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On Lagrange algorithm for reduced algebraic irrationalities*

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Abstract. In this paper the properties of Lagrange algorithm for expansion of algebraic number are refined. It has been shown that for reduced algebraic irrationalities the quantity of elementary arithmetic operations which needed for the computation of next incomplete quotient does not depend on the value of this incomplete quotient.

It is established that beginning with some index all residual fractions for an arbitrary reduced algebraic irrationality are the generalized Pisot numbers. An asymptotic formula for conjugate numbers to residual fractions is obtained.

The definition of generalized Pisot numbers differs from the definition of Pisot numbers by absence of the requirement to be integer.

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1 Introduction

The continued fraction expansion of algebraic irrationalities is one of the most difficult questions in the modern number theory. Various aspects of this theory can be seen in the papers [1–9, 11–13] Even in such developed theory as the theory of continued fractions of quadratic irrationalities one can find new interesting facts (see [10, 14]). The paper [17] describes the set of reduced algebraic irrationalities of n-th degree and asserts that this set has the property of rational convexity.

The aim of this paper is the refinement of properties of Lagrange algorithm for reduced algebraic irrationalities of n-th degree and for Pisot numbers in general case.

The case of the reduced algebraic irrationalities of n-th degree is very important for us. This case is connected with totally real algebraic fields of n-th degree which underly the construction of algebraic lattice used in quadrature formulas with weights in K. K. Frolov's method (see [5–7, 15, 16]).

2 Necessary definitions and facts

We begin with the definition of a reduced algebraic irrationality of n-th degree. Here we follow [8, 9, 17].

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Definition 1. Let

$$f(x)=\sum_{k=0}^{n} a_k x^k \in \mathbb{Z}[x], \quad a_n>0$$

be irreducible polynomial with integer coefficients [1] such that all its roots $\alpha^{(k)}$ ($k=1,2,\ldots,n$) are different real numbers satisfying the following condition

$$-1<\alpha^{(n)}<\ldots<\alpha^{(2)}<0, \quad \alpha^{(1)}>1.$$

The algebraic number $\alpha=\alpha^{(1)}$ is called a reduced algebraic irrationality of n-th degree.

Note that for minimal polynomial $f(x)$ that defines a reduced algebraic irrationality α of n-th degree we always have $a_0<0$, since $f(x)$ has only one root α belonging to $[0;\infty)$ and $f(x)>0$ for $x>\alpha$, so $f(0)<0$. Besides the following inequalities hold

$$\begin{gathered} a_n+a_{n-1}+\ldots+a_1+a_0=f(1)<0, \\ a_n-a_{n-1}+\ldots+(-1)^{n-1}a_1+(-1)^n a_0=(-1)^n f(-1)>0. \end{gathered}$$

For any real number α which is a reduced algebraic irrationality of n-th degree consider infinite continued fraction expansion

$$\alpha=\alpha_0=q_0+\cfrac{1}{q_1+\cfrac{1}{\ddots+\cfrac{1}{q_n+\cfrac{1}{\ddots}}}}=q_0+\cfrac{1}{q_1+\cfrac{1}{\ddots+\cfrac{1}{q_k+\cfrac{1}{\alpha_{k+1}}}}}.$$

As usually by P_k and Q_k we denote numerator and denominator of k-th order convergent of continued fraction and by α_k we denote its residual fraction of order k.

Thus $\alpha=\alpha_0$ and the equality

$$\alpha=\frac{\alpha_{k+1}P_k+P_{k-1}}{\alpha_{k+1}Q_k+Q_{k-1}}, \quad k\geq -1,$$

is valid if we assume as usually that $P_{-1}=1$, $P_{-2}=0$ and $Q_{-1}=0$, $Q_{-2}=1$.

It is easy to show that

$$\alpha_{k+1}=\frac{\alpha Q_{k-1}-P_{k-1}}{P_k-\alpha Q_k}, \quad k\geq -1.$$

[1] In particular, the irreducibility of a polynomial means that $(a_0,\ldots,a_n)=1$.

Lemma 1. *For an arbitrary reduced algebraic irrationality α of n-th degree its residual fractions α_1 is a reduced algebraic irrationality of n-th degree too that satisfies the irreducible polynomial*

$$f_1(x) = \sum_{k=0}^{n} a_{k,1} x^k \in \mathbb{Z}[x], \quad a_{n,1} > 0,$$

where

$$a_{k,1} = \frac{b_k}{d},\ d = (b_0, \ldots, b_n),\ b_k = -\sum_{m=n-k}^{n} a_m C_m^{m+k-n} q_0^{m+k-n}\ (0 \le k \le n).$$

Proof. See [8].

