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# Lacunary Ideal Convergence in Probabilistic Normed Space

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**Abstract.** The aim of this paper is to study the notion of lacunary $I$-convergence in probabilistic normed spaces as a variant of the notion of ideal convergence. Also lacunary $I$-limit points and lacunary $I$-cluster points have been defined and the relation between them has been established. Furthermore, lacunary Cauchy and lacunary $I$-Cauchy sequences are introduced and studied. Finally, we provided example which shows that our method of convergence in probabilistic normed spaces is more general.

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## 1 Introduction

Steinhaus [45] and Fast [13] independently introduced the notion of statistical convergence for sequences of real numbers. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Later on it was further investigated from various points of view. For example, statistical convergence has been investigated in summability theory by (Connor [7], Fridy [15], Šalát [40]), number theory and mathematical analysis by (Buck [1], Mitrinović et al. [37]), topological groups (Çakalli [2, 3]), topological spaces (Di Maio and Kočinac [34]), function spaces (Caserta and Kočinac [5]), measure theory (Cheng et al. [6], Connor and Swardson [8], Miller [36]). Fridy and Orhan [16] introduced the concept of lacunary statistical convergence. Some work on lacunary statistical convergence can be found in [2, 17, 20, 33].

Kostyrko, et al. [28] introduced the notion of $I$-convergence as a generalization of statistical convergence which is based on the structure of an admissible ideal $I$ of subset of natural numbers $\mathbb{N}$. Kostyrko et al. [29] gave some of basic properties of $I$-convergence and dealt with extremal $I$-limit points. Further details on ideal convergence can be found in [4, 11, 12, 21–25, 32, 41, 46], and many others. The notion of lacunary ideal convergence of real sequences was introduced in [47, 48], and Hazarika [18, 19] introduced the lacunary ideal convergent sequences of fuzzy real numbers and studied some properties. Debnath [10] introduced the notion of lacunary ideal convergence in intuitionistic fuzzy normed linear spaces. Recently, Yamanci and Gürdal [49] introduced the notion of lacunary ideal convergence in random $n$-normed space.

A family $I$ of subsets of $\mathbb{N}$, positive integers, i. e. $I \subset 2^{\mathbb{N}}$, is an ideal on $\mathbb{N}$ if and only if

(i) $\phi \in I$,

(ii) $A \cup B \in I$ for each $A, B \in I$,

(iii) each subset of an element of $I$ is an element of $I$.

A non-empty family of sets $F \subset 2^{\mathbb{N}}$ is a filter on $\mathbb{N}$ if and only if

(a) $\phi \notin F$,

(b) $A \cap B \in F$ for each $A, B \in F$,

(c) any superset of an element of $F$ is in $F$.

An ideal $I$ is called *non-trivial* if $I \neq \phi$ and $\mathbb{N} \notin I$. Clearly $I$ is a non-trivial ideal if and only if $F = F(I) = \{\mathbb{N} - A : A \in I\}$ is a filter in $\mathbb{N}$, called the filter associated with the ideal $I$.

A non-trivial ideal $I$ is called *admissible* if and only if $\{\{n\} : n \in \mathbb{N}\} \subset I$. A non-trivial ideal $I$ is *maximal* if there cannot exist any non-trivial ideal $J \neq I$ containing $I$ as a subset.

Recall that a sequence $x = (x_k)$ of points in $\mathbb{R}$ is said to be $I$-convergent to a real number $\ell$ if $\{k \in \mathbb{N} : |x_k - \ell| \geq \varepsilon\} \in I$ for every $\varepsilon > 0$ [28]. In this case we write $I - \lim x_k = \ell$.

By a lacunary sequence $\theta = (k_r)$, where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by $\theta$ will be denoted by $J_r = (k_{r-1}, k_r]$ and we let $h_r = k_r - k_{r-1}$. The space of lacunary strongly convergent sequences $\mathcal{N}_\theta$ was defined by Freedman et al. [14] as follows:

$$\mathcal{N}_\theta = \left\{ x = (x_k) : \lim_r \frac{1}{h_r} \sum_{k \in J_r} |x_k - L| = 0, \text{ for some } L \right\}.$$

Menger [35] proposed the probabilistic concept of the distance by replacing the number $d(p, q)$ as the distance between points $p, q$ by a probability distribution function $F_{p,q}(x)$. He interpreted $F_{p,q}(x)$ as the probability that the distance between $p$ and $q$ is less than $x$. This led to the development of the area now called probabilistic metric spaces. This is Šerstnev [44] who first used this idea of Menger to introduce the concept of a PN space. For an extensive view on this subject, we refer to [9, 26, 31, 42, 43]. Subsequently, Mursaleen and Mohiuddine [38] and Rahmat[39] studied the ideal convergence in probabilistic normed spaces and V. Kumar and K. Kumar [30] studied $I$-Cauchy and $I^*$-Cauchy sequences in probabilistic normed spaces.

