f \in C^\circ(G)$, then $P(f) \subseteq C^\circ(G)$.

Definition 1. A function $f \in C(G)$ is called a P-periodic function on a space G if the closure $\bar{P}(f)$ of the set $P(f)$ in the space $C(G)$ is a compact set.

Denote by P-$ap(G)$ the subspace of all P-periodic functions of a space G and P°-$ap(G) = P$-$ap(G) \cap C^{\circ}(G)$.

If the set P is finite, then P-$ap(G) = C(G)$.

Theorem 1. *Let P be a set of continuous translations of G. Then P-$ap(G)$ has the following properties:*

1. P-$ap(G)$ is a linear subspace of the linear space $C(G)$.

2. P-$ap(G)$ is a topological subring of the topological ring $C(G)$.

3. P-$ap(G)$ is a closed subspace of the complete metric space $C(G)$. In particular, P-$ap(G)$ is a complete metric space.

4. If $f \in C(G)$ is a constant function, then $f \in P$-$ap(G)$.

5. If $f \in P$-$ap(G)$, then for any $x \in G$ there exists a number $c(f,x) > 0$ such that $|f(\varphi(x))| \leq c(f,x)$ for any $\varphi \in P$.

6. If $f \in P$-$ap(G)$, $\psi \in \Pi(G)$ and $g(x) = f(\psi(x))$ for each $x \in G$, then $g \in P$-$ap(G)$. In particular, $P(f) \subseteq P$-$ap(G)$ and $f_{\psi} \in P$-$ap(G)$ for all $f \in P$-$ap(G)$ and $\psi \in \Pi(G)$.

7. P°-$ap(G)$ is a Banach algebra of continuous functions.

Proof. Fix $f, g \in P$-$ap(G)$. Since $\bar{P}(f)$, $\bar{P}(g)$, $-\bar{P}(f)$, $\bar{P}(f) + \bar{P}(g)$ and $\bar{P}(f) \cdot \bar{P}(g)$ are compact subsets of P-$ap(G)$ and $-\bar{P}(f) = \bar{P}(-f)$, $\bar{P}(f+g) \subseteq \bar{P}(f) + \bar{P}(g)$, $\bar{P}(f \cdot g) \subseteq \bar{P}(f) \cdot \bar{P}(g)$, then $-f, f+g, f \cdot g \in P$-$ap(G)$. Hence P-$ap(G)$ is a topological subring of the topological ring $C(G)$.

If $f \in P$-$ap(G)$ and $\lambda \in \mathbb{R}$, then the correspondence $t_{\lambda}(f) = \lambda f$ is a continuous mapping of $C(G)$ into $C(G)$ and $\bar{P}(\lambda f) = t_{\lambda}(\bar{P}(f))$. Hence $\lambda f \in P$-$ap(G)$ and P-$ap(G)$ is a linear subspace of the linear space $C(G)$.

Let $\{f_n \in P$-$ap(G) : n \in \mathbb{N}\}$ and $f = lim_{n \to \infty} f_n$. It is well known that $f \in C(G)$. Fix $\epsilon > 0$. There exist $n \in \mathbb{N}$ and a finite subset S of $C(G)$ such that:

– $|f_n(x) - f(x)| \leq \epsilon/3$ for each $x \in G$;

– $d(g, S) \leq \epsilon/3$ for each $g \in P(f_n)$.

Fix $\varphi \in P$. For a given $\epsilon > 0$ there exists $g \in S$ such that $|g(x) - f_n(\varphi(x)) \leq \epsilon/3 + \epsilon/3$ for each $x \in G$. Then $|g(x) - f(\varphi(x))| \leq |g(x) - f_n(\varphi(x))| + |f_n(\varphi(x)) - f(\varphi(x))| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$. Hence $d(h, S) \leq \epsilon$ for each $h \in P(f)$. By virtue of Lemma 1, $f \in P$-$ap(G)$. Hence, P-$ap(G)$ is a closed subspace of the complete metric space $C(G)$.

The Assertion 4 is obvious.

Assume that $f \in C(G)$, $b \in G$ and the set $\{f(\varphi(x)) : \varphi \in P\}$ is unbounded. Then there exists a sequence $\{\varphi_n \in P : n \in \mathbb{N}\}$ such that $|f(\varphi_1(b))| \geq 2 + |f(b)|$ and $|f(\varphi_{n+1}(b))| \geq 2 + |f(\varphi_n(b))|$ for each $n \in \mathbb{N}$. We put $g_n(x) = f(\varphi_n(x))$. Then $d(f, g_n) \geq 2n$ for each $n \in \mathbb{N}$. Hence $P(f)$ is an unbounded subset of $C(G)$ and $f \notin P$-$ap(G)$. The Assertion 5 is proved.

Fix $\psi \in \Pi(G)$. Consider the mapping $\Phi : C(G) \longrightarrow C(G)$, where $\Phi(h)(x) = h(\psi(x))$ for all $h \in C(G)$ and $x \in G$. We have $d(\Phi(f), \Phi(g)) \leq d(f,g)$ for all $f, g \in C(G)$. Fix now $f \in P$-$ap(G)$ and put $g(x) = f(\psi(x))$ for each $x \in G$. Let $\epsilon > 0$. Then there exists a finite subset S of $C(G)$ such that $d(h, S) \leq \epsilon$ for each $h \in P(f)$. We have $g_{\varphi}(x) = g(\varphi(x)) = f(\varphi(\psi(x)))$ for each $x \in G$ and each $\varphi \in P$.

Assume that $\varphi \in P$, $\delta > 0$, $h \in C(G)$ and $d(f_\varphi, h) \leq \delta$. Since $|f(\varphi(x) - h(x)| \leq \delta$ for any $x \in G$, we have $|f(\varphi(\psi(x)) - h(\psi(x))| \leq \delta$ for any $x \in G$. Hence, the set $\Phi(S)$ is finite and $d(h, \Phi(S)) \leq \epsilon$ for each $h \in P(g)$. By virtue of Lemma 1, the Assertion 6 is proved. The Assertion 7 is obvious. □

Corollary 1. *If P is a transitive set of translations of G, then any function $f \in P$-$ap(G)$ is bounded and P-$ap(G)$ is a Banach algebra of continuous functions.*

Theorem 2. *Let P be a set of continuous translations of G and F be a compact subset of the complete metric space P-$ap(G)$. Then the closure H of the set $P(F) = \cup\{P(f) : f \in F\}$ is a compact subset of the space P-$ap(G)$.*

Proof. Fix $\epsilon > 0$. There exists a finite subset S_1 of F such that $d(h, S_1) \leq \epsilon/2$ for each $h \in F$. For each $f \in F$ there exists a finite subset S_f of $P(f)$ such that $d(h, S_f) \leq \epsilon/2$ for each $h \in P(f)$. We put $S = \cup\{S_f : f \in S_1\}$. Fix $h \in F$ and $\varphi \in P$. There exists $f \in S_1$ such that $d(f, h) \leq \epsilon/2$. In continuation, there exists $g \in S_f$ such that $d(f_\varphi, g) \leq \epsilon/2$. Since $d(h_\varphi, f_\varphi) \leq d(h, f)$, we have $d(h_\varphi, g) \leq d(h_\varphi, f_\varphi) + d(f_\varphi, g) \leq \epsilon$. Hence $d(h, S) \leq \epsilon$ for each $h \in P(F)$. Lemma 1 completes the proof. □

Definition 2. Let G be a space and $\Gamma = \{P_\alpha : \alpha \in A\}$ be a non-empty family of non-empty subsets of the semigroup $\Pi(G)$. A function $f \in C(G)$ is called a Γ-periodic function of a space G if the function $f \in C(G)$ is P_α-periodic for any $\alpha \in A$.

Let G be a space and $\Gamma = \{P_\alpha : \alpha \in A\}$ be a non-empty family of non-empty subsets of the semigroup $\Pi(G)$. Denote by Γ-$ap(G)$ the subspace of all Γ-periodic functions of a space G. By definition, we have Γ-$ap(G) = \cap\{P_\alpha$-$ap(G) : \alpha \in A\}$.