Theorem 1. *For an arbitrary reduced algebraic irrationality α of n-th degree all its residual fractions α_m are reduced algebraic irrationalities of n-th degree, satisfying the irreducible polynomials*

$$f_m(x) = \sum_{k=0}^{n} a_{k,m} x^k \in \mathbb{Z}[x], \quad a_{n,m} > 0,$$

where

$$a_{k,m} = \frac{b_{k,m}}{d_m}, \quad d_m = (b_{0,m}, \ldots, b_{n,m}),$$
$$b_{k,m} = -\sum_{l=n-k}^{n} a_{l,m-1} C_l^{l+k-n} q_{m-1}^{l+k-n} \quad (0 \le k \le n).$$

Proof. See [8].

Theorem 2. *An incomplete quotient q_k is uniquely defined as an integer which satisfies the following condition*

$$f_k(q_k) < 0, \quad f_k(q_k + 1) > 0.$$

Proof. See [8].

It is not hard to see that to compute q_k we need to calculate $O(\ln q_k)$ values of polynomial $f_k(x)$. Indeed, consider the sequence $f_k(1)$, $f_k(2)$, …, $f_k(2^m)$, $f_k(2^{m+1})$, where $m = [\log_2(q_k)]$. It is clear that $f_k(2^j) < 0$ for all $0 \le j \le m$ and $f_k(2^{m+1}) > 0$. Further using the method of interval bisection contract the segment $[2^m; 2^{m+1}]$ to the segment $[q_k; q_k + 1]$, that will require to compute yet m values of $f_k(x)$. $\square$

Here in fact Lagrange algorithm of expansion for algebraic irrationality of arbitrary degree $n \ge 2$ is described.

Theorem 1 is generalized to the case for continued fraction of arbitrary totally real algebraic irrationality α of degree n. First we shall show Lemma on the transformation of the roots.

Lemma 2. *Let*

$$f(x) = \sum_{k=0}^{n} a_k x^k \in \mathbb{Z}[x], \quad a_n > 0$$

be irreducible polynomial with integral coefficients such that all its roots $\alpha^{(k)}$ $(k = 1, 2, \ldots, n)$ *are different real number satisfying the following condition*

$$\alpha^{(n)} < \ldots < \alpha^{(2)} < \alpha^{(1)},$$

and for integer number q *the following inequalities hold:*

$$\begin{cases} \alpha^{(k)} < q & \text{for } k \geq k_0, \\ q < \alpha^{(k)} < q + 1 & \text{for } k_0 > k \geq k_1, \\ \alpha^{(k)} > q + 1 & \text{for } k_1 > k \geq 1. \end{cases}$$

Then the polynomial

$$g(x) = -f\left(q + \frac{1}{x}\right) \cdot x^n = \sum_{k=0}^{n} b_k x^k.$$

has roots $\beta^{(k)} = \frac{1}{\alpha^{(k)} - q}$ $(k = 1, 2, \ldots, n)$ *satisfying the following inequalities*

$$\begin{cases} \beta^{(k)} < 0 & \text{for } k \geq k_0, \\ 1 < \beta^{(k)} & \text{for } k_0 > k \geq k_1, \\ 0 < \beta^{(k)} < 1 & \text{for } k_1 > k \geq 1. \end{cases}$$

Proof. See [8].

Theorem 3. *For an arbitrary totally real algebraic irrationality* α *of* n*-th degree all its residual fractions* α_m *are reduced algebraic irrationalities of* n*-th degree beginning with some index* $m_0 + 1$.

Proof. See [8].

3 Refinement of Lagrange algorithm for reduced algebraic irrationalities

Denote by $\mathbb{PZ}_n[x]$ the set of all irreducible polynomials with integer coefficients of n-th degree considered in Definition 1.