The notion of statistical convergence depends on the density (asymptotic or natural) of subsets of $\mathbb{N}$. A subset $E$ of $\mathbb{N}$ is said to have natural density $\delta(E)$ if

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} |\{k \leq n : k \in E\}| \text{ exists.}$$

**Definition 1.** A sequence $x = (x_k)$ is said to be *statistically convergent* to $\ell$ if for every $\varepsilon > 0$

$$\delta(\{k \in \mathbb{N} : |x_k - \ell| \geq \varepsilon\}) = 0.$$

In this case, we write $S - \lim x = \ell$ or $x_k \to \ell(S)$ and $S$ denotes the set of all statistically convergent sequences.

**Definition 2.** ([47,48]) Let $I \subset 2^{\mathbb{N}}$ be a non-trivial ideal. A real sequence $x = (x_k)$ is said to be *lacunary $I$-convergent* or *$I_\theta$-convergent* to $L \in \mathbb{R}$ if, for every $\varepsilon > 0$ the set

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} |x_k - L| \geq \varepsilon \right\} \in I.$$

$L$ is called the $I_\theta$-limit of the sequence $x = (x_k)$, and we write $I_\theta - \lim x = L$.

In this paper we study the concept of lacunary $I$-convergence in probabilistic normed spaces. We also define lacunary $I$-limit points and lacunary $I$-cluster points in probabilistic normed space and prove some interesting results.

## 2 Basic definitions and notations

Now we recall some notations and basic definitions that we are going to use in this paper.

**Definition 3.** A *distribution function* (briefly a d.f.) $F$ is a function from the extended reals $(-\infty, +\infty)$ into $[0, 1]$ such that

(a) it is non-decreasing;

(b) it is left-continuous on $(-\infty, +\infty)$;

(c) $F(-\infty) = 0$ and $F(+\infty) = 1$.

The set of all d.f.'s will be denoted by $\Delta$. The subset of $\Delta$ consisting of proper d.f's, namely of those elements $F$ such that $\ell^+ F(-\infty) = F(-\infty) = 0$ and $\ell^- F(+\infty) = F(+\infty) = 1$ will be denoted by $D$. A *distance distribution function* (briefly, d.d.f.) is a d.f. $F$ such that $F(0) = 0$. The set of all d.d.f.f's will be denoted by $\Delta^+$, while $D^+ := D \cap \Delta^+$ will denote the set of proper d.d.f.'s.

**Definition 4.** A *triangular norm* or, briefly, a *t-norm* is a binary operation $T : [0, 1] \times [0, 1] \to [0, 1]$ that satisfies the following conditions (see [27]):

(T1) $T$ is commutative, i. e., $T(s,t)=T(t,s)$ for all $s$ and $t$ in $[0,1]$;

(T2) $T$ is associative, i. e., $T(T(s,t),u)=T(s,T(t,u))$ for all $s,t$ and $u$ in $[0,1]$;

(T3) $T$ is nondecreasing, i. e., $T(s,t)\leq T(s',t)$ for all $t\in[0,1]$ whenever $s\leq s'$;

(T4) $T$ satisfies the boundary condition $T(1,t)=t$ for every $t\in[0,1]$.

$T^*$ is a continuous *t-conorm*, namely, a continuous binary operation on $[0,1]$ that is related to a continuous $t$-norm through $T^*(s,t)=1-T(1-s,1-t)$. Notice that by virtue of its commutativity, any $t$-norm $T$ is nondecreasing in each place. Some examples of $t$-norms $T$ and its $t$-conorms $T^*$ are: $M(x,y)=\min\{x,y\},\Pi(x,y)=x.y$ and $M^*(x,y)=\max\{x,y\},\Pi^*(x,y)=x+y-x.y$.

**Definition 5.** A Menger PN space under $T$ is a PN space $(X,\nu,\tau,\tau^*)$, denoted by $(X,\nu,T)$, in which $\tau=\tau_T$ and $\tau^*=\tau_{T^*}$, for some continuous $t$-norm $T$ and its $t$-conorm $T^*$.