From Theorem 1 follows

Corollary 2. *Let G be a space and $\Gamma = \{P_\alpha : \alpha \in A\}$ be a non-empty family of non-empty subsets of the semigroup $\Pi(G)$. Then Γ-$ap(G)$ has the following properties:*

1. Γ-$ap(G)$ is a linear subspace of the linear space $C(G)$.

2. Γ-$ap(G)$ is a topological subring of the topological ring $C(G)$.

3. Γ-$ap(G)$ is a closed subspace of the complete metric space $C(G)$. In particular, Γ-$ap(G)$ is a complete metric space.

4. If $f \in C(G)$ is a constant function, then $f \in \Gamma$-$ap(G)$.

5. If $f \in \Gamma$-$ap(G)$, $\psi \in \Pi(G)$ and $g(x) = f(\psi(x))$ for each $x \in G$, then $g \in \Gamma$-$ap(G)$. In particular, $f_\psi \in \Gamma$-$ap(G)$ for all $f \in \Gamma$-$ap(G)$ and $\psi \in \Pi(G)$.

6. Γ°-$ap(G)$ is a Banach algebra of continuous functions.

Let G be a space and $\Gamma = \{P_\alpha : \alpha \in A\}$ be a non-empty family of non-empty subsets of the semigroup $\Pi(G)$. A finite oriented set $(\alpha_1, \alpha_2, ..., \alpha_n)$, where $\alpha_1, \alpha_2, ..., \alpha_n \in A$ and $n \geq 1$, is called an A-cortege of the length n. For any A-cortege $\beta = (\alpha_1, \alpha_2, ..., \alpha_n)$ we put $B_\beta = \{\varphi_{\alpha_1} \circ \varphi_{\alpha_2} \circ ... \circ \varphi_{\alpha_n} : (\varphi_{\alpha_1}, \varphi_{\alpha_2}, ...\varphi_{\alpha_n}) \in P_{\alpha_1} \times P_{\alpha_2} \times ... \times P_{\alpha_n}\}$. Denote by A_∞ the set of all A-corteges and $\Gamma_\infty =$

$\{B_\beta : \beta \in A_\infty\}$. Then Γ_∞ is a non-empty family of non-empty subsets of the monoid $\Pi(G)$, $A \subseteq A_\infty$ and $\cup\{B_\beta : \beta \in A_\infty\}$ is a semigroup of the monoid $\Pi(G)$.

From Theorem 2 follows

Corollary 3. *Let G be a space and $\Gamma = \{P_\alpha : \alpha \in A\}$ be a non-empty family of non-empty subsets of the semigroup $\Pi(G)$. Then Γ_∞-ap$(G) = \Gamma$-ap(G).*

3 Almost periodicity on dynamical systems

A topological monoid is a topological space A with a continuous mapping $\cdot : A \times A \to A$ for which there exists a point $1 \in A$ such that $1 \cdot x = x \cdot 1 = x$ and $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for each $x, y, z \in X$. The element 1 is the unity of monoid A and we say that $xy = x \cdot y$ is the product of x, y.

A dynamical system is a triple (G, S, m), where S is a topological monoid, G is a Tychonoff space and $m : S \times G \to G$ is a continuous action on G, i. e. $m(s, m(t, x)) = m(st, x)$ and $m(1, x) = x$ for all $s, t \in S$ and $x \in G$. In theory of finite state machines and in automata theory the dynamical system (G, S, m) is called a semiautomaton, where S is called the input alphabet, G is called the set of states and m is the transition function.

Remark 1. Let G be a non-empty space. Then the semigroup $\Pi(G)$ is a monoid. Consider the evaluation action $e_G : \Pi(G) \times G \longrightarrow G$, where $e_G(\varphi) = \varphi(x)$ for all $x \in G$ and $\varphi \in \Pi(G)$. If S is a submonoid of the monoid $\Pi(G)$ and $m = e_G|S \times G$, then (G, S, m) is a dynamical system. In particular, $(G, \Pi(G), e_G)$ is a dynamical system.

Fix a discrete monoid S and a dynamical system (G, S, m). Then G is a topological universal algebra of the signature S. All operations from S are unary.

For any continuous real-valued function $f : G \to \mathbb{R}$ and any $s \in S$ we consider the function $f_s : G \to \mathbb{R}$, where $f_s(x) = f(m(s, x))$ for each $x \in G$, and put $S(f) = \{f_s : s \in S\}$.

A continuous function $f : X \to \mathbb{R}$ is called an almost periodic function of the dynamical system (G, S, m) if the closure $cl_{C(G)}S(f)$ is a compact subset of $C(G)$. Denote by $S(m)$-*ap*(G) the class of all almost periodic functions on G and $S(m)^\circ$-*ap*$(G) = S(m)$-*ap*$(G) \cap C^\circ(G)$.

Remark 2. Any element $s \in S$ generates the continuous mapping $m_s : G \longrightarrow G$, where $m_s(x) = m(s, x)$ for any point $x \in G$. We put $S_G = \{m_s : s \in G\}$. Then S_G is a submonoid of the monoid $\Pi(G)$. By construction, $f_s = f_{m_s}$ for all $s \in S$ and $f \in C(G)$. Hence $S(f) = S_G(f)$ for any function $f \in C(G)$. In particular, $S(m)$-*ap*$(G) = S_G$-*ap*(G).

The continuous action $m : S \times G \longrightarrow G$ generates the continuous action $m_C : S \times C(G) \longrightarrow C(G)$, where $m_C(s, f) = f_s$ for all $s \in S$ and $f \in C(G)$. Hence $(S, C(G), m_C)$ is a dynamical system generated by the continuous action $m : S \times G \longrightarrow G$. From Theorem 2 it follows that $m_C(S$-*ap*$(G) = S(m)$-*ap*(G). Therefore

$(S\text{-}ap(G), S, m_C)$ is a dynamical system too, generated by the continuous action $m : S \times G \longrightarrow G$.

From Theorem 1 follows

Corollary 4. *Let G be a space, S be a discrete monoid and (G, S, m) be a dynamical system. Then the space S-ap(G) has the following properties:*

1. $S(m)$-ap(G) is a linear subspace of the linear space $C(G)$.

2. $S(m)$-ap(G) is a topological subring of the topological ring $C(G)$.

3. $S(m)$-ap(G) is a closed subspace of the complete metric space $C(G)$. In particular, S-ap(G) is a complete metric space.

4. If $f \in C(G)$ is a constant function, then $f \in S$-ap(G).

5. If $f \in S(m)$-ap(G), $\psi \in \Pi(G)$ and $g(x) = f(\psi(x))$ for each $x \in G$, then $g \in S(m)$-ap(G). In particular, $S(f) \subseteq S(m)$-ap(G) for any $f \in S(m)$-ap(G).

6. $S(m)^\circ$-ap(G) is a Banach algebra of continuous functions.

If ρ is a pseudometric on G, $x \in G$ and $r > 0$, then $B(x, \rho, r) = \{y \in G : \rho(x, y) < r\}$ is the r-ball with the center x. The pseudometric ρ is continuous if the sets $B(x, \rho, r)$ are open in G.

A pseudometric ρ on G is totally bounded if for any real number $r > 0$ there exists a finite subset F of G such that $\rho(x, F) = min\{\rho(x, y) : y \in F\} < r$ for each $x \in G$.

A pseudometric ρ on (G, S, m) is totally S-bounded if it is totally bounded and for any real number $r > 0$ there exists a finite subset L of S such that: for each $s \in S$ there exists $s_r \in L$ such that $\rho(m(s, x), m(s_r, x)) < r$ for each $x \in G$.

A pseudometric $\rho : G \times G \to \mathbb{R}$ is S-invariant on (G, S, m) if ρ is continuous, $\rho(x, y) < \infty$ and $\rho(m(s, x), m(s, y)) \leq \rho(x, y)$ for all $x, y \in G$ and $s \in S$.

If $f : G \to \mathbb{R}$ is a function, then we put $\rho_f(x, y) = sup\{|f_s(x) - f_s(y)| : s \in S\}$ for all $x, y \in G$.