Lemma 3. *If polynomial*

$$f_0(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \in \mathbb{PZ}_n[x]$$

and $\alpha^{(1)} > \alpha^{(2)} > \ldots > \alpha^{(n)}$ *are its roots, then for the continued fraction expansion*

$$\alpha^{(1)} = \alpha_0 = q_0 + \cfrac{1}{q_1 + \cfrac{1}{\ddots + \cfrac{1}{q_n + \cfrac{1}{\ddots}}}}$$

we have

$$\left[-\frac{a_{n-1}}{a_n}\right] \le q_0 < -\frac{a_{n-1}}{a_n} + n - 1. \tag{1}$$

Proof. Indeed, using Viete's formula we have

$$-\frac{a_{n-1}}{a_n} = \alpha^{(1)} + \alpha^{(2)} + \ldots + \alpha^{(n)}.$$

Since $\alpha^{(1)}$ is a reduced algebraic irrationality of degree n, then

$$-1 < \alpha^{(n)} < \ldots < \alpha^{(2)} < 0, \quad \alpha^{(1)} > 1.$$

So

$$-n + 1 < \alpha^{(2)} + \ldots + \alpha^{(n)} < 0$$

and

$$-\frac{a_{n-1}}{a_n} < \alpha^{(1)} < -\frac{a_{n-1}}{a_n} + n - 1.$$

Since $q_0 < \alpha^{(1)} < q_0 + 1$ we get the statement of Lemma. □

Revise Lemma 1.

Lemma 4. *For a reduced algebraic irrationality α of degree n its residual fraction α_1 is a reduced algebraic irrationality of n-th degree too that satisfies the irreducible polynomial*

$$f_1(x) = \sum_{k=0}^{n} a_{k,1} x^k \in \mathbb{Z}[x], \quad a_{n,1} > 0,$$

where

$$a_{k,1} = \frac{b_k}{d_0},\ d_0 = (b_0, \ldots, b_n),\ b_k = -\sum_{m=n-k}^{n} a_m C_m^{m+k-n} q_0^{m+k-n}\ (0 \le k \le n).$$

The polynomial $f_1(x)$ has the roots

$$\alpha_1^{(j)} = \frac{1}{\alpha^{(j)} - q_0} \quad (1 \le j \le n)$$

and the following equality holds:

$$f_1(x) = \frac{-f_0(q_0)}{d_0} \prod_{j=1}^{n} \left(x - \frac{1}{\alpha^{(j)} - q_0}\right) \in \mathbb{PZ}_n[x].$$

Proof. Consider the polynomial

$$g(x) = -x^n f\left(q_0 + \frac{1}{x}\right).$$

We have:

$$g(x) = -a_n \prod_{j=1}^{n} \left(q_0 x + 1 - \alpha^{(j)} x\right) =$$
$$= -a_n \prod_{j=1}^{n} \left(q_0 - \alpha^{(j)}\right) \prod_{j=1}^{n} \left(x - \frac{1}{\alpha^{(j)} - q_0}\right) =$$
$$= -f_0(q_0) \prod_{j=1}^{n} \left(x - \frac{1}{\alpha^{(j)} - q_0}\right)$$

and $\alpha_1 = \frac{1}{\alpha^{(1)} - q_0}$.

On the other hand

$$g(x) = -\sum_{j=0}^{n} a_j (q_0 x + 1)^j x^{n-j} = -\sum_{j=0}^{n} a_j \sum_{\nu=0}^{j} C_j^{\nu} q_0^{\nu} x^{n-j+\nu} =$$
$$= -\sum_{k=0}^{n} x^k \sum_{m=n-k}^{n} a_m C_m^{k+m-n} q_0^{k+m-n} = \sum_{k=0}^{n} b_k x^k,$$

where

$$b_k = -\sum_{m=n-k}^{n} a_m C_m^{k+m-n} q_0^{k+m-n} \in \mathbb{Z} \quad (0 \le k \le n).$$

Since $1 \le q_0 < \alpha^{(1)} < q_0 + 1$ we obtain

$$b_n = -\sum_{m=0}^{n} a_m q_0^m = -f_0(q_0) > 0,$$
$$\frac{1}{\alpha^{(1)} - q_0} > 1, \quad -1 < \frac{1}{\alpha^{(j)} - q_0} < 0 \quad (2 \le j \le n).$$

So for $d_0 = (b_0, \ldots, b_n)$ the polynomial $f_1(x) = \frac{1}{d_0} g(x) \in \mathbb{PZ}_n[x]$ and Lemma is completely proved. □