**Definition 6.** Let $(X,\nu,T)$ be a PN space and $x=(x_k)$ be a sequence in $X$. We say that $(x_k)$ is *convergent* to $\ell\in X$ with respect to the probabilistic norm $\nu$ if for each $\varepsilon>0$ and $\alpha\in(0,1)$ there exists a positive integer $m$ such that $\nu_{x_k-\ell}(\varepsilon)>1-\alpha$ whenever $k\geq m$. The element $\ell$ is called the limit of the sequence $(x_k)$ and we shall write $\nu-\lim x_k=\ell$ or $x_k\stackrel{\nu}{\to}\ell$ as $k\to\infty$.

**Definition 7.** A sequence $(x_k)$ in $X$ is said to be *Cauchy* with respect to the probabilistic norm $\nu$ if for each $\varepsilon>0$ and $\alpha\in(0,1)$ there exists a positive integer $M=M(\varepsilon,\alpha)$ such that $\nu_{x_k-x_p}(\varepsilon)>1-\alpha$ whenever $k,p\geq M$.

**Definition 8.** Let $(X,\nu,T)$ be a probabilistic normed space, and let $r\in(0,1)$ and $x\in X$. The set
$$B\left(x,r;t\right)=\{y\in X:\nu_{y-x}(t)>1-r\}$$
is called the open ball with center $x$ and radius $r$ with respect to $t$.

Throughout the paper, we denote $I$ as an admissible ideal of subsets of $\mathbb{N}$ and $\theta=(k_r)$ as a fixed lacunary sequence, respectively, unless otherwise stated.

## 3 Main results

We now obtain our main results.

**Definition 9.** Let $I\subset 2^{\mathbb{N}}$ and $(X,\nu,T)$ be a PNS. A sequence $x=(x_k)$ in $X$ is said to be $I_\theta$-convergent to $L\in X$ with respect to the probabilistic norm $\nu$ if, for every $\varepsilon>0$ and $\alpha\in(0,1)$ the set
$$\left\{r\in\mathbb{N}:\ \frac{1}{h_r}\sum_{k\in J_r}\nu_{x_k-L}(\varepsilon)\leq 1-\alpha\right\}\in I.$$
$L$ is called the $I_\theta-$limit of the sequence $x=(x_k)$ in $X$, and we write $I_\theta^\nu-\lim x=L$.

**Example 1.** Let $(\mathbb{R},|.|)$ denote the space of all real numbers with the usual norm, and let $T(a,b)=ab$ for all $a,b\in[0,1]$. For all $x\in\mathbb{R}$ and every $t>0$, consider $\nu_x(t)=\frac{t}{t+|x|}$. Then $(\mathbb{R}\nu,T)$ is a PNS. If we take $I=\{A\subset\mathbb{N}:\ \delta(A)=0\}$, where $\delta(A)$ denotes the natural density of the set A, then $I$ is a non-trivial admissible ideal. Define a sequence $x=(x_k)$ as follows:

$$x_k=\begin{cases}1 & \text{if } k=i^2, i\in\mathbb{N},\\ 0 & \text{otherwise.}\end{cases}$$

Then for every $\alpha\in(0,1)$ and for any $\varepsilon>0$, the set

$$K=\left\{r\in\mathbb{N}:\ \frac{1}{h_r}\sum_{k\in J_r}\nu_{x_k}(\varepsilon)\le 1-\alpha\right\}$$

will be a finite set. Hence, $\delta(K)=0$ and consequently $K\in I$, i.e., $I_\theta^\nu-\lim x=0$.

**Lemma 1.** *Let $(X,\nu,T)$ be a PNS and $x=(x_k)$ be a sequence in $X$. Then, for every $\varepsilon>0$ and $\alpha\in(0,1)$ the following statements are equivalent:*
*(i)* $I_\theta^\nu-\lim x=L$,
*(ii)* $\left\{r\in\mathbb{N}:\ \frac{1}{h_r}\sum_{k\in J_r}\nu_{x_k-L}(\varepsilon)\le 1-\alpha\right\}\in I$,
*(iii)* $\left\{r\in\mathbb{N}:\ \frac{1}{h_r}\sum_{k\in J_r}\nu_{x_k-L}(\varepsilon)>1-\alpha\right\}\in F(I)$,
*(iii)* $I_\theta-\lim\nu_{x_k-L}(\varepsilon)=1$.