Theorem 3. *Fix a dynamical system (G, S, m) and $f \in S$-ap(G). Then:*

1. ρ_f is an S-invariant pseudometric.

2. ρ_f is a continuous pseudometric on G.

3. ρ_f is a totally bounded pseudometric if and only if the function f is bounded.

4. ρ_f is a totally S-bounded pseudometric provided the function f is bounded and for any real number $r > 0$ there exists a finite subset L of S such that: for each $s \in S$ there exists $s_r \in L$ such that $|f(m(ts, x) - f(m(ts_r, x))| < r$ for each $x \in G$ and every $t \in S$.

Proof. 1. Fix two points $x, y \in G$. By virtue of the Assertion 5 from Theorem 1, there exists a number $c > 0$ such that $|f_s(x))| \leq c$ and $|f_s(y))| \leq c$ for any $s \in S$. Hence $\rho_f(x, y) \leq 2c < \infty$. Let $\mu \in S$ and $g = f_\mu$. Then $g \in S$-ap(G), $g_s = f_{s\mu}$ for any $s \in G$ and $\rho_f(m(s, x), m(s, y)) = sup\{|g_s(x) - g_s(y)| : s \in S\} = sup\{|f_{s\mu}(x) - f_{s\mu}(y)| : s \in S\} \leq sup\{|f_s(x) - f_s(y)| : s \in S\} = \rho_f(x, y)$. Hence the pseudometric ρ_f is S-invariant.

2. Now fix a number $r > 0$ and a point $b \in G$. Then there exists a finite subset L of S such that $1 \in L$ and for each $s \in S$ there exists $l(s) \in L$ such

that $d(f_s, f_{l(s)}) < r/3$. Since the set L is finite, the set $U(b, L, r) = \{x \in G : |f_s(x) - f_s(b)| < r/3, s \in L\}$ is open in G. Hence $|f_s(x) - f_s(b)| \leq |f_s(x) - f_{s(l)}(x)| + |f_{s(l)}(x) - f_{s(l)}(b)| + |f_{s(l)}(b) - f_s(b)| < r$ for all $s \in L$ and $x \in U(b, L, r)$. Therefore $U(b, L, r) \subseteq B(b, \rho_f, r)$ and $B(b, \rho_f, r)$ is an open subset of G. Thus ρ_f is a continuous pseudometric on G. By construction, $\rho_f(m(s, x), m(l(s), x)) = sup\{|f_t(m(s, x)) - f_t m(l(s), x)| : t \in S\} = sup\{\rho_f((m(t \cdot s, x), m(t \cdot l(s), x)) : t \in S\}$.

3. Assume that the function f is bounded and $r > 0$. There exists a finite subset L of S such that $1 \in L$ and for each $s \in S$ there exists $l(s) \in L$ such that $d(f_s, f_{l(s)}) < r/3$. Since the functions f_s are bounded and the set L is finite, there exists a finite subset F of G such that for each $x \in G$ there exists $x(f) \in F$ such that $|f_s(x) - f_s(x(f))| < r/3$ for each $s \in L$. Hence $\rho_f(x, F) < r$ for each $x \in G$ and ρ_f is a totally bounded pseudometric.

4. Fix $b \in G$. Since $|f(x) - f(b)| \leq \rho_f(b, x)$ the function f is bounded if and only if the pseudometric ρ_f is bounded (i.e. $sup\{\rho_f(x, y) : x, y \in G\} < \infty$).

5. Assume that the function f is bounded and for any real number $r > 0$ there exists a finite subset L_r of S such that: for each $s \in S$ there exists $s_r \in L_r$ such that $|f(m(ts, x) - f(m(ts_r, x))| < r$ for each $x \in G$ and every $t \in S$.

Fix $r > 0$ and $s \in S$. Then $\rho_f(m(s, x), m(s_r, x)) = sup\{|f(m(ts, x) - f(m(ts_r, x))| : t \in S\} \leq r$. The proof is complete. □

If ρ is a bounded pseudometric on G and $a \in G$, then we put $f_{(\rho,a)}(x) = \rho(a, x)$ for any $x \in G$.

Theorem 4. *If ρ is a totally S-bounded S-invariant pseudometric on a dynamical system (G, S, m) and $a \in G$, then $f_{(\rho,a)} \in S(m)$-$ap(G)$ and the function $f_{(\rho,a)}$ is bounded for each $a \in G$.*

Proof. Fix $a \in G$ and $r > 0$. Let $g = f_{(\rho,a)}$. We have $g_s(x) = \rho(a, m(s, x))$ for all $x \in G$ and $s \in S$. Obviously, the function g is bounded. By assumption, there exists a finite subset L of S such that: for each $s \in S$ there exists $s_r \in L$ such that $\rho(m(s, x), m(s_r, x)) < r$ for each $x \in G$. We have $|g_s(x) - g_{s_r}(x)| = |\rho(a, m(s, x)) - \rho(a, m(s_r, x))| \leq \rho(m(s, x), m(s_r, x)) < r$. By virtue of Lemma 1, the assertion is proved. □

Theorem 5. *Fix a dynamical system (G, S, m). Then there exist a dynamical system $(\beta_{ap(S,m)}G, S, m_G)$ and a continuous mapping $\varphi : G \longrightarrow ap_{(S,m)}G$ such that:*

1. $\beta_{ap(S,m)}G$ is a compact space and the set $\varphi(G)$ is dense in $b_{ap(S,m)}G$.

2. φ is a homomorphism, i.e. $\varphi(m(s, x) = m(s, \varphi(x))$ for all $s \in S$ and $x \in G$.

3. $S(m)^\circ$-$ap(G) = \{g \circ \varphi : g \in C(\beta_{ap(S,m)}G)\}$.

4. $C(\beta_{ap(S,m)}G)\} = S(m)$-$ap(\beta_{ap(S,m)}G)\}$.

5. The topology of the space $\beta_{ap(S,m)}G$ is induced by the family of all S-invariant pseudometrics on the dynamical system $(\beta_{ap(S,m)}G, S, m_G)$.

Proof. Let $F = S(m)^\circ$-$ap(G)$. Consider the mapping $e_F : G \to \mathbb{R}^F$, where $e_F(x) = (f(x) : f \in F)$. Denote by $b_F G = \beta_{ap(S,m)}G$ the closure of the set $e_F(G)$ in $\mathbb{R}^F$. We put $\varphi = e_F$. Then $(b_F G, e_F)$ is a compactification of G. For any $f \in F$ consider the pseudometric $\rho_f(x,y) = sup\{|f_s(x) - f_s(y)| : s \in S\}$ for all $x, y \in G$. By virtue of Theorem 3, the pseudometric ρ_f is continuous, stable and totally bounded on (G, S, m). Since $|f(y) - f(x)| \leq \rho_f(x,y)$ for all $x, y \in G$, there exists a continuous pseudometric $\bar{\rho_f}$ on $b_F G$ such that $\rho_f(x,y) = \bar{\rho_f}(\varphi(x), \varphi(y))$ for all $x, y \in G$. We say that $\bar{\rho_f}$ is the continuous extension of ρ_f on $b_F G$. The topology of the compact space is induced by the pseudometrics $\{\bar{\rho_f} : f \in F\}$.

For every $f \in F$ there exists a unique function $\bar{f} \in C(b_F G)$ such that $f = \bar{f} \circ \varphi$. Hence $\bar{F} = \{\bar{f} : f \in F\}$ is a closed subalgebra of the Banach algebra $C(b_F G)$.

Fix $s \in S$. The mapping $m_s : G \to G$, where $m_s(x) = m(s,x)$ for every $x \in G$ is continuous. If $x \in G$, then we put $\mu_s(\varphi(x)) = \varphi(m_s(x))$. For $x, y \in G$ with $\varphi(x) = \varphi(y)$ we have $0 \leq \rho_f(m_s(x), m_s(y)) \leq \rho_f(x,y) = 0$ for any $f \in F$ and $\varphi(m_s(x)) = \varphi(m_s(y))$. Therefore μ_s is a single-valued continuous mapping of $\varphi(G)$ into $\varphi(G)$.