Theorem 4. *Let $\alpha = \alpha_0$ be a reduced algebraic irrationality of n-th degree satisfying the irreducible polynomial*

$$f_0(x) = \sum_{k=0}^{n} a_{k,0} x^k \in \mathbb{Z}[x], \quad a_{n,0} > 0.$$

And let a sequence of the polynomials $f_m(x)$ $(m \ge 1)$ and a sequence of natural numbers q_m $(m \ge 0)$ define the recurrence relations

$$f_{m-1}(q_{m-1}) < 0, \quad f_{m-1}(q_{m-1} + 1) > 0 \quad (m \ge 1), \tag{2}$$
$$\left[-\frac{a_{n-1,m-1}}{a_{n,m-1}}\right] \le q_{m-1} < -\frac{a_{n-1,m-1}}{a_{n,m-1}} + n - 1 \quad (m \ge 1), \tag{3}$$

$$f_m(x) = \sum_{k=0}^{n} a_{k,m} x^k \in \mathbb{Z}[x], \quad a_{n,m} > 0,$$

$$a_{k,m} = \frac{b_{k,m}}{d_{m-1}},\ d_{m-1} = (b_{0,m}, \dots, b_{n,m}),$$

$$b_{k,m} = -\sum_{\nu=n-k}^{n} a_{\nu,m-1} C_{\nu}^{\nu+k-n} q_{m-1}^{\nu+k-n}\ (0 \le k \le n). \tag{4}$$

Then:

(1) the polynomials $f_m(x)$ have the roots

$$\alpha_m^{(j)} = \frac{\alpha^{(j)} Q_{m-2} - P_{m-2}}{P_{m-1} - \alpha^{(j)} Q_{m-1}} \quad (1 \le j \le n); \tag{5}$$

(2)

$$f_m(x) = \frac{-f_{m-1}(q_{m-1})}{d_{m-1}} \prod_{j=1}^{n} \left(x - \alpha_m^{(j)}\right) \in \mathbb{PZ}_n[x];$$

(3) α has the following continued fraction expansion

$$\alpha = \alpha_0 = q_0 + \cfrac{1}{q_1 + \cfrac{1}{\ddots + \cfrac{1}{q_n + \cfrac{1}{\ddots}}}}.$$

Proof. The proof is by induction on m.

For $m = 1$ the statements of theorem are valid by Lemma 4 and the equalities $Q_0 = 1$, $P_0 = q_0$, $Q_{-1} = 0$ and $P_{-1} = 1$.

Suppose the statements are valid for $m \ge 1$, applying Lemma 4 to reduced algebraic irrationality $\alpha_m^{(1)}$ we get the statements (2) – (4).

Further we obtain

$$\alpha_{m+1}^{(j)} = \frac{1}{\alpha_m^{(j)} - q_m} = \frac{1}{\frac{\alpha^{(j)} Q_{m-2} - P_{m-2}}{P_{m-1} - \alpha^{(j)} Q_{m-1}} - q_m} = \frac{\alpha^{(j)} Q_{m-1} - P_{m-1}}{P_m - \alpha^{(j)} Q_m}$$

and the statement (5) holds.

By (5) numbers $\alpha_m^{(1)}$ are the residual fractions for α ($m = 0, 1, \dots$), so a sequence $q_0, q_1, \dots$ is a sequence of incomplete quotients for α. This completes the proof. $\square$

It is easy to show that we need to calculate $O(\ln n)$ values of $f_m(x)$ for the computation of q_m. Indeed, for $A = \left[-\frac{a_{n-1,m}}{a_{n,m}}\right]$ consider a sequence of $f_m(A)$, $f_m(A+1)$, $\dots$, $f_m(A+n-1)$ consisting of n members. It is clear that if $f_m(A+n-1) < 0$ then $q_m = A+n-1$. Otherwise using the method of interval bisection contract the segment $[A; A+n-1]$ to the segment $[q_m; q_m+1]$ that will require to compute yet $O(\ln n)$ values of $f_m(x)$.

Thus the new version of Lagrange algorithm for expansion of an algebraic irrationality of arbitrary degree $n \geq 2$ in the case of reduced algebraic irrationality of n-th degree has a new property: for the computation of next incomplete quotient of continued fraction expansion of this irrationality we need to calculate at most $O(\ln n)$ values of polynomial $f_m(x)$. Since for the computation of coefficients of a polynomial $f_m(x)$ via the coefficients of a polynomial $f_{m-1}(x)$ we need at most $O\left(n^2\right)$ elementary arithmetic operations then the quantity of operations for the computation of next incomplete quotient does not depend on the value of this incomplete quotient.