**Theorem 1.** *Let $(X,\nu,T)$ be a PNS and if a sequence $x=(x_k)$ in $X$ is $I_\theta$-convergent to $L\in X$ with respect to the probabilistic norm $\nu$, then $I_\theta^\nu-\lim x$ is unique.*

*Proof.* Suppose that $I_\theta^\nu-\lim x=L_1$ and $I_\theta^\nu-\lim x=L_2$ $(L_1\neq L_2)$. Given $\alpha>0$ and choose $\beta\in(0,1)$ such that

$$T(1-\beta,1-\beta)>1-\alpha. \tag{1}$$

Then for $\varepsilon>0$, define the following sets:

$$K_1=\left\{r\in\mathbb{N}:\ \frac{1}{h_r}\sum_{k\in J_r}\nu_{x_k-L_1}\left(\frac{\varepsilon}{2}\right)\le 1-\beta\right\},$$

$$K_2=\left\{r\in\mathbb{N}:\ \frac{1}{h_r}\sum_{k\in J_r}\nu_{x_k-L_2}\left(\frac{\varepsilon}{2}\right)\le 1-\beta\right\}.$$

Since $I_\theta^\nu-\lim x=L_1$, using Lemma 1, we have $K_1\in I$. Also, using $I_\theta^\nu-\lim x=L_2$, we get $K_2\in I$. Now let

$$K=K_1\cup K_2.$$

Then $K \in I$. This implies that its complement $K^c$ is a non-empty set in $F(I)$. Now if $r \in K^c$, let us consider $r \in K_1^c \cap K_2^c$. Then we have

$$\frac{1}{h_r}\sum_{k\in J_r}\nu_{x_k-L_1}\left(\frac{\varepsilon}{2}\right) > 1-\beta \text{ and } \frac{1}{h_r}\sum_{k\in J_r}\nu_{x_k-L_2}\left(\frac{\varepsilon}{2}\right) > 1-\beta.$$

Now, we choose an $s \in \mathbb{N}$ such that

$$\nu_{x_s-L_1}\left(\frac{\varepsilon}{2}\right) > \frac{1}{h_r}\sum_{k\in J_r}\nu_{x_k-L_1}\left(\frac{\varepsilon}{2}\right) > 1-\beta$$

and

$$\nu_{x_s-L_2}\left(\frac{\varepsilon}{2}\right) > \frac{1}{h_r}\sum_{k\in J_r}\nu_{x_k-L_2}\left(\frac{\varepsilon}{2}\right) > 1-\beta$$

e. g., consider $\max\left\{\nu_{x_k-L_1}\left(\frac{\varepsilon}{2}\right), \nu_{x_k-L_2}\left(\frac{\varepsilon}{2}\right) :\ k \in J_r\right\}$ and choose that $k$ as $s$ for which the maximum occurs. Then from (1), we have

$$\nu_{L_1-L_2}(\varepsilon) \geq T\left(\nu_{x_s-L_1}\left(\frac{\varepsilon}{2}\right), \nu_{x_s-L_2}\left(\frac{\varepsilon}{2}\right)\right) > T\left(1-\beta, 1-\beta\right) > 1-\alpha.$$

Since $\alpha > 0$ is arbitrary, we have $\nu_{L_1-L_2}(\varepsilon) = 1$ for all $\varepsilon > 0$, which implies that $L_1 = L_2$. Therefore, we conclude that $I_\theta^\nu - \lim x$ is unique. □

Here, we introduce the notion of $\theta$-convergence in a PNS and discuss some properties.

**Definition 10.** Let $(X, \nu, T)$ be a PNS. A sequence $x = (x_k)$ in $X$ is $\theta$-convergent to $L \in X$ with respect to the probabilistic norm $\nu$ if, for $\alpha \in (0,1)$ and every $\varepsilon > 0$, there exists $r_o \in \mathbb{N}$ such that

$$\frac{1}{h_r}\sum_{k\in J_r}\nu_{x_k-L}(\varepsilon) > 1-\alpha$$

for all $r \geq r_o$. In this case, we write $\nu^\theta - \lim x = L$.

**Theorem 2.** *Let $(X, \nu, T)$ be a PNS and let $x = (x_k)$ in $X$. If $x = (x_k)$ is $\theta$-convergent with respect to the probabilistic norm $\nu$, then $\nu^\theta - \lim x$ is unique.*

*Proof.* Suppose that $\nu^\theta - \lim x = L_1$ and $\nu^\theta - \lim x = L_2$ $(L_1 \neq L_2)$. Given $\alpha \in (0,1)$ and choose $\beta \in (0,1)$ such that $T(1-\beta, 1-\beta) > 1-\alpha$. Then for any $\varepsilon > 0$, there exists $r_1 \in \mathbb{N}$ such that

$$\frac{1}{h_r}\sum_{k\in J_r}\nu_{x_k-L_1}(\varepsilon) > 1-\alpha$$

for all $r \geq r_1$. Also, there exists $r_2 \in \mathbb{N}$ such that

$$\frac{1}{h_r}\sum_{k\in J_r}\nu_{x_k-L_2}(\varepsilon) > 1-\alpha$$

for all $r \geq r_2$. Now, consider $r_o = \max\{r_1, r_2\}$. Then for $r \geq r_o$, we will get an $s \in \mathbb{N}$ such that

$$\nu_{x_s - L_1}\left(\frac{\varepsilon}{2}\right) > \frac{1}{h_r}\sum_{k \in J_r} \nu_{x_k - L_1}\left(\frac{\varepsilon}{2}\right) > 1 - \beta$$

and

$$\nu_{x_s - L_2}\left(\frac{\varepsilon}{2}\right) > \frac{1}{h_r}\sum_{k \in J_r} \nu_{x_k - L_2}\left(\frac{\varepsilon}{2}\right) > 1 - \beta.$$