We have $\rho_f(m_s(x), m_s(y)) \leq \rho_f(x,y)$ for all $x, y \in G$. Hence the mapping m_s is uniformly continuous for every pseudometric $\bar{\rho_f}$, $f \in F$. Therefore there exists a continuous extension $\nu_s : b_F G \longrightarrow b_F G$ of μ_s. By construction, $\nu_s \circ \nu_t = \nu_{s \cdot t}$. We prove that $(\beta_{ap(S,m)}G, S, m_G)$, where $m_G(s,x) = \nu_s(x)$ for each $x \in b_F G = \beta_{ap(S,m)}G$, is a dynamical system.

By construction, φ is a homomorphism.

The mapping $\psi : F \longrightarrow C(b_F G)$, where $\psi(f) = \bar{f}$ for each $f \in F$ is an isometrical embedding. Hence $\psi(F) \subseteq S(m)\text{-}ap(\beta_{ap(S,m)}G)\}$. It is obvious that $g \circ \varphi \in F$ for any $g \in S(m)\text{-}ap(\beta_{ap(S,m)}G)\}$. Therefore $S(m)^{\circ}\text{-}ap(G) = \{g \circ \varphi : g \in S(m) - ap(\beta_{ap(S,m)}G)\}$.

Since $\{\bar{f} : f \in F\} = \{g|\varphi(G) : g \in S(m)\text{-}ap(\beta_{ap(S,m)}G)\}$, by Stone-Weierstrass theorem ([8], Theorem 3.2.21), we have $S(m)\text{-}ap(\beta_{ap(S,m)}G) = C(\beta_{ap(S,m)}G)$. The topology of the space $\beta_{ap(S,m)}G$ is induced by the family of S-invariant pseudometrics $\{\rho_g : g \in C(\beta_{ap(S,m)}G)\}$ on the dynamical system $(\beta_{ap(S,m)}G, S, m_G)$. The proof is complete. $\square$

Remark 3. We say that the dynamical system $(\beta_{ap(S,m)}G, S, m_G)$ is the maximal a-compactification of the dynamical system (G, S, m).

4 Almost periodicity on universal algebras

Fix a discrete signature $\Omega = \oplus\{\Omega_n : n \in N = \{0, 1, 2, ...\}\}$, where $\{\Omega_n : n \in N\}$ is a non-empty family of pairwise disjoint discrete spaces.

Let $P(\Omega)$ be a minimal set of operations on Ω-algebras for which:

P1. $\Omega \subseteq P(\Omega)$.

P2. If $n \geq 1$, $\omega \in \Omega_n$, $p_1, ..., p_n \in P(E)$, p_i is an m_i-ary operation and $m = m_1 + ... + m_n$, then $p = \omega(p_1, ..., p_n)$ is an m-ary operation, $p(x_1, ..., x_m) = \omega(p_1(x_1, ..., x_m), ..., p_n(x_{m-m_n+1}, ..., x_m))$.

P3. If $u_0(x) = x$ for any Ω-algebra G and every $x \in G$, then $u_0 \in P(\Omega)$.

The set $P(\Omega)$ is called the set of Ω-polynomials. If G is a topological Ω-algebra and $p \in P(\Omega)$ is an n-ary polynomial, then $p : G^n \to G$ is a continuous operation.

Let $\lambda : G^n \to G$ be an n-ary operation. If $n = 0$, then we put $\lambda(x) = \lambda(G^0)$ for each $x \in G$ and $T_\lambda(G) = \{\lambda\}$. If $n = 1$, then $T_\lambda(G) = \{\lambda\}$. Let $n \geq 2$ and $1 \leq i \leq n$. For every $a = (a_1, ..., a_n) \in G^n$ we put $t_{ia\lambda}(x) = \lambda(a_1, ..., a_{i-1}, x, a_{i+1}, ..., a_n)$ for each $x \in G$. We put $T_{i\lambda}(G) = \{t_{ia\lambda} : a \in G^n\}$ and $T_\lambda(G) = \cup\{T_{i\lambda}(G) : i \leq n\}$. Therefore $T_\lambda(G)$ is a set of translations on a space G. If λ is a continuous operation, then $T_\lambda(G) \subseteq \Pi(G)$.

Now we put $T_\Omega(G) = \cup\{T_\omega(G) : \omega \in P(\Omega)\}$ for any topological Ω-algebra G. By construction, $T_\Omega(G)$ is a monoid of continuous translations of the space G and $T_\Omega(G) \subseteq \Pi(G)$.

If G is a topological Ω-algebra and $m_\Omega = e_G|T_\Omega(G) \times G$, then $(G, T_\Omega(G), m_\Omega)$ is a dynamical system, generated by the structure of Ω-algebra on G.

Definition 3. Let G be a topological Ω-algebra. The set Ω-$AP(G) = T_\Omega(G)(m_\Omega)$-$ap(G)$ is called the algebra of almost periodic continuous functions on the topological Ω-algebra G.

All statements proved in the above two Sections are true for almost periodic continuous functions on the topological Ω-algebras. The set Ω°-$AP(G) = C^\circ(G) \cap (\Omega$-$AP(G))$ is a Banach algebra of continuous functions on G.

Definition 4. An Ω-algebra G is called Ω-finite if there exists a finite subset $F \subseteq P(\Omega)$ such that $T_\Omega(G) = \cup\{T_\omega(G) : \omega \in F\}$.

Any finite Ω-algebra is Ω-finite. If Ω is a structure of a semigrup, or of a monoid, or a group on G, then G is is Ω-finite.

Definition 5. An Ω-algebra G is called a right (left) Mal'cev algebra if there exists a ternary operation $p \in P(\Omega)$ such that $p(x, x, y) = y$ (respectively $p(y, x, x) = y$) for all $x, y \in G$. If $p(x, x, y) = p(y, x, x) = y$, then G is called a Mal'cev algebra [4, 10].

Proposition 1. *Let G be a right (left) Mal'cev topological Ω-algebra. Then the monoid $T_\Omega(G)$ is transitive on G. Moreover, any almost periodic function $f \in \Omega$-$AP(G)$ is bounded and Ω-$AP(G)$ is a Banach algebra of continuous functions on G.*

Proof. Assume that $p \in P(\Omega)$ is a ternary operation and $p(x, x, y) = y$ for all $x, y \in G$. Fix $a, b \in G$. If $\varphi(x) = p(x, a, b)$, then $\varphi \in T_\Omega(G)$ and $\varphi(a) = b$. Hence the monoid $T_\Omega(G)$ is transitive on G. Corollary 1 completes the proof. □

A pseudometric $\rho : G \times G \to \mathbb{R}$ is stable on a topological Ω-algebra G if ρ is continuous, $\rho(x, y) < \infty$ and $\rho(\omega(x_1, ..., x_n), \omega(y_1, ..., y_n)) \leq \Sigma\{\rho(x_i, y_i) : i \leq n\}$ for all $x_1, y_1, ..., x_n, y_n \in G$, $n \geq 1$ and $\omega \in \Omega$.

If ρ is a stable pseudometric on a topological Ω-algebra G, $n \leq 1$ and $p \in P(\Omega)$ is an n-ary polynomial, then $\rho(p(x_1, ..., x_n), p(y_1, ..., y_n)) \leq \Sigma\{\rho(x_i, y_i) : i \leq n\}$ for all $x_1, y_1, ..., x_n, y_n \in G$.

In [5] the following theorem was proved:

Theorem 6. *Let ρ be a continuous pseudometric on a topological Ω-algebra G. The pseudometric ρ is stable if and only if it is $T_\Omega(G)$-invariant.*

From Theorems 6 and 5 follows

Corollary 5. *Fix a topological Ω-algebra G. Then there exist an Ω-algebra $\beta_{ap(\Omega)}G$ and a continuous homomorphism $\alpha_G : G \longrightarrow \beta ap_(\Omega)G$ such that:*

1. $\beta_{ap(\Omega)}G$ is a compact Ω-algebra and the set $\alpha_G(G)$ is dense in $\beta_{ap(\Omega)}G$.

2. Ω°-$AP(G) = \{g \circ \varphi : g \in C(\beta_{ap(\Omega)}G)\}$.