Make an essential remark. If we will not use the greatest common divisor d_{m-1} in formula (4), then all coefficients will be increased and time for practical realisation using symbolic arithmetic will increase too. The calculation of d_{m-1} requires additional time, but it is compensated by range extension for calculations of incomplete quotients. On the other hand, even establishing that $d_{m-1} = 1$ requires time which depends on the polynomial coefficients, but it does not depend on the value of incomplete quotient.

4 The case of generalized Pisot numbers

Now we give the definition of generalized Pisot numbers.

Definition 2. Let

$$f(x) = \sum_{k=0}^{n} a_k x^k \in \mathbb{Z}[x], \quad a_n > 0,$$

be an arbitrary irreducible polynomial with integer coefficients such that its roots $\alpha^{(k)}$ $(k = 1, 2, \ldots, n)$ satisfy the following conditions

$$|\alpha^{(j)}| < 1 \ (2 \leq j \leq n), \quad \alpha^{(1)} > 1.$$

The algebraic number $\alpha = \alpha^{(1)}$ is called a generalized Pisot number of n-th degree.

It is easy to see that the definition of generalized Pisot numbers differs from the definition of Pisot numbers by absence of the requirement to be integer.

Theorem 5. *Let $\alpha = \alpha_0$ be a real algebraic irrationality of n-th degree satisfying the irreducible polynomial*

$$f_0(x) = \sum_{k=0}^{n} a_{k,0} x^k \in \mathbb{Z}[x], \quad a_{n,0} > 0,$$

and α have the following continued fraction expansion

$$\alpha = \alpha_0 = q_0 + \cfrac{1}{q_1 + \cfrac{1}{\ddots + \cfrac{1}{q_n + \cfrac{1}{\ddots}}}}.$$

Let a sequence of the polynomials $f_m(x)$ $(m \geq 1)$ be defined by the recurrence relations

$$\begin{aligned} f_m(x) &= \sum_{k=0}^{n} a_{k,m} x^k \in \mathbb{Z}[x], \quad a_{n,m} > 0, \\ a_{k,m} &= \frac{b_{k,m}}{d_{m-1}},\ d_{m-1} = (b_{0,m}, \ldots, b_{n,m}), \\ b_{k,m} &= -\sum_{\nu=n-k}^{n} a_{\nu,m-1} C_{\nu}^{\nu+k-n} q_{m-1}^{\nu+k-n} \ \ (0 \leq k \leq n). \end{aligned} \tag{6}$$

Then:

(1) the polynomials $f_m(x)$ have the following roots

$$\alpha_m^{(j)} = \frac{\alpha^{(j)} Q_{m-2} - P_{m-2}}{P_{m-1} - \alpha^{(j)} Q_{m-1}} \quad (1 \leq j \leq n). \tag{7}$$

(2)

$$f_m(x) = \frac{-f_{m-1}(q_{m-1})}{d_{m-1}} \prod_{j=1}^{n} \left(x - \alpha_m^{(j)} \right). \tag{8}$$

(3) beginning with some index m_0 all residual fractions $\alpha_m^{(1)}$ are generalized Pisot numbers $(m \geq m_0)$.

Proof. Consider a sequence of the polynomials

$$g_m(x) = -x^n f_{m-1}\left(q_{m-1} + \frac{1}{x} \right) \quad (m \geq 1).$$

Repeating arguments of Lemma 4 and Theorem 4 we get (7) and (8).

To prove the last statement of Theorem, transforming expression (7) we obtain:

$$\alpha_m^{(j)} = \frac{Q_{m-2}}{Q_{m-1}} \frac{\alpha^{(j)} - \frac{P_{m-2}}{Q_{m-2}}}{\frac{P_{m-1}}{Q_{m-1}} - \alpha^{(j)}} \quad (1 \leq j \leq n). \tag{9}$$

For $j = 1$ we have the obvious inequality $\alpha_m^{(1)} > 1$ using the definition of a residual fraction.