Then, we have

$$\nu_{L_1 - L_2}(\varepsilon) \geq T\left(\nu_{x_s - L_1}\left(\frac{\varepsilon}{2}\right), \nu_{x_s - L_2}\left(\frac{\varepsilon}{2}\right)\right) > T(1-\beta, 1-\beta) > 1 - \alpha.$$

Since $\alpha > 0$ is arbitrary, we have $\nu_{L_1 - L_2}(\varepsilon) = 1$ for all $\varepsilon > 0$, which implies that $L_1 = L_2$. □

**Theorem 3.** *Let $(X, \nu, T)$ be a PNS and let $x = (x_k)$ in $X$. If $\nu^\theta - \lim x = L$, then $I_\theta^\nu - \lim x = L$.*

*Proof.* Let $\nu^\theta - \lim x = L$, then for every $\varepsilon > 0$ and given $\alpha \in (0,1)$, there exists $r_0 \in \mathbb{N}$ such that

$$\frac{1}{h_r}\sum_{k \in J_r} \nu_{x_k - L}(\varepsilon) > 1 - \alpha$$

for all $r \geq r_0$. Therefore the set

$$B = \left\{ r \in \mathbb{N} : \ \frac{1}{h_r}\sum_{k \in J_r} \nu_{x_k - L}(\varepsilon) \leq 1 - \alpha \right\} \subseteq \{1, 2, ..., n_0 - 1\}.$$

But, with $I$ being admissible, we have $B \in I$. Hence $I_\theta^\nu - \lim x = L$. □

**Theorem 4.** *Let $(X, \nu, T)$ be a PNS and $x = (x_k), y = (y_k)$ be two sequence in $X$.*
*(i) If $I_\theta^\nu - \lim x_k = L_1$ and $I_\theta^\nu - \lim y_k = L_2$, then $I_\theta^\nu - \lim(x_k \pm y_k) = L_1 \pm L_2$;*
*(ii) If $I_\theta^\nu - \lim x_k = L$ and $a$ be a non-zero real number, then $I_\theta^\nu - \lim ax_k = aL$. If $a = 0$, then result is true only if $I$ is admissible of $N$.*

*Proof.* (i) We shall prove, if $I_\theta^\nu - \lim x_k = L_1$ and $I_\theta^\nu - \lim y_k = L_2$, then $I_\theta^\nu - \lim(x_k + y_k) = L_1 + L_2$, only. The proof of the other part follows similarly.

Take $\varepsilon > 0, \alpha \in (0,1)$ and choose $\beta \in (0,1)$ such that the condition (1) holds. If we define

$$A_1 = \left\{ r \in \mathbb{N} : \ \frac{1}{h_r}\sum_{k \in J_r} \nu_{x_k - L_1}\left(\frac{\varepsilon}{2}\right) \leq 1 - \beta \right\}$$

and

$$A_2 = \left\{ r \in \mathbb{N} : \ \frac{1}{h_r}\sum_{k \in J_r} \nu_{y_k - L_2}\left(\frac{\varepsilon}{2}\right) \leq 1 - \beta \right\},$$

then $A_1^c \cap A_2^c \in F(I)$. We claim that

$$A_1^c \cap A_2^c \subset \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{(x_k - L_1) + (y_k - L_2)}(\varepsilon) > 1 - \alpha \right\}.$$

Let $n \in A_1^c \cap A_2^c$. Now, using (1), we have

$$\frac{1}{h_r} \sum_{n \in J_r} \nu_{(x_n - L_1) + (y_n - L_2)}(\varepsilon) \geq T\left( \frac{1}{h_r} \sum_{n \in J_r} \nu_{x_n - L_1}\left(\frac{\varepsilon}{2}\right), \frac{1}{h_r} \sum_{n \in J_r} \nu_{y_n - L_2}\left(\frac{\varepsilon}{2}\right) \right)$$

$$> T(1-\beta, 1-\beta) > 1 - \alpha.$$

Hence

$$A_1^c \cap A_2^c \subset \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{(x_k - L_1) + (y_k - L_2)}(\varepsilon) > 1 - \alpha \right\}.$$

As $A_1^c \cap A_2^c \in F(I)$, so

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{(x_k - L_1) + (y_k - L_2)}(\varepsilon) \leq 1 - \alpha \right\} \in I.$$

Therefore $I_\theta^\nu - \lim(x_k + y_k) = L_1 + L_2$.