3. $C(\beta_{ap(\Omega)}G)\} = \Omega$-$AP(\beta_{ap(S,m)}G)$.

4. The topology of the space $\beta_{ap(\Omega)}G)$ is induced by the family of all stable pseudometrics on the topological Ω-algebra $\beta_{ap(\Omega)}G)$.

5. The a-compactification $(\beta_{ap(\Omega)}G, \alpha G) = (b_F G, e_F)$, where $F = \Omega^\circ$-$AP(G)$.

Remark 4. We say that the topological Ω-algebra $\beta_{ap(\Omega)}G$ is the maximal almost periodic a-compactification of the topological Ω-algebra G.

Lemma 2. *Let G be a topological Ω-algebra and ρ be a stable totally bounded pseudometric on G. If $\omega \in P(\Omega)$, $c \in G$ and $h(x) = \rho(c,x)$ for any $x \in G$, then the function h is bounded and the closure of the set $\{h_\varphi : \varphi \in T_\omega(G)\}$ in $C^\circ(G)$ is a compact set.*

Proof. Since ρ is totally bounded, by construction, $h \in C^\circ(G)$. Fix $\epsilon > 0$. If ω is n-ary polynomial and $n \le 1$, then the assertion of Lemma is obvious. Assume that $n \ge 2$ and ω is an n-ary polynomial. There exists a finite subset $L \subseteq G$ such that $\rho(x, L) < \epsilon/2$ for any $x \in G$. For every $i \le n$ we put $T_{(i,\omega,L)} = \{t_{ia\lambda} : a = (a_1, ..., a_n) \in L^n\}$ and $T_{(\omega,L)}(G) = \cup\{T_{(i,\omega,L)}(G) : i \le n\}$. Obviously, the set $T_{(\omega,L)}(G)$ is finite. Fix $\varphi \in T_\omega(G)$. Then $\varphi(x) = \omega(x_1, ..., x_{i-1}, x, x_{i+1}, ..., x_n)$ for some $i \le n$ and $x = (x_1, ..., x_n) \in G^n$. There exists $a = (a_1, ..., a_n) \in L^n$ such that $\rho(x_j, a_j) < \epsilon/n$ for each $j \le n$. Let $\psi = t_{ia\omega}$. Then $\psi \in T_{(\omega,L)}(G)$ and $h_\varphi(x) - h_\psi(x) < \Sigma\{\rho(x_j, a_j) : j \le n, j \ne i\} < \epsilon$. Lemma 1 completes the proof. □

Lemma 3. *Let G be a a compact topological Ω-algebra, $n \in \mathbb{N}$ and $\omega \in P(\Omega)$ be an n-ary polynomial. If $h \in C(G)$, then the set $\{h_\varphi : \varphi \in T_\omega(G)\}$ in $C^\circ(G)$ is a compact set.*

Proof. If $n \le 1$, then the assertion of Lemma is obvious. Assume that $n \ge 2$. Let $h \in C(G)$. Fix $i \le n$. Let $G_k = G$ for any k and $Z_i = \Pi\{G_j : j \le n, j \ne i\}$. For any $z = (z_1, ..., z_{i-1}, ..., z_{i+1}, ..., z_n) \in Z_i$ we put $\Psi_i(z)(x) = h(\omega(z_1, ..., z_{i-1}, x, z_{i+1}, ..., z_n))$ for each $x \in G$. Then $\Psi_i : Z_i \longrightarrow C(G)$ is a continuous mapping. Since $\Psi_i(Z_i) = \{h_\psi : \psi \in T_{i\omega}\}$, the set $\{h_\psi : \psi \in T_{i\omega}\}$ is compact. Hence the set $\{h_\varphi : \varphi \in T_\omega(G)\}$ is compact too. □

Corollary 6. *Let G be an Ω-finite topological Ω-algebra and ρ be a stable totally bounded pseudometric on G. If $c \in G$ and $h(x) = \rho(c,x)$ for any $x \in G$, then $h \in \Omega^\circ$-$AP(G)$.*

Corollary 7. *Let G be a compact Ω-finite topological Ω-algebra. Then:*

1. The topology of G is induced by a family of stable pseudometrics.

2. Ω°-$AP(G) = C(G)$.

Corollary 8. *Let G be an Ω-finite topological Ω-algebra. For any bounded continuous pseudometric ρ on G we put $C(G,\rho) = \{a + b\cdot\rho(z,x) : z \in G, a, d \in \mathbb{R}\}$. Then the set $\cup\{C(G,\rho) : \rho$ is a totally bounded stable pseudometric on $G\}$ is a dense subset of the Banach algebra Ω°-$AP(G)$.*

5 Weakly almost periodic functions on algebras

Fix a discrete signature $\Omega = \oplus\{\Omega_n : n \in N = \{0, 1, 2, ...\}\}$.

Definition 6. Let G be a topological Ω-algebra. A function $f \in C(G)$ is called a weakly almost periodic function on G if the closure of the set $\{f_t = f \circ t : t \in T_\omega(G)\}$ in $C(G)$ is compact for every $\omega \in P(\Omega)$.

If $\Gamma(\Omega) = \{T_\omega(G) : \omega \in P(\Omega)\}$, then Ω-$wAP(G) = \Gamma(\Omega)$-$ap(G)$ is the algebra of weakly almost periodic continuous functions on the topological Ω-algebra G. Hence Corollary 2 is true for the algebra of weakly almost periodic continuous functions on the topological Ω-algebra G. Moreover, if $\Gamma_0(\Omega) = \{T_\omega(G) : \omega \in \Omega\}$, then from Corollary 3 it follows that Ω-$wAP(G) = \Gamma_0(\Omega)$-$ap(G)$. Obviously, Ω-$AP(G) \subseteq \Omega$-$wAP(G)$. Let Ω°-$wAP(G) = \Omega$-$wAP(G) \cap C^\circ(G)$.

Theorem 7. *Let G be a a compact topological Ω-algebra. Then Ω-$wAP(G) = C(G)$.*

Proof. Follows from Lemma 3. □

Example 1. Let G be the compact space of all complex numbers z with $|z| = 1$. Relatively to the multiplicative operation $\{\cdot\}$ and inverse operation $\{^{-1}\}$ the space G is a compact commutative group with the unite 1. Let $g : G \longrightarrow G$ be a homeomorphism and $\omega_g(x,y) = x \cdot y$ for all $x, y \in G$. Then (G, ω_g) is a topological quasigroup. Denote by $P(g)$ the translations of the topological quasigroup (G, ω_g). Obviously, $g \in P(g)$.

In [6] such homeomorphism g_0 was constructed for which only constant functions are continuous almost periodic on (G, ω_{g_0}) and every stable pseudometric ρ on (G, ω_{g_0}) is trivial ($\rho(x,y) = 0$ for all $x, y \in G$). Let $\Omega_1 = \{^{-1}, g_0, g_0^{-1}\}$, $\Omega_2 = \{\cdot\}$ and $\Omega = \Omega_1 \cup \Omega_2$. Then $\omega_{g_0}, ^{-1}, g_0, g_0^{-1} \in P(\Omega)$ and only constant functions are continuous almost periodic on the Ω-algebra G. In particular, every stable pseudometric ρ on the Ω-algebra G is trivial. Therefore the Ω-algebra G is not Ω-finite. Since G is a compact space, then, by virtue of Theorem 7, we have Ω-$wAP(G) = C(G)$.

Definition 7. Let $\{\rho_\mu : \mu \in M\}$ be a family of pseudometrics on an Ω-algebra G. The family $\{\rho_\mu : \mu \in M\}$ is called a stable set of pseudometrics if the set M is non-empty and for every $\alpha \in M$, every $n \geq 1$ and every $\lambda \in \Omega_n$ there exists $\beta = \beta(\lambda,\alpha) \in M$ such that $\rho_\alpha(x_1, y_1) \leq \rho_\beta(x_1, y_1)$ and $\rho_\alpha(\lambda(x_1, ...x_n), \lambda(y_1, ..., y_n)) \leq \sum\{\rho_\beta(x_i, y_i) : i \leq n\}$ for all $x_1, y_1, ..., x_n, y_n \in G$.