Let $2 \leq j \leq n$, then

$$\alpha_m^{(j)} = \frac{Q_{m-2}}{Q_{m-1}} \left(-1 + \frac{\frac{P_{m-1}}{Q_{m-1}} - \frac{P_{m-2}}{Q_{m-2}}}{\frac{P_{m-1}}{Q_{m-1}} - \alpha^{(j)}} \right) = \frac{Q_{m-2}}{Q_{m-1}} \left(-1 + \frac{\frac{(-1)^m}{Q_{m-1}Q_{m-2}}}{\frac{P_{m-1}}{Q_{m-1}} - \alpha^{(j)}} \right) =$$
$$= \frac{Q_{m-2}}{Q_{m-1}} \left(-1 + \frac{(-1)^m}{Q_{m-1}Q_{m-2}\left(\frac{P_{m-1}}{Q_{m-1}} - \alpha^{(j)}\right)} \right). \tag{10}$$

Since

$$\lim_{m\to\infty} \left| \frac{P_{m-1}}{Q_{m-1}} - \alpha^{(j)} \right| = \left| \alpha^{(1)} - \alpha^{(j)} \right|,$$

and all roots are distinct, it follows that

$$|\alpha_m^{(j)}| \leq \frac{Q_{m-2}}{Q_{m-1}} \left(1 + \frac{2}{Q_{m-1}Q_{m-2}\delta} \right) = \frac{Q_{m-2}}{Q_{m-1}} + \frac{2}{Q_{m-1}^2\delta} < 1, \tag{11}$$

for $m > m_0$, where

$$\delta = \min_{2\leq j\leq n} \left| \alpha^{(1)} - \alpha^{(j)} \right| > 0.$$

By (11) we obtain that beginning with index m_0 all residual fractions $\alpha_m^{(1)}$ are generalized Pisot numbers. This completes the proof. □

The importance of generalized Pisot numbers for Lagrange algorithm of continued fraction expansion of an algebraic number is explained by the following generalization of Lemma 3.

Lemma 5. *If*

$$f_0(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 \in \mathbb{Z}[x]$$

is a minimal polynomial for generalized Pisot number $\alpha^{(1)} = \alpha_0$, then for the continued fraction expansion

$$\alpha^{(1)} = \alpha_0 = q_0 + \cfrac{1}{q_1 + \cfrac{1}{\ddots + \cfrac{1}{q_n + \cfrac{1}{\ddots}}}}$$

the following inequality holds

$$\left[-\frac{a_{n-1}}{a_n} \right] + 1 - n \leq q_0 < -\frac{a_{n-1}}{a_n} + n - 1. \tag{12}$$

Proof. Indeed, using Viete's formula we have:

$$-\frac{a_{n-1}}{a_n} = \alpha^{(1)} + \alpha^{(2)} + \ldots + \alpha^{(n)}.$$

Since a minimal polynomial $f_0(x)$ is irreducible it follows that

$$\alpha^{(2)} + \alpha^{(3)} + \ldots + \alpha^{(n)} \neq 0,$$

for otherwise we have $\alpha^{(1)} = -\frac{a_{n-1}}{a_n} \in \mathbb{Q}$ and get a contradiction with the irreducibility of minimal polynomial $f_0(x)$.

As $\alpha^{(1)}$ is a generalized Pisot number then

$$|\alpha^{(j)}| < 1 \quad (2 \leq j \leq n).$$

So

$$0 < |\alpha^{(2)} + \ldots + \alpha^{(n)}| < n - 1$$

and

$$-\frac{a_{n-1}}{a_n} + 1 - n < \alpha^{(1)} < -\frac{a_{n-1}}{a_n} + n - 1.$$

Since $q_0 < \alpha^{(1)} < q_0 + 1$ we obtain the statement of Lemma. □

Thus, from Theorem 5 and Lemma 5 it follows that beginning with some index m_0 all incomplete quotients q_m $(m \geq m_0)$ require for their calculations at most $O(\ln n)$ computations of values of polynomial $f_m(x)$.

5 Conclusion

The results of this paper show that reduced algebraic irrationalities in the case of totally real algebraic fields and generalized Pisot numbers in general case play a fundamental role in the continued fraction expansion of algebraic irrationalities. Beginning with some index all residual fractions are the reduced algebraic numbers in the first case and generalized Pisot numbers in the second case.

The formulas (10) and (11) imply that beginning with some index m_0 a peculiar asymptotic formula for the conjugate to residual fractions takes place

$$\alpha_m^{(j)} = -\frac{Q_{m-2}}{Q_{m-1}} + O\left(\frac{2}{Q_{m-1}^2 \delta}\right).$$

Hence beginning with index m_0 some more powerful analog of Lemma 3 holds, which is valid for any real irrationality. The next article will be devoted to the study of this phenomenon.

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