(ii) Suppose $a \neq 0$. Since $I_\theta^\nu - \lim x_k = L$, for each $\varepsilon > 0$ and $\alpha \in (0,1)$, the set

$$A(\varepsilon, \alpha) = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L}(\varepsilon) < 1 - \alpha \right\} \in F(I).$$

If $n \in A(\varepsilon, \alpha)$, then we have

$$\frac{1}{h_r} \sum_{k \in J_r} \nu_{ax_k - aL}(\varepsilon) = \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L}\left(\frac{\varepsilon}{|a|}\right)$$

$$\geq T\left( \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L}(\varepsilon), \nu_0\left(\frac{\varepsilon}{|a|} - \varepsilon\right) \right)$$

$$\geq T\left( \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L}(\varepsilon), 1 \right) \geq \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L}(\varepsilon) > 1 - \alpha.$$

Hence

$$A(\varepsilon, \alpha) \subset \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{ax_k - aL}(\varepsilon) > 1 - \alpha \right\}$$

and

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{ax_k - aL}(\varepsilon) > 1 - \alpha \right\} \in F(I).$$

It follows that

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{ax_k - aL}(\varepsilon) \leq 1 - \alpha \right\} \in I.$$

Hence $I_\theta^\nu - \lim ax_k = aL$.

Next suppose that $a = 0$. Then for each $\varepsilon > 0$ and $\alpha \in (0,1)$, we have

$$\frac{1}{h_r} \sum_{k \in J_r} \nu_{0x_k - 0L}(\varepsilon) = \frac{1}{h_r} \sum_{k \in J_r} \nu_0(\varepsilon) = 1 > 1 - \alpha,$$

it follows that $\nu^\theta - \lim x = \ell$. Hence from Theorem 3, $I_\theta^\nu - \lim x = \ell$. $\square$

**Theorem 5.** *Let $(X, \nu, T)$ be a PNS and let $x = (x_k)$ in $X$. If $\nu^\theta - \lim x = L$, then there exists a subsequence $(x_{m_k})$ of $x = (x_k)$ such that $\nu - \lim x_{m_k} = L$.*

*Proof.* Let $\nu^\theta - \lim x = L$. Then, for every $\varepsilon > 0$ and given $\alpha \in (0,1)$, there exists $r_0 \in \mathbb{N}$ such that

$$\frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L}(\varepsilon) > 1 - \alpha$$

for all $r \geq r_0$. Clearly, for each $r \geq r_0$, we can select an $m_k \in J_r$ such that

$$\nu_{x_{m_k} - L}(\varepsilon) > \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L}(\varepsilon) > 1 - \alpha.$$

It follows that $\nu - \lim x_{m_k} = L$. $\square$

**Definition 11.** Let $(X, \nu, T)$ be a PNS and let $x = (x_k)$ be a sequence in $X$. Then,

(1) An element $L \in X$ is said to be $I_\theta$-limit point of $x = (x_k)$ if there is a set $M = \{m_1 < m_2 < ... < m_k < ...\} \subset \mathbb{N}$ such that the set $M^\imath = \{r \in \mathbb{N} : \ m_k \in J_r\} \notin I$ and $\nu^\theta - \lim x_{m_k} = L$.

(2) An element $L \in X$ is said to be $I_\theta$-cluster point of $x = (x_k)$ if for every $\varepsilon > 0$ and $\alpha \in (0,1)$, we have

$$\left\{ r \in \mathbb{N} : \ \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L}(\varepsilon) > 1 - \alpha \right\} \notin I.$$

Let $\Lambda_\nu^{I_\theta}(x)$ denote the set of all $I_\theta$-limit points and $\Gamma_\nu^{I_\theta}(x)$ denote the set of all $I_\theta$-cluster points in $X$, respectively.