Remark 5. Let $T(R)$ be the topology induced by a stable set of pseudometrics $R = \{\rho_\mu : \mu \in M\}$ on an Ω-algebra G. Then for each $n \geq 1$ and $\omega \in \Omega_n$ the operation ω is continuous relative to the topology $T(R)$.

Lemma 4. *Let $\{\rho_\mu : \mu \in M\}$ be a stable net of pseudometrics on an Ω-algebra G. Then for every $\alpha \in M$, every $n \geq 1$ and every n-ary polynomial $\lambda \in P(\Omega_)$ there exists $\beta = \beta(\lambda, \alpha) \in M$ such that $\rho_\alpha(x_1, y_1) \leq \rho_\beta(x_1, y_1)$ and $\rho_\alpha(\lambda(x_1, ...x_n), \lambda(y_1, ..., y_n)) \leq \sum\{\rho_\beta(x_i, y_i) : i \leq n\}$ for all $x_1, y_1, ..., x_n, y_n \in G$.*

Proof. Assume that $n, m_1, m_2, ..., m_n \geq 1$, $\lambda \in \Omega_n$, $p_1, p_2, ..., p_n \in P(\Omega)$ and for each $i \leq n$ the polynomial p_i is m_i-ary and for every $\alpha \in M$ there exists $\beta_i = \beta(p_i, \alpha) \in M$ such that $\rho_\alpha(x_1, y_1) \leq \rho_{\beta_i}(x_1, y_1)$ and $\rho_\alpha(p_i(x_1, ...x_{m_i}), p_i(y_1, ..., y_{m_i})) \leq \sum\{\rho_{\beta_i}(x_i, y_i) : i \leq m_i\}$ for all $x_1, y_1, ..., x_{m_i}, y_{m_i} \in G$. Put $p = \lambda(p_1, ..., p_n)$ and $m = m_1 + ... + m_n$. Then p is m-ary polynomial.

Fix $\alpha \in M$. We put $\alpha_1 = \beta(p_1, \alpha)$, $\alpha_2 = \beta(p_2, \alpha_1)$,..., $\alpha_n = \beta(p_n, \alpha_{n-1})$ and $\beta = \beta(\lambda, \alpha_n)$. Then $\rho_\alpha(x_1, y_1) \leq \rho_\beta(x_1, y_1)$ and $\rho_\alpha(p(x_1, ...x_m), p(y_1, ..., y_m)) \leq \sum\{\rho_\beta(x_i, y_i) : i \leq m\}$ for all $x_1, y_1, ..., x_m, y_m \in G$. The proof is complete. □

Lemma 5. *Let A be a non-empty set and $\{\rho_\mu : \mu \in M_\alpha\}$ be a stable set of pseudometrics on an Ω-algebra G for each $\alpha \in A$. If $M = \cup\{M_\alpha : \alpha \in A\}$, then the family $\{\rho_\mu : \mu \in M\}$ is a stable set of pseudometrics on the Ω-algebra G.*

Proof. It is obvious. □

Proposition 2. *Let $R = \{\rho_\mu : \mu \in M\}$ be a stable set of continuous totally bounded pseudometrics on a topological Ω-algebra G. Then there exist a compact topological Ω-algebra G/R, a continuous homomorphism $p_R : G \longrightarrow G/R$ and a stable set of continuous totally bounded pseudometrics $\bar{R} = \{\bar{\rho}_\mu : \mu \in M\}$ on a topological Ω-algebra G/R such that:*

1. *The topology of the space G/R is induced by the family of pseudometrics $\bar{R}$.*
2. *$\bar{\rho}_\mu(p_R(x), p_R(y)) = \rho_\mu(x, y)$ for all $x, y \in G$ and $\mu \in M$.*
3. *$(G/R, p_R)$ is an a-compactification of the topological Ω-algebra G.*

Proof. Fix $\mu \in M$. Then there exists a metric space (Y_μ, d_μ) and a mapping $p_\mu : G \to Y_\mu$ of G onto Y_μ such that $d_\mu(p_\mu(x), p_\mu(y)) = \rho_\mu(x, y)$ for all $x, y \in G$. Denote by $(G_\mu, \bar{d_\mu})$ the completion of the metric space (Y_μ, d_μ). Since the metric d_μ is totally bounded, G_μ is a compact space.

Consider the continuous mapping $p_R : G \longrightarrow \Pi\{G_\mu : \mu \in M\}$, where $p_R(x) = (p_\mu(x) : \mu \in M\}$ for each point $x \in G$. We put $Y = p_R(G)$ and by G/R denote the closure of Y in the compact space $\Pi\{G_\mu : \mu \in M\}$. For each $\mu \in M$ on G/R there exists a continuous pseudometric $\bar{\rho_\mu}$ such that $\bar{\rho}_\mu(p_R(x), p_R(y)) = \rho_\mu(x, y)$ for all $x, y \in G$.

Fix $n \geq 1$ and $\omega \in \Omega_n$. Let $a = (a_1, ..., a_n) \in Y^n$. Fix $b = (b_1, ..., b_n) \in G^n$ such that $p_R(b_i) = a_i$ for any $i \leq n$. We put $\omega(a) = p(\omega(b))$. We affirm that the mapping $\omega : Y^n \longrightarrow Y$ is single-valued. Let $c = (c_1, ..., c_n) \in G^n$ and $p_R(c_i) = a_i$ for any $i \leq n$. Suppose that $p_R(\omega(c) \neq p_R(\omega(b))$. Then there exists $\alpha \in M$ such that $\rho_\alpha(\omega(c), \omega(b)) > 0$. Since R is a stable set of pseudometrics, there exists $\beta = \beta(\omega, \alpha) \in M$ such that $\rho_\alpha(x_1, y_1) \leq \rho_\beta(x_1, y_1)$ and $\rho_\alpha(\omega(x_1, ...x_n), \omega(y_1, ..., y_n)) \leq \sum\{\rho_\beta(x_i, y_i) : i \leq n\}$ for all $x_1, y_1, ..., x_n, y_n \in G$. In particular, $0 < \rho_\alpha(\omega(c), \omega(b)) \leq \sum\{\rho_\beta(c_i, b_i) : i \leq n\}$. Thus $\rho_\beta(c_i, b_i) > 0$ for some $i \leq n$. Since $p_R(c_i) = p_R(a_i)$, we have $\rho_\mu(c_i, b_i) = 0$, a contradiction. Thus $\omega : Y^n \to Y$ is an n-ary operation on Y and on Y there exists the structure of Ω-algebra relative to which p_R is a homomorphism.

By construction, the pseudometrics $\bar{R}$ forms a stable set of pseudometrics on Y. Hence Y is a topological algebra and p_R is a continuous homomorphism of G onto Y.

Let $U(\bar{R})$ be the uniformity generated by the pseudometrics $\bar{R}$ on G/R and $(Y, U(\bar{R})_Y)$ be the uniform subspace of the uniform space $(G/R, U(\bar{R}))$. By the definition of a stable set of pseudometrics, the operation $\omega : Y^n \longrightarrow G/R$ is a uniformly continuous mapping for each $n \geq 1$ and every $\omega \in \Omega_n$. Hence the operation ω is continuous extendable on G/R^n and on G/R there exists a structure of topological Ω-algebra such that Y is a subalgebra of the compact Ω-algebra G/R. The proof is complete. □

Assume that v is a unary operation and $v(x) = x$ for each Ω-algebra G and any point $x \in G$. Let M_Ω be the family of all finite ordered subsets of $\Omega \cup \{v\}$ such that v is the first element in each $\alpha \in M_\Omega$. If $\alpha = (\alpha_1, ..., \alpha_n), \beta = (\beta_1, ..., \beta_m) \in M_\Omega$, then:

– $\alpha \leq \beta$ if and only if $n \leq m$ and $\alpha_i \beta_i$ for any $i \leq n$;

– $c(\alpha) = n$ and $c(\beta) = m$.