**Theorem 6.** *Let $(X,\nu,T)$ be a PNS. For each sequence $x=(x_k)$ in $X$, we have $\Lambda_{\nu}^{I_\theta}(x)\subset\Gamma_{\nu}^{I_\theta}(x)$.*

*Proof.* Let $L\in\Lambda_{\nu}^{I_\theta}(x)$, then there exists a set $M\subset\mathbb{N}$ such that $M^{\imath}\notin I$, where $M$ and $M^{\imath}$ are as in Definition 5, satisfies $\nu^{\lambda}-\lim x_{m_k}=L$. Thus, for every $\varepsilon>0$ and $\alpha\in(0,1)$, there exists $r_0\in\mathbb{N}$ such that

$$\frac{1}{h_r}\sum_{k\in J_r}\nu_{x_{m_k}-L}(\varepsilon)>1-\alpha$$

for all $r\geq r_0$. Therefore,

$$B=\left\{r\in\mathbb{N}:\ \frac{1}{h_r}\sum_{k\in J_r}\nu_{x_k-L}(\varepsilon)>1-\alpha\right\}\supseteq M^{\imath}\setminus\{m_1,m_2,...,m_{n_0}\}.$$

Now, with $I$ being admissible, we must have $M^{\imath}\setminus\{m_1,m_2,...,m_{k_0}\}\notin I$ and as such $B\notin I$. Hence $L\in\Gamma_{\nu}^{I_\theta}(x)$. □

**Theorem 7.** *Let $(X,\nu,T)$ be a PNS. For each sequence $x=(x_k)$ in $X$, the set $\Gamma_{\nu}^{I_\theta}(x)$ is a closed set in $X$ with respect to the usual topology induced by the probabilistic norm $\nu^{\theta}$.*

*Proof.* Let $y\in\overline{\Gamma_{\nu}^{I_\theta}(x)}$. Take $\varepsilon>0$ and $\alpha\in(0,1)$. Then there exists $L_0\in\Gamma_{\nu}^{I_\theta}(x)\cap B(y,\alpha,\varepsilon)$. Choose $\delta>0$ such that $B(L_0,\delta,\varepsilon)\subset B(y,\alpha,\varepsilon)$. We have

$$G=\left\{r\in\mathbb{N}:\ \frac{1}{h_r}\sum_{k\in J_r}\nu_{x_k-y}(\varepsilon)>1-\alpha\right\}$$

$$\supseteq\left\{r\in\mathbb{N}:\ \frac{1}{h_r}\sum_{k\in J_r}\nu_{x_k-L_0}(\varepsilon)>1-\delta\right\}=H.$$

Thus $H\notin I$ and so $G\notin I$. Hence $y\in\Gamma_{\nu}^{I_\theta}(x)$. □

**Theorem 8.** *Let $(X,\nu,T)$ be a PNS and let $x=(x_k)$ in $X$. Then the following statements are equivalent:*

(1) *$L$ is an $I_\theta-$limit point of $x$,*

(2) *There exist two sequences $y$ and $z$ in $X$ such that $x=y+z$ and $\nu^{\theta}-\lim y=L$ and $\left\{r\in\mathbb{N}:\ k\in J_r,\ z_k\neq\overline{\theta}\right\}\in I$, where $\overline{\theta}$ is the zero element of $X$.*

*Proof.* Suppose that (1) holds. Then there exist sets $M$ and $M^{\imath}$ as in Definition 11 such that $M^{\imath}\notin I$ and $\nu^{\theta}-\lim x_{m_k}=L$. Define the sequences $y$ and $z$ as follows:

$$y_k=\begin{cases}x_k & \text{if } k\in J_r;\ r\in M^{\imath},\\ L & \text{otherwise}\end{cases}$$

and

$$z_k = \begin{cases} \overline{\theta} & \text{if } k \in J_r;\, r \in M^{\imath}, \\ x_k - L & \text{otherwise.} \end{cases}$$

It sufficies to consider the case $k \in J_r$ such that $r \in \mathbb{N} \diagdown M^{\imath}$. Then for each $\alpha \in (0,1)$ and $\varepsilon > 0$, we have $\nu_{y_k - L}(\varepsilon) = 1 > 1 - \alpha$. Thus, in this case,

$$\frac{1}{h_r} \sum_{k \in J_r} \nu_{y_k - L}(\varepsilon) = 1 > 1 - \alpha.$$

Hence $\nu^{\theta} - \lim y = L$. Now $\{r \in \mathbb{N} : \; k \in J_r, \; z_k \neq \theta\} \subset \mathbb{N} \diagdown M^{\imath}$ and so $\{r \in \mathbb{N} : \; k \in J_r, \; z_k \neq \theta\} \in I$.

Now, suppose that (2) holds. Let $M^{\imath} = \{r \in \mathbb{N} : \; k \in J_r, \; z_k = \theta\}$. Then, clearly $M^{\imath} \in F(I)$ and so it is an infinite set. Construct the set $M = \{m_1 < m_2 < ... < m_k < ...\} \subset \mathbb{N}$ such that $m_k \in J_r$ and $z_{m_k} = \overline{\theta}$. Since $x_{m_k} = y_{m_k}$ and $\nu^{\theta} - \lim y = L$ we obtain $\nu^{\theta} - \lim x_{m_k} = L$. This completes the proof. ☐

**Theorem 9.** *Let $(X, \nu, T)$ be a PNS and $x = (x_k)$ be a sequence in X. Let I be an admissible ideal in N. If there is an $I_{\theta}^{\nu}$-convergent sequence $y = (y_k)$ in X such that $\{k \in N : y_k \neq x_k\} \in I$ then x is also $I_{\theta}^{\nu}$-convergent.*

*Proof.* Suppose that $\{k \in \mathbb{N} : y_k \neq x_k\} \in I$ and $I_{\theta}^{\nu} - \lim y = \ell$. Then for every $\alpha \in (0,1)$ and $\varepsilon > 0$, the set

$$\left\{ r \in \mathbb{N} : \; \frac{1}{h_r} \sum_{k \in J_r} \nu_{y_k - L}(\varepsilon) \leq 1 - \alpha \right\} \in I.$$

For every $0 < \alpha < 1$ and $\varepsilon > 0$, we have

$$\left\{ r \in \mathbb{N} : \; \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L}(\varepsilon) \leq 1 - \alpha \right\} \tag{2}$$

$$\subseteq \{k \in \mathbb{N} : y_k \neq x_k\} \cup \left\{ r \in \mathbb{N} : \; \frac{1}{h_r} \sum_{k \in J_r} \nu_{y_k - L}(\varepsilon) \leq 1 - \alpha \right\}.$$

As the both sets of right-hand side of (2) are in $I$, therefore we have that

$$\left\{ r \in \mathbb{N} : \; \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L}(\varepsilon) \leq 1 - \alpha \right\} \in I.$$

This completes the proof of the theorem. ☐

**Definition 12.** Let $(X, \nu, T)$ be a PNS. A sequence $x = (x_k)$ in $X$ is said to be $\theta$-Cauchy sequence with respect to the probabilistic norm $\nu$ if, for every $\varepsilon > 0$ and $\alpha \in (0, 1)$, there exist $r_0, m \in \mathbb{N}$ satisfying

$$\frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - x_m}(\varepsilon) > 1 - \varepsilon$$

for all $r \geq r_0$.

**Definition 13.** Let $I$ be an admissible ideal of $\mathbb{N}$. Let $(X, \nu, T)$ be a PNS. A sequence $x = (x_k)$ in $X$ is said to be $I_\theta$-Cauchy sequence with respect to the probabilistic norm $\nu$ if, for every $\varepsilon > 0$ and $\alpha \in (0, 1)$, there exists $m \in \mathbb{N}$ satisfying

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - x_m}(\varepsilon) > 1 - \varepsilon \right\} \in F(I).$$

**Definition 14.** Let $I$ be an admissible ideal of $\mathbb{N}$. Let $(X, \nu, T)$ be a PNS. A sequence $x = (x_k)$ in $X$ is said to be $I_\theta^*$-Cauchy sequence with respect to the probabilistic norm $\nu$ if there exists a set $M = \{m_1 < m_2 < ... < m_k < ...\} \subset \mathbb{N}$ such that the set $M^\imath = \{r \in \mathbb{N} : m_k \in J_r\} \in F(I)$ and the subsequence $(x_{m_k})$ of $x = (x_k)$ is a $\theta$-Cauchy sequence with respect to the probabilistic norm $\nu$.

The following theorem is an analogue of Theorem 3, so the proof is omitted.

**Theorem 10.** *Let $I$ be an admissible ideal of $N$. Let $(X, \nu, T)$ be a PNS. If a sequence $x = (x_k)$ in $X$ is $\theta$-Cauchy sequence with respect to the probabilistic norm $\nu$, then it is $I_\theta$-Cauchy sequence with respect to the same norm.*

The proof of the following theorem is similar to that of Theorem 5.

**Theorem 11.** *Let $(X, \nu, T)$ be a PNS. If a sequence $x = (x_k)$ in $X$ is $\theta$-Cauchy sequence with respect to the probabilistic norm $\nu$, then there is a subsequence of $x = (x_k)$ which is ordinary Cauchy sequence with respect to the same norm.*

The following theorem can be proved easily using similar techniques as in the proof of Theorem 6.

**Theorem 12.** *Let $I$ be an admissible ideal of $N$. Let $(X, \nu, T)$ be a PNS. If a sequence $x = (x_k)$ in $X$ is $I_\theta^*$-Cauchy sequence with respect to the probabilistic norm $\nu$, then it is $I_\theta$-Cauchy sequence as well.*

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