The set $\{v\}$ is the minimal element of the set M_Ω and $c(\{v\}) = 1$. If $\lambda \in \Omega$, then (λ), (λ, λ),..., $(\lambda, \lambda, ..., \lambda)$ are distinct elements.

Let $\alpha \in M_\Omega$ and $c(\alpha) = 1$. Then $\{v\} \subseteq \alpha \subseteq \{v\} \cup \Omega_0$. We put $P(\alpha) = \alpha \cup \{v(\omega) : \omega \in \alpha\}$.

Assume that $\alpha, \beta \in M_\Omega$, $\alpha \leq \beta$, $c(\beta) = c(\alpha) + 1$ and the polynomials $P(\alpha)$ are constructed. Then $P(\beta) = \beta \cup P(\alpha) \cup \{\omega(p_1, p_2, ..., p_n) : p_1, p_2, ..., p_n \in P(\alpha) \cup \beta, \omega \in \beta \cap \Omega_n, n \geq 1\}$. By induction, the set $P(\alpha)$ is constructed for each $\alpha \in M_\Omega$. Any set $P(\alpha)$ is finite, $P(\alpha) \subseteq P(\beta)$ for $\alpha \leq \beta$ and $P(\Omega) = \cup\{P(\beta) : \beta \in M_\Omega\}$. Let $T(\alpha) = \cup\{T(\lambda) : \lambda \in \alpha\}$ for each $\alpha \in M_\Omega$.

Assume that f is a function on an Ω-algebra G. For each $\alpha \in M_\Omega$ we put $\rho_{(f,\alpha)}(x, y) = sup\{|f_t(y) - f_t(x)| : t \in T(\alpha)\}$ for all $x, y \in G$.

Proposition 3. *Let G be a topological Ω-algebra and $f \in \Omega$-$wAP(G)$. Then:*

1. $R(f) = \{\rho_{(f,\alpha)} : \alpha \in M_\Omega\}$ is a stable set of continuous pseudometrics on G.

2. If the function f is bounded, then the pseudometrics $\{\rho_{(f,\alpha)}\}$ are totally bounded.

3. $\rho_{(f,\alpha)}(x, y) \geq |f(x) - f(y)$ for all $x, y \in G$.

Proof. 1. Since $v \in \alpha$, we have $\rho_{(f,\alpha)}(x, y) \geq |f(x) - f(y)|$ for all $x, y \in G$.

2. Since f is a weakly almost periodic continuous function and the set of polynomials $P(\alpha)$ is finite for any $\alpha \in M_\Omega$, the closure of the set $\alpha(f) = \{t_f : t \in T(\alpha)\}$

in $C(G)$ is a compact set. From this fact it follows that the pseudometric $\rho_{(f,\alpha)}$ is continuous and $\rho_{(f,\alpha)}(x,y) < \infty$ for all $\alpha \in M_\Omega$ and $x, y \in G$.

3. Fix $\alpha \in M_\Omega$, $n \geq 1$ and $\omega \in \Omega_n$. Assume that $\alpha = (\alpha_1, ..., \alpha_m)$ for some $m \geq 1$. We put $\beta = (\alpha_1, ..., \alpha_n, \omega)$. Then $\alpha < \beta$ and $c(\beta) = c(\alpha) + 1$.

Since $T(\alpha) \subseteq T(\beta)$, we have $\rho_{(f,\alpha)}(x,y) \leq \rho_{(f,\beta)}(x,y)$ for all $x, y \in G$.

Fix $x_1, y_1, ..., x_n, y_n \in G$. Since $\varphi \circ \psi \in T(\beta)$ for any $\varphi \in T(\alpha)$ and each $\alpha\psi \in T(\omega)$, we have $\rho_{(f,\alpha)}(\omega(x_1, ...x_m), \omega(y_1, ..., y_m)) \leq \sum\{\rho_{(f,\beta)}(x_i, y_i) : i \leq m\}$. Hence $R(f)$ is a stable set of continuous pseudometrics on G.

4. Assume now that the function f is bounded. Fix $\epsilon > 0$ and $\alpha \in M_\Omega$.

Since the closure of the set $\alpha(f) = \{f_t : t \in T(\alpha)\}$ in $C(G)$ is a compact set, there exists a finite set $L = \{t_1, t_2, ..., t_k\} \subseteq T(\alpha)$ such that for each $t \in T(\alpha)$ there exists $i \leq k$ such that $d(f_t, f_{t_i}) < \epsilon/3$. Assume that $v \in L$.

We put $g(x) = \Sigma\{|f_t(x)| : t \in L\}$. The function g is continuous and bounded. There exists a finite subset F of G such that $min\{|g(x) - g(y)| : y \in F\} < \epsilon/6$ for any $x \in G$. Hence for each $x \in G$ there exists $x(f) \in F$ such that $|f_t(x) - f_t(x(f))| < \epsilon/3$ for any $t \in L$. We affirm that $d_{(f,\alpha)}(x, x(f)) < \epsilon$. Suppose that $x \in G$ and $d_{(f,\alpha)}(x, x(f)) \geq \epsilon > 0$. Then there exist $\varphi \in T(\alpha)$ and $t \in L$ such that $|f_\varphi(x) - f_\varphi(x(f))| > \epsilon$ and $d(f_\varphi, f_t) < \epsilon/3$. By construction, we have $|f_\varphi(x) - f_\varphi(x(f))| = |f_\varphi(x) - f_t(x) + f_t(x) - f_t(x(f)) + f_t(x(f)) - f_\varphi(x(f))| \leq |f_\varphi(x) - f_t(x)| + |f_t(x) - f_t(x(f))| + |f_t(x(f)) - f_\varphi(x(f))| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$, a contradiction. Therefore the pseudometrics $\{\rho_{(f,\alpha)}\}$ are totally bounded. The proof is complete. □

Corollary 9. *Let G be a topological Ω-algebra. Then the maximal a-compactification $(\beta_\Omega G, \beta_G) = (b_F G, e_F)$, where $F = \Omega^\circ$-$wAP(G)$.*

Corollary 10. *Let G be a compact Ω-finite topological Ω-algebra. Then Ω-$wAP(G)$ $= \Omega$-$AP(G)$.*

Remark 6. Let G be a topological Ω-algebra and F be a closed subalgebra of the algebra Ω°-$wAP(G)$ with the following proprieties:

– if f is a constant function, then $f \in F$;

– if $\in F$ and $t \in T(\Omega)$, then $f_t \in F$.

Then $(b_F G, e_F)$ is an a-compactification of G. Any a-compactification can be constructed in this way.

6 Cartesian product of topological algebras

Let $\Omega = \oplus\{\Omega_n : n \in N\}$ be a discrete signature.

For any nulary polynomial $\omega \in P(\Omega)$ and any Ω-algebra G there exists a unique neutral element $\omega_G \in G$ such that $e_{0G}(\omega, G^0) = \omega_G$.

Fix a class $\mathcal{K}$ of topological Ω-algebras with the following properties:

1. If $A \in \mathcal{K}$, then A is a Tychonoff space.
2. The Cartesian product of algebras from $\mathcal{K}$ is an algebra from $\mathcal{K}$.
3. There exists a nulary polynomial $1 \in P(\Omega)$ such that for the point $1_G = e_{0G}(1, G^0)$, each $n \geq 1$ and every $\lambda \in \Omega_n$ we have $\lambda(1_G, ...1_G) = 1_G$ for every $G \in K$.

4. There exists a ternary polynomial $p \in P(\Omega)$ such that $p(x,x,y) = p(y,x,x) = y$ for all $G \in \mathcal{K}$ and $x, y \in G$.

5. There exists a binary polynomial $v \in P(\Omega)$ such that $v(1_G, x) = v(x, 1_G) = x$ for all $G \in \mathcal{K}$ and $x \in G$.

6. If G is a Tychonoff topological Ω-algebra with the properties 3-5, then $G \in \mathcal{K}$.

We may assume that $1 \in \Omega_0$, $p \in \Omega_3$ and $v \in \Omega_2$.

A mapping $\varphi : X \to Y$ is injective if $f(x) \neq f(y)$ for every two distinct points $x, y \in X$.

Lemma 6. *Let $\varphi : A \to B$ be a homomorphism of a topological Ω-algebra $A \in \mathcal{K}$ into an Ω-algebra B, A_1 be a dense subset of A and $\varphi_1 = \varphi|A_1 : A_1 \to B$ be an injective mapping. Then φ is injective too.*

Proof. We may consider that $B = \varphi(A)$. On B we consider the quotient topology $\{U \subseteq B : \varphi^{-1}(U) \textit{ is open in } A\}$. Since $A \in \mathcal{K}$, B is a topological Ω-algebra and $\varphi : A \to B$ is an open continuous mapping (see [4]). Suppose that $a, b \in A$, $a \neq b$ and $\varphi(a) = \varphi(b)$. We fix two open subsets U, V of A for which $a \in U$, $b \in V$ and $U \cap V = \emptyset$. Then the set $W = \varphi(U) \cap \varphi(V)$ is open in B, $\varphi(A_1)$ is a dense subset of B, $\varphi(a) = \varphi(b) \in W$ and $W \cap \varphi(A_1) = \emptyset$, a contradiction. The proof is complete. □

Lemma 7. *Let $A \in \mathcal{K}$ and A be a dense subalgebra of the topological Ω-algebra B. Then $B \in \mathcal{K}$.*

Proof. Is obvious. □

Theorem 8. *Let $\{G_\mu \in \mathcal{K} : \mu \in M\}$ be a non-empty family of topological Ω-algebras and $G = \Pi\{G_\mu \in \mathcal{K} : \mu \in M\}$. Then:*

1. $\beta_{ap(\Omega)}G = \Pi\{\beta_{ap(\Omega)}G_\mu : \mu \in M\}$ and $\alpha_G(x) = (\alpha_{G_\mu}(x_\mu) : \mu \in M)$ for each point $x = (x_\mu) : \mu \in M) \in G$.

2. $(\beta_\Omega G, \beta_G) = \Pi\{\beta_\Omega G_\mu : \mu \in M\}$ and $\beta_G(x) = (\beta_{G_\mu}(x_\mu) : \mu \in M)$ for each point $x = (x_\mu) : \mu \in M) \in G$.

Proof. From Lemma 7 it follows that $\beta_{ap(\Omega)}A \in \mathcal{K}$ for any $A \in \mathcal{K}$.

Let $M = \{1, 2\}$. Then $G = G_1 \times G_2$. There exists a continuous homomorphism $\psi : \beta_{ap(\Omega)}G \longrightarrow \beta_{ap(\Omega)}G_1 \times \beta_{ap(\Omega)}G_2$ such that $\psi(\alpha_G(x,y) = (\alpha_{G_1}(x), \alpha_{G_2}(y))$ for every point $(x, y) \in G$.

We can identify $x \in G_1$ with $(x, 1_{G_2}) \in G$ and $y \in G_2$ with $(1_{G_1}, y) \in G$. In this case $1_G = (1_{G_1}, 1_{G_2})$ and G_1, G_2 are subalgebras of the algebra G. If $h \in \Omega\text{-}AP(G)$, then:

– for each $y \in G_2$ there exists $h_y \in \Omega\text{-}AP(G_1)$ such that $h_y(x) = h(x, y)$ for each $x \in G_1$;

– for each $x \in G_1$ there exists $h_x \in \Omega\text{-}AP(G_2)$ such that $h_x(y) = h(x, y)$ for each $y \in G_2$.

Hence $\psi|\alpha_G(G)$ is an injective mapping. From Lemma 6 it follows that ψ is an isomorphism. Hence the assertions 1 of theorem are true for any finite set M.

Suppose that the set M is infinite. If $B \subseteq M$, then we put $G_B = \Pi\{G_\mu : \mu \in B\}$. Let $G = G_M$ and $\pi_B : G \to G_B$ be the natural projection. We identify G_B with the subalgebra $\{x = (x_\mu : \mu \in M) \in G : x_\mu = 0_{G_\mu} \textit{ for any } \mu \in M \setminus B\}$. In this case $\pi_B : G \to G_B$ is the retraction.

Let $\overline{r}_E G_B = \Pi\{r_E G_\mu : \mu \in M\}$ and identity $\overline{r}_E G_B$ with the subalgebra $\{x = (x_\mu : \mu \in M) : x_\mu = 0_{r_E G_\mu}$ for every $\mu \in M \backslash B\}$ of the algebra $\overline{r}_E G = \overline{r}_E G_M$. Let $\overline{\pi}_B : \overline{r}_E G \to \overline{r}_E G_B$ be the natural projection. We put $G' = \cup\{G_B \subseteq G : B$ is a finite subset of $M\}$. Then G' is a dense subalgebra of the topological Ω-algebra G. If $B \subseteq M$, then $G''_B = r_B(G_B)$ and $G'' = \cup\{G''_B : B$ is a finite subset of $M\} = r_\mu(G')$. For every finite subset $B \subseteq M$ the mapping $\nu_M | G''_B : G''_B \to \overline{r}_E G_B$ is a topological isomorphism. Hence $\nu_M : G'' \to \overline{r}_E G_M$ is an injection. Lemma 6 completes the proof of Assertions 1. The proof of Assertions 2 is similar. The proof is complete. □

Theorem 9. *Let $G \in \mathcal{K}$ be a pseudocompact topological Ω-algebra B. Then:*

1. On βG there exists a structure of topological Ω-algebra such that $\beta G \in \mathcal{K}$ and G is a dense subalgebra of the Ω-algebra βG.

2. Ω-$wAP(G) = C(G) = C^\circ(G)$.

Proof. In [12] it was proved that for any pseudocompact topological Mal'cev E-algebra G and each $n \in \mathbb{N}$ the space G^n is pseudocompact. From the I. Glicksberg's theorem ([8], Problem 3.12.20 (d), p. 299) it follows that $\beta(G^n) = (\beta G)^n$ for each $n \in \mathbb{N}$. Thus for each $n \in \mathbb{N}$ and every $\omega \in \Omega_n$ there exists a continuous extension $\omega : \beta(G^n) \longrightarrow \beta G$ of the mapping $\omega : G^n \longrightarrow G$. Therefore on βG there exists a structure of topological Ω-algebra such that G is a dense subalgebra of the Ω-algebra βG. From Lemma 7 it follows that $\beta G \in \mathcal{K}$. Theorem 7 completes the proof. □

References

[1] Alfsen E. M., Holm P. *A note on compact representations and almost periodicity in topological groups.* Math. Scand., 1962, **10**, 127–136.

[2] Belousov V. D. *Foundations of the Theory of Quasigroups and Loups.* Nauka, Moscow, 1967 (in Russian).

[3] Berglund J. F., Junghenn H. D., Milnes P. *Compact Right Topological Semigroups and Generalizations of Almost Periodicity.* Lecture Notes Math., 663, Springer, Berlin, 1978.

[4] Choban M. M. *Some topics in topological algebra.* Topology and Appl., 1993, **54**, 183–202.

[5] Choban M. M. *The theory of stable metrics.* Math. Balcanika, 1988, **2**, No.4, 357–373.

[6] Cioban M., Pavel D. *Almost periodic functions on quasidroups.* Analele Universitati din Oradea. Fascicola Matematica, 2006, **XIII**, 99–124.

[7] Corduneanu C. *Almost Periodic Functions.* Interscience Publ., New York, 1969.

[8] Engelking R. *General Topology.* PWN, Warsawa, 1977.

[9] Hart J. K., Kunen K. *Bohr compactifications of discrete structures.* Fund. Math., 1999, **160**, 101–151.

[10] Mal'cev A. I. *Free topological algebras.* Izvestia Akad. Nauk SSSR, 1957, **21**, 171–198.

[11] von Neumann J. *Almost periodic functions in a group.* Trans. Amer. Math. Soc., 1934, **30**, 445–492.

[12] Reznicenko R. A., Uspenskij V. V. *Pseudocompact Mal'cev spaces.* Topology Appl., 1998, **86**, 83–104.

[13] Weil A. *L'integration dans les groups topologiques et ses applications.* Herman, Paris, 1951.

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